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TILES WITH NO SPECTRA IN DIMENSION 4

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Abstract

We show by a counterexample that the "tiling $\Rightarrow$ spectral" part of Fuglede's Spectral Set Conjecture fails already in Z^4 and R^4.

1. Introduction

The *Spectral Set Conjecture* of Fuglede [1] relates the class of *tiling* sets of R^d to some Fourier analytic property, called *spectrality*. To be able to state the conjecture precisely we recall the appropriate setting. Let G be a locally compact Abelian group (we will only consider Z^d, R^d and finite commutative groups), the dual group is denoted by $\widehat{G}$. Once for all we fix a Haar-measure on G, and $\widehat{f}$ will stand for the Fourier transform of a function $f : G \to \mathsf{C}$. $Z(f)$ denotes the zero set of the function f. Further we use the notation χ_T for the characteristic function of the set $T \subseteq G$.

Definition. An open set $T \subseteq G$ is called *spectral* with spectrum $L \subseteq \widehat{G}$ if L is a complete orthogonal system in $L_2(T)$.

Definition. An open subset T of G is said to be a *tiling set* (or simply *tile*), if the whole group G can be covered by translated disjoint copies of T up to a set of zero measure. That is there exists a set $T' \subseteq G$, called a *tiling complement of* T such that $T' + T$ is the whole of G except a set of zero measure and for all $t \neq s$, $t, s \in T'$ we have $(t + T) \cap (s + T) = \emptyset$.

Remark 1. It is easy to see – and will be used throughout – that the latter *packing condition* is equivalent to $(T - T) \cap (T' - T') = \{0\}$. In fact, for a finite group G tiling is equivalent to $|G| = |T| \cdot |T'|$ and $(T - T) \cap (T' - T') = \{0\}$.

Now, the Spectral Set Conjecture reads as follows.

A domain $\Omega \subseteq \mathsf{R}^d$ is spectral if and only if it can tile R^d by translations.

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Although there were many results supporting the conjecture (already Fuglede himself proved it in case the tiling complement or the spectrum is assumed to be a lattice), Tao [15] has recently come up with a counterexample, disproving the "spectral $\Rightarrow$ tiling" part in dimension 5 and higher. Matolcsi [11] has reduced this dimension to 4, and later Kolountzakis and Matolcsi [6] disproved this part in dimension 3. They also clarified a method that could be used to give counterexamples in lower dimensions. Concerning the other, "tiling $\Rightarrow$ spectral" direction of the conjecture Kolountzakis and Matolcsi [7] have given a counterexample in dimension larger or equal to 5. Our aim is to prove

THEOREM 1. *There exists a tiling set in R^4 which is not spectral.*

The constructions of Tao [15] and Kolountzakis, Matolcsi [7] are based on examples in finite commutative groups. Let us describe the now automatic transition mechanism of transferring a counterexample from a finite Abelian group to Z^d and R^d by quoting the following two results of Kolountzakis and Matolcsi from [7]. (Hereafter Z_n denotes the cyclic group of n elements, for convenience regarded as $\mathsf{Z}/n\mathsf{Z}$.)

THEOREM 2 (Kolountzakis-Matolcsi). *Let $\mathbf{n} = (n_1, \ldots, n_d) \in \mathsf{N}^d$ and consider a set $A \subseteq G = \mathsf{Z}_{n_1} \times \cdots \times \mathsf{Z}_{n_d}$. For the set*

(1)
$$T = T(\mathbf{n}, k) = \{0, n_1, 2n_1, \ldots, (k-1)n_1\} \times \cdots \times \{0, n_d, 2n_d, \ldots, (k-1)n_d\}$$

define $B(k) = A + T$. Then, for large enough values of k, the set $B(k) \subset \mathsf{Z}^d$ is spectral in Z^d if and only if A is spectral in G.

THEOREM 3 (Kolountzakis-Matolcsi). *Suppose $B \subseteq \mathsf{Z}^d$ is a finite set and $Q = (0, 1)^d$. Then B is a spectral set in Z^d if and only if $B + Q$ is a spectral set in R^d.*

Note that obviously in the above constructions for a tile $A \subset G$ we must also have that $B = B(k) \subset \mathsf{Z}^d$ tiles Z^d (for any $k \in \mathsf{N}$) and for $B \subset \mathsf{Z}^d$ tiling Z^d also $B + Q$ tiles R^d. Whence it is now straightforward that our task is reduced to exhibit a counterexample in a finite group G.

2. Proof of the result

We are going to prove Theorem 1 at the end of this section. First we start by constructing a counterexample in a finite group, which indeed suffices, as described in the introduction.

To exhibit a counterexample in R^4, we follow the idea of Kolountzakis and Matolcsi [7], which is based on arguments in Z_6^5 and the extension of the finite counterexamples to Z^5 and R^5. However, to go down with the dimension to 4, we have to modify the starting point, and to construct an example of "tiling $\not\Rightarrow$ spectral" first in the group Z^4, based on considerations in Z_6^4.

When working with $d \times r$ matrices over a finite commutative group G, the column and row vectors are regarded as elements of G^d and $\widehat{G^r}$, respectively. Particularly for cyclic groups $G = \mathsf{Z}_n$ the duality pairing between G^d and $\widehat{G^d}$ in this identification takes the following form

$$\gamma(g) = e^{\frac{2\pi i}{n}\gamma \cdot g} \qquad \text{for} \quad g \in G^d = \mathsf{Z}_n^d, \gamma \in \widehat{G^d} = (\mathsf{Z}_n^d)^\top.$$

We will also "identify" any matrix with the set of its columns or rows; the meaning should be obvious from the context. For example, consider the mod 6 matrices

$$T := \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix} \qquad \text{and} \qquad L := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 4 \\ 2 & 0 & 4 & 4 \\ 2 & 4 & 0 & 2 \\ 4 & 4 & 2 & 0 \\ 4 & 2 & 4 & 2 \end{pmatrix}.$$

Then $T \subseteq \mathsf{Z}_6^4$ is a spectral set with spectrum $L \subseteq \widehat{\mathsf{Z}_6^4}$. This is so because $L \cdot T = K$ holds mod 6, with

$$K := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 & 4 & 4 \\ 0 & 2 & 0 & 4 & 4 & 2 \\ 0 & 2 & 4 & 0 & 2 & 4 \\ 0 & 4 & 4 & 2 & 0 & 2 \\ 0 & 4 & 2 & 4 & 2 & 0 \end{pmatrix},$$

and $\frac{1}{6}K$ (considered now as a matrix of real numbers) is a log-Hadamard matrix. (Recall that Matolcsi [11] calls a square matrix $H = [h_{jk}]_{j,k=1,\dots,n}$ a *log-Hadamard matrix* if the entrywise exponential of $2\pi i H$, that is, $[e^{2\pi i h_{jk}}]_{j,k=1,\dots,n}$, is a complex Hadamard matrix, i.e., a complex matrix with orthogonal rows and all entries having absolute value 1). The matrix K first appeared in the context of the Spectral Set Conjecture in Tao [15]. Later, Kolountzakis and Matolcsi [7] used it to construct a counterexample to the "tiling $\Rightarrow$ spectral" part of Fuglede's conjecture, and also the above decomposition (originally mod 3) was utilized in [11] to bring down the dimension in the disproof of the other, "spectral $\Rightarrow$ tiling" direction of the conjecture.

In finite groups G there is a very straightforward way of justifying that a subset $T \subseteq G$ is not a tile. Namely, if the number of elements $|T|$ does not divide the order $|G|$ of the group, then T cannot be a tile. Unfortunately we have no such immediate evidence for being not spectral. However, a convenient reformulation of being a spectrum is the following.

PROPOSITION 1 (Kolountzakis [5] p. 37, Kolountzakis–Matolcsi [7]). *The set* $S \subseteq \widehat{G}$ *is a spectrum of the set* $R \subseteq G$ *if and only if* $|S| = |R|$ *and* $S - S \subseteq Z(\widehat{\chi}_R) \cup \{0\}$.

Since we want to find a tile which is not spectral, we will use the above proposition together with a duality argument (see [7]).

LEMMA 2. *Let* $R \subseteq G$ *be a subset in* G *and suppose that there is a subset* $L \subseteq \widehat{G}$ *with* $|R|\cdot|L| = |G|$ *such that* L *is not a tile in* $\widehat{G}$ *and* $Z(\widehat{\chi}_R) \cap (L - L) = \emptyset$. *Then* R *can not be spectral.*

PROOF. If S was a spectrum of R, then $|S| = |R|$ and $S - S \subseteq Z(\widehat{\chi}_R) \cup \{0\}$ in view of Proposition 1, and hence the packing condition $(S - S) \cap (L - L) = \{0\}$ would hold. Since by condition we also have $|\widehat{G}| = |G| = |R| \cdot |L| = |S| \cdot |L|$, this packing condition and Remark 1 ensures that $S + L$ is in fact a *tiling* of $\widehat{G}$, which is impossible by assumption.

Therefore, ultimately, our goal is to establish the situation presented in the above lemma, i.e., to construct a *tiling* set $R \subset G$ together with a corresponding $L \subset \widehat{G}$ satisfying the above assumptions.

REMARK 2. Suppose that the conditions of Lemma 2 are fulfilled and moreover that R is tiling (this is what we are aiming at). Then for any tiling complement T of R we have $|R|\cdot|T| = |G|$ and also $\chi_{R+T} = \chi_G$, $\widehat{\chi}_G = \widehat{\chi}_R \cdot \widehat{\chi}_T$, thus $Z(\widehat{\chi}_R) \cup Z(\widehat{\chi}_T) \cup \{0\} = \widehat{G}$. Hence the assumption $Z(\widehat{\chi}_R) \cap (L - L) = \emptyset$ leads to $Z(\widehat{\chi}_T) \cup \{0\} \supseteq (L - L)$ and so by $|L| = |G|/|R| = |T|$ we find that L is a spectrum of T according to Proposition 1. That is, T is tiling with complement R, and is spectral with spectrum L, but also $Z(\widehat{\chi}_R) \cap (L - L) = \emptyset$ is satisfied. This shows that possible examples of R, T and L satisfying the condition in Lemma 2 have to be such that R is tiling with complement T whose spectrum is L.

So as a first step, we construct a set $T \subseteq \mathsf{Z}_6^4$ which is tiling and spectral with some spectrum L and further for each element $\mathbf{z} \in L^\top - L^\top$ ($\mathbf{z}$ is 4-dimensional column vector) there exists a tiling complement $R_\mathbf{z}$ of T such that $\mathbf{z}^\top \notin Z(\widehat{\chi}_{R_\mathbf{z}})$. Then with the help of these $R_\mathbf{z}$s and L, in the end we will construct a larger finite group $\mathscr{G}$, so that the above described situation will finally be achieved for some (other) $\mathscr{R}$ and $\mathscr{L}$.

Given a set T, the easiest way to produce a tiling complement of T is to apply the pull-back procedure described in the next lemma.

LEMMA 3 (Szegedy [14]). *Let* G *be a finite Abelian group,* $T \subseteq G$ *and suppose that there exists a homomorphism* $\varphi : G \to H$ *such that* φ *is injective* *on T and $\varphi(T)$ is a tile in H. Then T tiles also G, and a tiling complement is given by $\varphi^{-1}(\tilde{T})$ where $\tilde{T}$ is a complement of $\varphi(T)$.*

So we have to define a group homomorphism $\varphi : \mathsf{Z}_6^4 \to H$ with some group H such that $\varphi(T)$ tiles H and φ is injective on T. Then one can apply Lemma 3 to pull back the tiling complement of $\varphi(T)$ into Z_6^4 showing T to be a tile. Kolountzakis and Matolcsi [7] have applied this method with one-dimensional group homomorphisms $\varphi : \mathsf{Z}_6^5 \to \mathsf{Z}_6$, in connection with a 5-dimensional decomposition of the matrix K. Their construction led to a counterexample in dimension 5.

To reduce the dimension to 4 we need to give a suitable 4-dimensional decomposition of K. The above, most straightforward, choice $K = L \cdot T$ could be a good candidate, since as remarked T is spectral and also tiling. However, executing some calculations it turns out that in this case there exist some vectors $\mathbf{z} \in L^\top - L^\top$ for which there is no one-dimensional homomorphism producing a tiling complement R' of T such that it satisfies the above non-vanishing requirement $\widehat{\chi}_{R'}(\mathbf{z}^\top) \neq 0$.

Now, there are two possibilities, if we are sticking to Lemma 3. Either we look for non-one-dimensional homomorphisms or we choose a different T. Let us observe the instructive number theoretic reason of lacking such good one-dimensional φ-s: the last column of T is $0 \bmod 2$. Thus we modify the above T so that this obstacle vanishes. To do this, we will keep the above L and K and alter only T. Since there are only even entries in L, we can freely add 3 to any of the elements of T, while $K = L \cdot T \bmod 6$ will still hold, showing T to be spectral in Z_6^4 with the same spectrum L. First of all, we fix $T \bmod 3$ as in (2), so we have to specify it mod 2. Let

$$T := \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \mod 2, \tag{2}$$

and hence

$$T = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 5 \end{pmatrix} \mod 6. \tag{3}$$

We claim the following

LEMMA 4. *Consider $T \subseteq \mathsf{Z}_6^4$ given in (3). Then for all $\mathbf{z}^\top \in L - L$ we find a tiling complement $R_{\mathbf{z}}$ of T such that $\mathbf{z}^\top \notin Z(\widehat{\chi}_{R_{\mathbf{z}}})$ and $R_{\mathbf{z}}$ is a subgroup of G.*

Proof. Let us fix $\mathbf{z}^\top \in L - L$. Our idea, as described in the preceding discussion, is to produce the tiling complement $R_{\mathbf{z}}$ as $\ker\varphi$ for some one-dimensional homomorphism, i.e., we look for the homomorphism in the form $\varphi(\mathbf{x}) = \mathbf{v}^\top \cdot \mathbf{x}$ with some $\mathbf{v} \in \mathsf{Z}_6^4$ to be chosen appropriately (column vector, hence $\mathbf{v}^\top$ is a row vector). Then for the Fourier transform

$$\widehat{\chi}_{R_{\mathbf{z}}}(\mathbf{w}^\top) = \sum_{\mathbf{x}\in R_{\mathbf{z}}} e^{\frac{2\pi i}{6}\mathbf{w}^\top\cdot\mathbf{x}}, \tag{4}$$

and a choice of $\mathbf{v}$ satisfying $\alpha\mathbf{v} = \mathbf{z}$ with some $\alpha \in \mathsf{Z}_6$ will ensure

$$\widehat{\chi}_{R_{\mathbf{z}}}(\mathbf{z}^\top) = \sum_{\mathbf{x}\in R_{\mathbf{z}}} e^{\frac{2\pi i}{6}\alpha\mathbf{v}^\top\cdot\mathbf{x}} = \sum_{\mathbf{x}\in R_{\mathbf{z}}} e^{\frac{2\pi i}{6}\alpha\varphi(\mathbf{x})} = \sum_{\mathbf{x}\in R_{\mathbf{z}}} e^0 = |R_{\mathbf{z}}| > 0. \tag{5}$$

So the homomorphism φ should be given in such a way that $\mathbf{z}^\top$ becomes a scalar multiple of $\mathbf{v}^\top$.

To find a suitable $\mathbf{v}$ we let $\mathbf{k} := \mathbf{z}^\top \cdot T \in K - K$. Notice that $\mathbf{z}$ has even coordinates, and $\mathbf{k}$ is a permutation of $(0, 0, 2, 2, 4, 4) \bmod 6$. Now "divide" $\mathbf{k}$ by 2 (mod 6) (this is because of the above consideration with α); then for each entry we have two possibilities, as $0 = 2 \cdot 0 = 2 \cdot 3$, $2 = 2 \cdot 1 = 2 \cdot 4$, $4 = 2 \cdot 2 = 2 \cdot 5$.

So we fix $\mathbf{e}$ among the possible "halves" of $\mathbf{k}$ such that it will be a permutation of $(0, 1, 2, 3, 4, 5)$ and, moreover that the matrix equation $\mathbf{v}^\top \cdot T = \mathbf{e}$ has a solution mod 6 in $\mathbf{v}$. Actually, it is enough to solve $\mathbf{v}^\top \cdot T = \mathbf{e}$ mod 3 and mod 2, and then the mod 6 solution is easily recovered. These assumptions will ensure that the homomorphism $\varphi : \mathsf{Z}_6^4 \to \mathsf{Z}_6$ defined by $\mathbf{v}$ is surjective (hence tiling) and injective on T. Observe that $K - K$ consists of all the vectors with first coordinate 0 and the rest 5 coordinates being any permutation of $(0, 2, 2, 4, 4)$. Thus for any choice of $\mathbf{e}$, by $2\cdot 2 = 1 \bmod 3$ we will have $\mathbf{e} = 2\mathbf{k} \bmod 3$. That is, a solution $\mathbf{v}_3$ of $\mathbf{v}^\top \cdot T = \mathbf{e} \bmod 3$ undoubtedly exists, because $L \cdot T = K \bmod 3$, hence $2 \cdot (L - L) \cdot T$ covers $2(K - K)$ containing $\mathbf{e} = 2\mathbf{k} \bmod 3$. We show that with an appropriate choice of $\mathbf{e}$ one also finds a mod 2 solution. Clearly the first coordinate e_1 of $\mathbf{e} = (e_1, e_2, e_3, e_4, e_5, e_6)$ can be fixed as 0. Among the coordinates of $\mathbf{k}$ there are exactly two falling into each of the mod 3 classes. These we call *pairs*. Now we have to distinguish between these pairs mod 2. Notice that we can choose $\mathbf{e}$ such that among e_2, e_3 and e_4 exactly two are odd. Indeed, among these three elements there is either a pair from the same mod 3 class, or all three elements differ mod 3. In either case we can prescribe e_2, e_3 and e_4 such that $e_2 = 1 \bmod 2$, and e_3 and e_4 have different parity, while for the rest two coordinates e_5 and e_6 of $\mathbf{e}$ the only restriction is that the mod 3 pairs have to be mod 2 different. Choosing $\mathbf{e}$ in such a way and using (2) an easy calculation shows that $\mathbf{v}^\top \cdot T = (0, v_1, v_2, v_3, v_4, v_2 + v_3 + v_4) = \mathbf{e} \bmod 2$ has a solution $\mathbf{v}_2 \bmod 2$.

Now the desired $\mathbf{v}$ can be computed from $\mathbf{v}_2$ and $\mathbf{v}_3$ because the moduli are relatively primes.

It remains only to show that $\mathbf{z}^\top \notin Z(\widehat{\chi}_{R_\mathbf{z}})$, but this is obvious by construction. In fact, let $\mathbf{x} \in R_\mathbf{z}$: this means $\mathbf{v}^\top \cdot \mathbf{x} = 0$. On the other hand, $2\mathbf{v}^\top \cdot T = 2\mathbf{e}$ and $\mathbf{z}^\top \cdot T = \mathbf{k} = 2\mathbf{e}$. From this $2\mathbf{v} = \mathbf{z}$ follows, whence $0 = 2\mathbf{v}^\top \cdot \mathbf{x} = \mathbf{z}^\top \cdot \mathbf{x}$, so keeping (5) in mind gives $\widehat{\chi}_{R_\mathbf{z}}(\mathbf{z}^\top) > 0$.

REMARK 3. Let us make the above proof more comprehensible by means of a particular example of constructing $\mathbf{v}$ and the corresponding homomorphism. E.g., let $\mathbf{z}^\top := (0, 2, 2, 4) \in L - L$. Then $\mathbf{k} = \mathbf{z}^\top \cdot T = (0, 0, 2, 2, 4, 4) \bmod 6$. So $\mathbf{e} = \mathbf{k}/2 = (0, 0, 1, 1, 2, 2) \bmod 3$, and as described above we can choose $\mathbf{e} = (0, 1, 1, 0, 1, 0) \bmod 2$, resulting in $\mathbf{e} = (0, 3, 1, 4, 5, 2) \bmod 6$. The solution vectors $\mathbf{v}_2$ and $\mathbf{v}_3$ are $\mathbf{v}_2^\top = (1, 1, 0, 1) \bmod 2$ and $\mathbf{v}_3^\top = (0, 1, 1, 2) \bmod 3$, hence $\mathbf{v}^\top = (3, 1, 4, 5)$.

Using the above T, its tiling complements $R_\mathbf{z} := \ker \varphi$ (with the φ above *depending* on $\mathbf{z}$) and also its spectrum L, we are now in the position to construct our final counterexample to the "tiling ⇒ spectral" part of Fuglede's Conjecture in dimension 4.

PROOF OF THEOREM 1. Let $L^\top - L^\top = \{\mathbf{z}_j : j = 1, \dots, k\}$, say ($\mathbf{z}_j$ is a column vector). Take $\mathscr{L} \subseteq \widehat{\mathscr{G}} := \mathsf{Z}_6^4 \times \mathsf{Z}_p$ to be the set of the elements of L extended by a 0 in the fifth coordinate (i.e., considering $L \subseteq \widehat{G} \cong \widehat{G} \times \{0\} =: \widehat{\mathscr{G}}_0$ as imbedded into $\mathscr{G}$, which trivial identification – as well as the similar, dual imbedding of G into $\mathscr{G}$ – we do not mention further on). We put together the desired tiling but not spectral set from the above constructed tiling complements $R_j := R_{\mathbf{z}_j}$ of $T \times \{0\}$. So let $p \geq k$ be relatively prime to 6, and let us augment the sequence $R_1, \dots, R_k$ by listing the R_js and then repeating R_k additionally $p - k$ times. Consider the group $\mathscr{G} = \mathsf{Z}_6^4 \times \mathsf{Z}_p$ (which is, on the other hand, isomorphic to $\mathsf{Z}_6^3 \times \mathsf{Z}_{6p}$) and the set

$$\mathscr{R} = \bigcup_{j=1}^{p} \left(R_j + (0, 0, 0, 0, j)^\top\right).$$

Consider now the sets $\mathscr{R}$ and $\mathscr{L}$. First, $\mathscr{R} + T \times \{0\}$ is a tiling, as for all $j = 1, \dots, p$ $\left(R_j + (0, 0, 0, 0, j)^\top\right) + T \times \{0\}$ is a tiling of the translated subgroups $\mathscr{G}_0 + (0, 0, 0, 0, j)^\top$ of $\mathscr{G}_0 := \mathsf{Z}_6^4 \times \{0\}$. Hence $\mathscr{R}$ is a tile of $\mathscr{G}$ with the tiling complement $T \times \{0\}$.

Moreover, L is a spectrum of T, hence we get $|\mathscr{L}| = |L| = |T| = |\mathscr{G}|/|\mathscr{R}|$. (It can also bee seen easily that $\mathscr{L}$ is a spectrum of $T \times \{0\}$, but we do not need this here.) We need to show that also $Z(\widehat{\chi}_{\mathscr{R}}) \cap (\mathscr{L} - \mathscr{L}) = \emptyset$. So let $0 \neq \mathbf{z} \in \mathscr{L}^\top - \mathscr{L}^\top$ be any element; it corresponds to $\mathbf{z}_j$ for some index $j \leq p$. Then the Fourier transform of $\chi_{\mathscr{R}}$ evaluated at $\mathbf{z}^\top$ is

$$\widehat{\chi}_{\mathscr{R}}(\mathbf{z}^\top) = \widehat{\chi}_{R_1}(\mathbf{z}^\top) + \cdots + \widehat{\chi}_{R_{k-1}}(\mathbf{z}^\top) + (p - k + 1)\widehat{\chi}_{R_k}(\mathbf{z}^\top) > 0,$$

because all the terms are non-negative (all R_ms being subgroups), and by construction the j^{th} term is strictly positive in view of Lemma 4. So $\mathscr{R}$ and $\mathscr{L}$ fulfill the initial requirements for a pair of sets for a counterexample.

Furthermore, $\mathscr{L}$ is not a tile. To see this note that $\mathscr{L} \subset \widehat{\mathscr{G}}_0$, hence $\mathscr{L}$ can be a tile if only it tiles also the subgroup $\widehat{\mathscr{G}}_0$, that is, if L tiles $\widehat{G}$. But since L consists of vectors with all coordinates even, it is in fact a subset of the subgroup $E \leq \widehat{G}$ with even coordinates, hence in order to tile $\widehat{G}$, it has to tile even E. However, this is not possible since $|L| = 6$, which does not divide $|E| = 3^4$. Thus we see that the sets $\mathscr{R}$ and $\mathscr{L}$ provide all the properties of the construction we were aiming at, whence $\mathscr{R}$ is tiling $\mathscr{G}$ while being non spectral.

Having a counterexample in $\mathscr{G} \cong \mathsf{Z}_6^3 \times \mathsf{Z}_{6p}$, the counterexample in Z^4 and R^4 is obtained by an application of Theorems 2 and 3.

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