MATH. SCAND. 98 (2006), 262–281

HOLOMORPHIC FOCK SPACES FOR POSITIVE LINEAR TRANSFORMATIONS

R. FABEC, G. ÓLAFSSON, and A. N. SENGUPTA*

Abstract

Suppose A is a positive real linear transformation on a finite dimensional complex inner product space V. The reproducing kernel for the Fock space of square integrable holomorphic functions on V relative to the Gaussian measure $d\mu_A(z) = \frac{\sqrt{\det A}}{\pi^n} e^{-\operatorname{Re}\langle Az,z\rangle}\,dz$ is described in terms of the linear and antilinear decomposition of the linear operator A. Moreover, if A commutes with a conjugation on V, then a restriction mapping to the real vectors in V is polarized to obtain a Segal-Bargmann transform, which we also study in the Gaussian-measure setting.

Introduction

The classical Segal-Bargmann transform is an integral transform which defines a unitary isomorphism of $L^2(\mathsf{R}^n)$ onto the space $\mathsf{F}(\mathsf{C}^n)$ of entire functions on C^n which are square integrable with respect to the Gaussian measure $\mu = \pi^{-n} e^{-\|z\|^2} dxdy$, where $dxdy$ stands for the Lebesgue measure on $\mathsf{R}^{2n} \simeq \mathsf{C}^n$, see [1], [3], [4], [5], [11], [12]. There have been several generalizations of this transform, based on the heat equation or the representation theory of Lie groups [6], [10], [13]. In particular, it was shown in [10] that the Segal-Bargmann transform is a special case of *the restriction principle*, i.e., construction of unitary isomorphisms based on the polarization of a restriction map. This principle was first introduced in [10], see also [9], where several examples were explained from that point of view. In short the restriction principle can be explained in the following way. Let M_{C} be a complex manifold and let $M \subset M_{\mathsf{C}}$ be a totally real submanifold. Let $\mathsf{F} = \mathsf{F}(M_{\mathsf{C}})$ be a Hilbert space of holomorphic functions on M_{C} such that the evaluation maps $\mathsf{F} \ni F \mapsto F(z) \in \mathsf{C}$ are continuous for all $z \in M_{\mathsf{C}}$, i.e., F is a *reproducing kernel Hilbert space*. There exists a function $K : M_{\mathsf{C}} \times M_{\mathsf{C}} \to \mathsf{C}$ holomorphic in the first variable, anti-holomorphic in the second variable, and such that the following hold:

(a) $K(z, w) = \overline{K(w, z)}$ for all $z, w \in M_{\mathsf{C}}$;

*The research of G. Ólafsson was supported by DMS-0070607, DMS-0139473 and DMS-0402068. The research of A. Sengupta was supported by DMS-0201683. The authors would like to thank the referee for valuable comments and remarks.

Received January 31, 2004; in revised form March 20, 2005

(b) If $K_w(z) := K(z, w)$ then $K_w \in \mathsf{F}$ and

$$F(w) = \langle F, K_w \rangle_{\mathsf{F}}, \quad \forall F \in \mathsf{F}, z \in M_{\mathsf{C}}.$$

The function K is the *reproducing kernel* for the Hilbert space F. Let $D : M \to \mathsf{C}^*$ be measurable. Then the restriction map $R : F \mapsto RF := DF|_M$ is injective. Assume that there is a measure μ on M such that $RF \in L^2(M, \mu)$ for all F in a dense subset of F. Assuming that R is closeable, $\mathrm{Im}(R)$ is dense in $L^2(M, \mu)$, and by polarizing R^*, we can write

$$R^* = U|R^*|$$

where $U : L^2(M, \mu) \to \mathsf{F}$ is a unitary isomorphism and $|R^*| = \sqrt{RR^*}$. Using the fact that F is a reproducing kernel Hilbert space we get

$$Uf(z) = \langle Uf, K_z \rangle_{\mathsf{F}} = \langle f, U^* K_z \rangle_{L^2} = \int_M f(m) \overline{(U^* K_z)(m)} \, d\mu(m).$$

Thus U is always an integral operator. We notice also that the formula for U shows that the important object in this analysis is the reproducing kernel $K(z, w)$.

We will use the following notation through this article: Let $\langle z, w \rangle = z_1 \overline{w_1} + \cdots + z_n \overline{w_n}$ be the standard inner product on C^n and let $(z, w) = \mathrm{Re}(\langle z, w \rangle)$ be the corresponding inner product on C^n viewed as a $2n$-dimensional *real* vector spaces. Notice that $(x, y) = \langle x, y \rangle = x_1 y_1 + \cdots + x_n y_n$ for $x, y \in \mathsf{R}^n$. We write $z^2 = z_1^2 + \cdots + z_n^2$ for $z \in \mathsf{C}^n$.

The reproducing kernel for the classical Fock space is given by $K(z, w) = e^{\langle z, w \rangle}$, where $z, w \in \mathsf{C}^n$. By taking $D(x) := (2\pi)^{-n/4} e^{-x^2/2}$, for $x \in \mathsf{R}^n$, which is closely related to the heat kernel, we arrive at the classical Segal-Bargmann transform, given as the holomorphic continuation of

$$Ug(x) = (2/\pi)^{n/4} e^{x^2/2} \int_{\mathsf{R}^n} g(y) e^{-(x-y)^2} \, dy.$$

Notice that $\mathsf{R}^n \ni x \mapsto Ug(x) \in \mathsf{C}$ has a unique holomorphic extension to C^n.

The same principle can be used to construct the Hall-transform for compact Lie groups, [6]. In [2], Driver and Hall, motivated by application to quantum Yang-Mills theory, introduced a Fock space and Segal-Bargmann transform depending on two parameters $r, s > 0$, giving different weights to the x and y directions, where $z = x + iy \in \mathsf{C}^n$ (this was also studied in [13]). Thus F is now the space of holomorphic functions $F(z)$ on C^n which are square-integrable with respect to the Gaussian measure $dM_{r,s}(z) = \frac{1}{(\pi r)^{n/2} (\pi s)^{n/2}} e^{-\frac{x^2}{r} - \frac{y^2}{s}} \, dx dy$. A Segal-Bargmann transform for this Fock space is given in [13] and in Theorem 3 of [7]. We show this is a very special case of a larger family of Fock spaces and associated Segal-Bargmann tranforms. Indeed, if A is a real linear positive definite matrix A on a complex inner product space V, then

$$d\mu_A(z) = \frac{\sqrt{\det(A)}}{\pi^n} e^{-(Az,z)}\,|dz| \tag{0.1}$$

gives rise to a Fock space F_A. We find an expression for the reproducing kernel $K_A(z, w)$. We use the restriction principle to construct a natural generalization of the Segal-Bargmann transform for this space, with a certain natural restriction on A. We study this also in the Gaussian setting, and indicate a generalization to infinite dimensions.

We will fix the following notation for the types of bilinear pairings that we shall be using in this paper:

(i) $\langle z, w\rangle$ denotes a Hermitian inner product on a complex vector space V, i.e., a pairing which is complex-linear in z, complex-conjugate-linear in w, and $\langle z, z\rangle > 0$ if $z \neq 0$. We denote by $\|z\| = \sqrt{\langle z, z\rangle}$ the corresponding norm;

(ii) (x, y) denotes an inner product on V viewed as a *real* vector space. The standard choice is $(x, y) = \mathrm{Re}\langle x, y\rangle$;

(iii) $z \cdot w$ denotes a complex-bilinear pairing. In the standard situation we have $z \cdot w = \langle z, \overline{w}\rangle$. We set $z^2 = z \cdot z$.

1. The Fock space and the restriction principle

In this section we recall some standard facts about the classical Fock space of holomorphic functions on C^n. We refer to [5] for details and further information. Let μ be the measure $d\mu(z) = \pi^{-n} e^{-\|z\|^2} dxdy$ and let F be the classical Fock-space of holomorphic functions $F : \mathsf{C}^n \to \mathsf{C}$ such that

$$\|F\|_{\mathsf{F}}^2 := \int_{\mathsf{C}^n} |F(z)|^2\, d\mu(z) < \infty\,.$$

The space F is a reproducing Hilbert space with inner product

$$\langle F, G\rangle_{\mathsf{F}} = \int_{\mathsf{C}^n} F(z)\overline{G(z)}\, d\mu(z)$$

and reproducing kernel $K(z, w) = e^{\langle z,w\rangle}$.

Thus

$$F(w) = \int_{\mathsf{C}^n} F(z)\overline{K(z, w)}\, d\mu(z) = \langle F, K_w\rangle_{\mathsf{F}}$$

where $K_w(z) = K(z, w)$. The function $K(z, w)$ is holomorphic in the first variable, anti-holomorphic in the second variable, and $K(z, w) = \overline{K(w, z)}$. Notice that $K(z, z) = \langle K_z, K_z \rangle_{\mathsf{F}}$. Hence $\|K_z\|_{\mathsf{F}} = e^{\|z\|^2/2}$. Finally the linear space of finite linear combinations $\sum c_j K_{z_j}$, $z_j \in \mathbb{C}^n$, $c_j \in \mathbb{C}$, is dense in F. An orthonormal system in F is given by the monomials $e_\alpha(z) = z_1^{\alpha_1} \cdots z_n^{\alpha_n} / \sqrt{\alpha_1! \cdots \alpha_n!}$, $\alpha \in \mathbb{N}_0^n$. If the reference to the Fock space is clear, then we simply write $\langle \cdot, \cdot \rangle$ instead of $\langle \cdot, \cdot \rangle_{\mathsf{F}}$, and similarly for the corresponding norm.

View $\mathbb{R}^n \subset \mathbb{C}^n$ as a totally real submanifold of $\mathbb{C}^n$. We will now recall the construction of the classical Segal-Bargmann transform using the *restriction principle*, see [9], [10]. For constructing a restriction map as explained in the introduction we need to choose the function $D(x)$. One motivation for the choice of D is the heat kernel, but another one, more closely related to representation theory, is that the restriction map should commute with the action of $\mathbb{R}^n$ on the Fock space and $L^2(\mathbb{R}^n)$. Indeed, take

$$T(x)F(z) = m(x, z)F(z - x)$$

for F in F where $m(x, z)$ has properties sufficient to make $x \mapsto T(x)$ a unitary representation of $\mathbb{R}^n$ on F. Namely, we need a multiplier m satisfying

$$|m(x, z)| = \left(\frac{d\mu(z - x)}{d\mu(z)} \right)^{\frac{1}{2}} = e^{(\mathrm{Re}\langle z, x \rangle - x^2/2)} .$$

We take $m(x, z) := e^{\langle z, x \rangle - x^2/2}$. Set

$$D(x) = (2\pi)^{-n/4} m(0, x) = (2\pi)^{-n/4} e^{-x^2/2}$$

and define $R : \mathsf{F} \to C^\infty(\mathbb{R}^n)$ by

$$RF(x) := D(x)F(x) = (2\pi)^{-n/4} e^{-x^2/2} F(x).$$

Then

$$RT(y)F(x) = RF(x - y).$$

Since F is holomorphic, the map R is injective. Furthermore, the holomorphic polynomials $p(z) = \sum a_\alpha z^\alpha$ are dense in F and obviously $Rp \in L^2(\mathbb{R}^n)$. Thus, we may and will consider R as a densely defined operator from F into $L^2(\mathbb{R}^n)$. The Hermite functions $h_\alpha(x) = (-1)^{|\alpha|} \left(D^\alpha e^{-\|x\|^2} \right) e^{x^2/2}$ are images under R of polynomials and thus are in the image of the operator R. Hence, $\mathrm{Im}(R)$ is dense in $L^2(\mathbb{R}^n)$. Using continuity of the evaluation maps $F \mapsto F(z)$, it can be checked that R is a closed operator. Hence, R has a densely-defined adjoint

$R^* : L^2(\mathsf{R}^n) \to \mathsf{F}$. For $z, w \in \mathsf{C}^n$, recall that $z \cdot w = \sum z_j w_j$. Then, for any g in the domain of R^*, we have:

$$\begin{aligned} R^*g(z) = \langle R^*g, K_z\rangle = \langle g, RK_z\rangle &= (2\pi)^{-n/4} \int_{\mathsf{R}^n} g(y)e^{-\|y\|^2/2}e^{z\cdot y}\,dy \\ &= (2\pi)^{-n/4}e^{z^2/2} \int_{\mathsf{R}^n} g(y)e^{-(z-y)^2/2}\,dy \\ &= (2\pi)^{n/4}e^{z^2/2} g * p(z) \end{aligned}$$

where $p(z) = (2\pi)^{-n/2}e^{-z^2/2}$ is holomorphic. Applying the map $R : \mathsf{F} \to C^\infty(\mathsf{R}^n)$, we have

$$RR^*g(x) = g * p(x)\,. \tag{1.1}$$

Since $p \in L^1(\mathsf{R}^n)$ and $g \in L^2(\mathsf{R}^n)$, it follows that $g * p \in L^2(\mathsf{R}^n)$, and so the preceding equation shows that R^*g is in the domain of the operator R, and so g is in the domain of the operator composite RR^*. This argument also shows that RR^* is a bounded operator, on its domain, with operator norm $\|RR^*\|\|p\|_1$. Moreover, for every g in the domain of R^*, we have

$$\langle R^*g, R^*g\rangle = \langle RR^*g, g\rangle \le \|RR^*\|\,\|g\|_2\,.$$

Thus R^* is a bounded operator. Being an adjoint, it is also closed. Therefore, the domain of R^* is in fact the full space $L^2(\mathsf{R}^n)$. So for any $f \in D(R)$, we have

$$\langle Rf, Rf\rangle = \langle R^*(Rf), f\rangle \le \|R^*\|\,\|Rf\|_2\|f\|_2,$$

which implies that the operator R is bounded. Again, being a closed, densely-defined bounded operator, R is, in fact, defined on all of F. In summary,

LEMMA 1.1. *The linear operators R and R^* are everywhere defined and continuous.*

Let $p_t(x) = (2\pi t)^{-n/2}e^{-x^2/2t}$ be the heat kernel on R^n. Then $(p_t)_{t>0}$ is a convolution semigroup and $p = p_1$. Hence $\sqrt{RR^*} = p_{1/2}*$ or

$$RUg(x) = \left|R^*\right| g(x) = p_{1/2} * g(x) = \pi^{-n/2} \int_{\mathsf{R}^n} g(y)e^{-(x-y)^2}\,dy\,.$$

It follows that

$$Ug(x) = (2/\pi)^{n/4}e^{x^2/2} \int_{\mathsf{R}^n} g(y)e^{-(x-y)^2}\,dy$$

for $x \in \mathsf{R}^n$. But the function on the right hand side is holomorphic in x. Analytic continuation gives the following classical Segal-Bargmann tranform.

THEOREM 1.2. *The map* $U : L^2(\mathsf{R}^n) \to \mathsf{F}$ *given by*

$$Ug(z) = (2/\pi)^{n/4} \int_{\mathsf{R}^n} g(y) \exp(-y^2 + 2\langle z, y\rangle - z^2/2)\, dy$$

is a unitary isomorphism.

2. Twisted Fock spaces

Let $V \simeq \mathsf{C}^n$ be a finite dimensional complex vector space of complex dimension n and let $\langle \cdot, \cdot\rangle$ be a complex Hermitian inner-product.

We will also consider V as a real vector space with real inner product defined by $(z, w) = \mathrm{Re}\langle z, w\rangle$. Notice that $(z, z) = \langle z, z\rangle$ for all $z \in \mathsf{C}^n$. Let J be the real linear transformation of V given by $Jz = iz$. Note that $J^* = -J = J^{-1}$ and thus J is a skew symmetric real linear transformation. Fix a real linear transformation A. Then $A = H + K$ where

$$H := \frac{A + J^{-1}AJ}{2} \qquad \text{and} \qquad K := \frac{A - J^{-1}AJ}{2}.$$

We have $HJ = \frac{1}{2}(AJ - J^{-1}A) = \frac{1}{2}J(J^{-1}AJ + A) = JH$ and $KJ = \frac{1}{2}(AJ + J^{-1}A) = \frac{1}{2}J(J^{-1}AJ - A) = -JK$. Thus H is complex linear and K is conjugate linear.

We now assume that A is symmetric and positive definite relative to the real inner product $(\cdot, \cdot)$.

LEMMA 2.1. *The complex linear transformation H is self adjoint, positive with respect to the inner product $\langle \cdot, \cdot\rangle$, and invertible.*

PROOF. Since A is positive and invertible as a real linear transformation, we have $(Az, z) > 0$ for all $z \neq 0$. But J is real linear and skew symmetric. Hence $(JAJ^{-1}z, z) > 0$ for all $z \neq 0$. In particular $H = \frac{1}{2}(A + JAJ^{-1})$ is complex linear, symmetric with respect to the real inner product $(\cdot, \cdot)$, and positive. Consequently $\mathrm{Re}\langle Hiv, w\rangle = \mathrm{Re}\langle iv, Hw\rangle$. This implies $\mathrm{Im}\langle Hv, w\rangle = \mathrm{Im}\langle v, Hw\rangle$ and hence $\langle Hv, w\rangle = \langle v, Hw\rangle$. Thus H is complex self adjoint and $\langle Hz, z\rangle > 0$ for $z \neq 0$.

LEMMA 2.2. *Let $w \in V$. Then $\langle Aw, w\rangle = (Aw, w) + i\,\mathrm{Im}\langle Kw, w\rangle$ and $(Aw, w) = (Hw, w) + (Kw, w)$.*

PROOF. The first statement follows from

$$\begin{aligned}\langle Aw, w\rangle = \langle Hw, w\rangle + \langle Kw, w\rangle &= (Hw, w) + (Kw, w) + i\,\mathrm{Im}\langle Kw, w\rangle \\ &= (Aw, w) + i\,\mathrm{Im}\langle Kw, w\rangle.\end{aligned}$$

Taking the real part in the second line gives the second claim, which also follows directly from bilinearity of $(\cdot,\cdot)$.

Denote by $\det_{\mathsf{R}}(\cdot)$ the determinant of a R-linear map on $V \simeq \mathsf{C}^n \simeq \mathsf{R}^{2n}$ and by $\det(\cdot)$ the determinant of a complex linear map of V. Let μ_A be the measure defined by $d\mu_A(z) = \pi^{-n}\sqrt{\det_{\mathsf{R}} A}\, e^{-(Az,z)}dxdy$ and let F_A be the space of holomorphic functions $F : \mathsf{C}^n \to \mathsf{C}$ such that

$$\|F\|_A^2 := \int |F(z)|^2 \, d\mu_A(z) < \infty\,. \tag{2.1}$$

Our normalization of μ is chosen so that $\|1\|_A = 1$. Just as in the classical case one can show that F_A is a reproducing kernel Hilbert space, but this will also follow from the following Lemma. We notice that all the holomorphic polynomials $p(z)$ are in F. To simplify the notation, we let $T_1 = H^{-1/2}$. Then T_1 is symmetric, positive definite and complex linear. Let $c_A = \sqrt{\det_{\mathsf{R}}(A^{1/2}T_1)} = (\det_{\mathsf{R}}(A)/\det_{\mathsf{R}}(H))^{1/4}$.

LEMMA 2.3. *Let $F : V \to \mathsf{C}$ be holomorphic. Then $F \in \mathsf{F}_A$ if and only if $F \circ T_1 \in \mathsf{F}$. Moreover, the map $\Psi : \mathsf{F}_A \to \mathsf{F}$ given by*

$$\Psi(F)(w) := c_A \exp\left(-\overline{\langle KT_1w, T_1w\rangle}/2\right) F(T_1w)$$

is a unitary isomorphism. In particular

$$\Psi^* F(w) = \Psi^{-1}F(w) = c_A^{-1} \exp\left(\overline{\langle Kw, w\rangle}/2\right) F(\sqrt{H}w)\,.$$

PROOF. Note F is holomorphic if and only if $F \circ T_1$ is holomorphic as T_1 is complex linear and invertible. Moreover, unitarity follows from

$$\begin{aligned}
\|\Psi F\|^2 &= \pi^{-n} \int_V |\Psi F(w)|^2 \, e^{-\langle w,w\rangle} \, dw \\
&= \pi^{-n}\sqrt{\det{}_{\mathsf{R}} A} \int_V |F(w)|^2 e^{-(Kw,w)} e^{-\langle \sqrt{H}w, \sqrt{H}w\rangle} \, dw \\
&= \pi^{-n}\sqrt{\det{}_{\mathsf{R}} A} \int_V |F(w)|^2 e^{-(Kw,w)} e^{-\langle Hw,w\rangle} \, dw \\
&= \pi^{-n}\sqrt{\det{}_{\mathsf{R}} A} \int_V |F(w)|^2 e^{-((H+K)w,w)} \, dw \\
&= \pi^{-n}\sqrt{\det{}_{\mathsf{R}} A} \int_V |F(w)|^2 e^{-(Aw,w)} \, dw \\
&= \|F\|_A^2.
\end{aligned}$$

THEOREM 2.4. *The space* F_A *is a reproducing kernel Hilbert space with reproducing kernel*

$$K_A(z, w) = c_A^{-2} e^{\frac{1}{2}\overline{\langle Kz,z\rangle}} e^{\langle Hz,w\rangle} e^{\frac{1}{2}\langle Kw,w\rangle} .$$

PROOF. By Lemma 2.3 we get

$$\begin{aligned} c_A \exp(-\overline{\langle KT_1w, T_1w\rangle}/2)F(T_1w) &= \Psi(F)(w) \\ &= (\Psi(F), K_w)_{\mathsf{F}} \\ &= (F, \Psi^*(K_w))_{\mathsf{F}_A} . \end{aligned}$$

Hence

$$K_A(z, w) = c_A^{-1} \exp\left(\overline{\langle Kw, w\rangle}/2\right) \Psi^*(K_{\sqrt{H}w}) = c_A^{-2} e^{\frac{1}{2}\overline{\langle Kz,z\rangle}} e^{\langle Hz,w\rangle} e^{\frac{1}{2}\langle Kw,w\rangle}.$$

3. The Restriction Map

We continue to assume $A > 0$. We notice that Lemma 2.3 gives a unitary isomorphism $\Psi^*U : L^2(\mathsf{R}^n) \to \mathsf{F}_A$, where U is the classical Segal-Bargmann transform. But this is not the natural transform that we are looking for. As H is positive definite there is an orthonormal basis $e_1, \dots, e_n$ of V and positive numbers $\lambda_j > 0$ such that $He_j = \lambda_j e_j$. Let $V_{\mathsf{R}} := \sum \mathsf{R}e_k$. Set $\sigma(\sum a_i e_i) = \sum \bar{a}_i e_i$. Then σ is a conjugation with $V_{\mathsf{R}} = \{z : \sigma z = z\}$. For $z \in V$ we will when convenient write $\bar{z}$ for $\sigma(z)$. We say that a vector is *real* if it belongs to V_{R}. As $He_j = \lambda_j e_j$ with $\lambda_j \in \mathsf{R}$ it follows that $HV_{\mathsf{R}} \subseteq V_{\mathsf{R}}$. We note that for the complex linear mapping H that $\det_{\mathsf{R}} H = (\det H)^2$ and that $\det H$ is equal to the determinant of the real linear transformation $H|_{V_{\mathsf{R}}}$.

LEMMA 3.1. $\langle Kz, w\rangle = \langle Kw, z\rangle$.

PROOF. Note that σK is complex linear. Since $J^* = -J$, $K = \frac{1}{2}(A - JAJ^{-1})$ is real symmetric. Thus $(Kw, z) = (w, Kz) = (Kz, w)$. Also note

$$(iKz, w) = (JKz, w) = -(KJz, w) = -(Jz, Kw) = -(iz, Kw).$$

Hence $\mathrm{Re}\langle iKz, w\rangle = -\mathrm{Re}\langle iz, Kw\rangle$. So $-\mathrm{Im}\langle Kz, w\rangle = \mathrm{Im}\langle z, Kw\rangle$. This gives $\mathrm{Im}\langle Kw, z\rangle = \mathrm{Im}\langle Kz, w\rangle$. Hence $\langle Kz, w\rangle = \langle Kw, z\rangle$.

LEMMA 3.2. $(\sigma K)^* = K\sigma$.

PROOF. We have $\langle \sigma z, \sigma w\rangle = \langle w, z\rangle$. Hence

$$\langle \sigma Kz, w\rangle = \langle \sigma w, \sigma^2 Kz\rangle = \langle \sigma w, Kz\rangle = \langle z, K\sigma w\rangle .$$

COROLLARY 3.3. *If* $x, y \in V_{\mathsf{R}}$, *then* $\langle Hx, y\rangle$ *is real and* $\langle Ax, y\rangle = \langle Ay, x\rangle$.

PROOF. Clearly $\langle\cdot,\cdot\rangle$ is real on $V_{\mathsf{R}} \times V_{\mathsf{R}}$. Since $HV_{\mathsf{R}} \subseteq V_{\mathsf{R}}$, we see $\langle Hx, y\rangle$ is real. Next, $\langle Ax, y\rangle = \langle Hx, y\rangle + \langle Kx, y\rangle$. The term $\langle Hx, y\rangle$ equals $\langle Hy, x\rangle$ because $\langle Hx, y\rangle$ is real and H is self-adjoint. On the other hand, $\langle Kx, y\rangle = \langle Ky, x\rangle$ by Lemma 3.1. So $\langle Ax, y\rangle = \langle Ay, x\rangle$.

As before we would like to have a multiplier $m : V_{\mathsf{R}} \times V \to \mathsf{C}^*$ such that

$$T(x)F(z) = m(x, z)F(z - x)$$

is a unitary representation of F_A that commutes with translation on $L^2(V_{\mathsf{R}})$. It turns out the multipliers we construct are co-boundaries under translation by elements of V_{R} on V.

DEFINITION 3.4. A function m is a *co-boundary* on $V \cong \mathsf{C}^n$ under translation by V_{R} if there is a nonzero complex valued function b on V with

$$m(x, z) = b(z - x)b(z)^{-1} \qquad \text{for } x \in V_{\mathsf{R}} \text{ and } z \in V.$$

It is well known and easy to verify that every co-boundary m on V under translation by V_{R} is a multiplier.

LEMMA 3.5. *The function*

$$m(x, z) = e^{\langle Hz,x\rangle} e^{\langle K\bar{z},x\rangle} e^{-\langle Ax,x\rangle/2} = e^{\langle Ax,\bar{z}\rangle - \langle Ax,x\rangle/2}$$

is a co-boundary.

PROOF. Define $b(z) = e^{-\langle Hz+K\bar{z},\bar{z}\rangle/2}$. Then

$$\begin{aligned} b(z - x)b(z)^{-1} &= e^{-\langle H(z-x)+K(\bar{z}-x),\bar{z}-x\rangle/2} e^{\langle Hz+K\bar{z},\bar{z}\rangle/2} \\ &= e^{(\langle Hx+Kx,\bar{z}\rangle + \langle Hz+K\bar{z},x\rangle)/2} e^{-\langle Hx+Kx,x\rangle/2} \\ &= e^{\langle Hz,x\rangle} e^{\langle K\bar{z},x\rangle} e^{-\langle Ax,x\rangle/2} \\ &= e^{\langle Hx,\bar{z}\rangle + \langle Kx,\bar{z}\rangle} e^{-\langle Ax,x\rangle/2} \\ &= e^{\langle Ax,\bar{z}\rangle - \langle Ax,x\rangle/2} \end{aligned}$$

since $A = H + K$, $\langle Hx, \bar{z}\rangle = \langle z, \sigma Hx\rangle = \langle z, Hx\rangle = \langle Hz, x\rangle$, and $\langle Kx, \bar{z}\rangle = \langle \sigma\bar{z}, \sigma Kx\rangle = \langle K\sigma z, x\rangle = \langle K\bar{z}, x\rangle$.

COROLLARY 3.6. *Let* $m(x, z) = e^{\langle Ax,\bar{z}\rangle - \langle Ax,x\rangle/2}$. *Set* $T_x F(z) := m(x, z)F(z - x)$ *for* $x \in V_{\mathsf{R}}$. *Then* $x \mapsto T_x$ *is a representation of the abelian group* V_{R} *on* F_A. *It is unitary if and only if* $KV_{\mathsf{R}} \subseteq V_{\mathsf{R}}$, *or equivalently* $AV_{\mathsf{R}} \subseteq V_{\mathsf{R}}$.

PROOF. Since m is a multiplier, we have $T_x T_y = T_{x+y}$. For each T_x to be unitary, we need $|m(x,z)| = e^{(Az,x)-(Ax,x)/2}$. But

$$|m(x,z)| = e^{(Hz,x)} e^{(K\bar{z},x)} e^{-(Ax,x)/2} = e^{(Az,x)-(Ax,x)/2} e^{(K\bar{z}-Kz,x)}.$$

Thus T_x is unitary for all x if and only if the real part of every vector $K\bar{z} - Kz$ is 0. Since $\bar{z} - z$ runs over iV_{R} as z runs over V, T_x is unitary for all x if and only if $K(iV_{\mathsf{R}}) \subset iV_{\mathsf{R}}$, which is equivalent to $K(V_{\mathsf{R}}) \subset V_{\mathsf{R}}$. But since $A = H + K$ and H leaves V_{R} invariant, this is equivalent to V_{R} being invariant under A.

REMARK. There is no uniqueness in the choice of a real vector space V_{R} such that $HV_{\mathsf{R}} \subseteq V_{\mathsf{R}}$ and $V = V_{\mathsf{R}} \oplus iV_{\mathsf{R}}$. Indeed, any orthonormal basis $e_1, e_2, \dots, e_n$ of eigenvectors for H gives such a subspace. But since A is only real linear on V, an interesting question is when one can choose V_{R} with $AV_{\mathsf{R}} \subseteq V_{\mathsf{R}}$, and in this case how unique is the choice of V_{R}? This probably depends on the degree of non complex linearity of the tranformation A.

Recall that $\det_{\mathsf{R}} H = (\det H)^2$. To simplify some calculations later on we define $c := (2\pi)^{-n/4} \left(\frac{\det_{\mathsf{R}} A}{\det H}\right)^{1/4}$. We remark for further reference:

LEMMA 3.7. *$c_A^{-2} c^2 = \frac{\sqrt{\det H}}{(2\pi)^{n/2}}$ and $c^{-1} \frac{\sqrt{\det(H)}}{\pi^{n/2}} = \left(\frac{2}{\pi}\right)^{n/4} \frac{(\det H)^{3/4}}{(\det_{\mathsf{R}} A)^{1/4}}$.*

Let $D(x) = c\, m(x,0) = c\, e^{-\langle Ax,x\rangle/2}$ and define $R : \mathsf{F}_A \to C^\infty(V_{\mathsf{R}})$ by

$$RF(x) := D(x)F(x).$$

Since m is holomorphic on V^2, D has a holomorphic extension to V.

LEMMA 3.8. *The restriction map R intertwines the action of V_{R} on F_A and the left regular action L on functions on V_{R}.*

PROOF. For all $x, y \in V_{\mathsf{R}}$, we have

$$\begin{aligned} R(T_y F)(x) &= c\, m(x,0) T_y F(x) \\ &= c\, m(x,0) m(y,x) F(x-y) \\ &= c\, m(x,0) m(-y,-x) F(x-y) \\ &= c\, m(x-y,0) F(x-y) \\ &= L_y RF(x). \end{aligned}$$

4. The Generalized Segal-Bargmann Transform

As for the classical space, R specifies a densely defined closed operator $\mathsf{F}_A \to L^2(V_{\mathsf{R}})$. It also has dense image in $L^2(V_{\mathsf{R}})$. To see this, let $\{h_\alpha\}_\alpha$ be the orthonormal basis of $L^2(V_{\mathsf{R}})$ given by the Hermite functions. Then $\left\{\det(A)^{\frac{1}{4}} h_\alpha(\sqrt{A}x)\right\}_\alpha$

is an orthonormal basis of $L^2(V_{\mathsf{R}})$ which is contained in the image of the set of polynomial functions under R. It follows again that R has a densely defined adjoint and

$$R^*h(z) = \langle R^*h, K_{A,z}\rangle = \langle h, RK_{A,z}\rangle$$

where $K_{A,z}(w) = K_A(w,z) = c_A^{-2} e^{\frac{1}{2}\overline{\langle Kw,w\rangle}} e^{\langle Hw,z\rangle} e^{\frac{1}{2}\langle Kz,z\rangle}$. Thus

$$\begin{aligned}
R^*h(z) &= c \int h(x) e^{-\langle Ax,x\rangle/2} \overline{K_A(x,z)}\, dx \\
&= c_A^{-2} c \int h(x) e^{-\langle Ax,x\rangle/2} e^{\frac{1}{2}\overline{\langle Kz,z\rangle}} e^{\langle z,Hx\rangle} e^{\frac{1}{2}\langle Kx,x\rangle}\, dx \\
&= c_A^{-2} c\, e^{\frac{1}{2}\overline{\langle Kz,z\rangle}} \int h(x) e^{-\langle Hx,x\rangle/2} e^{-\langle Kx,x\rangle/2} e^{\langle z,Hx\rangle} e^{\frac{1}{2}\langle Kx,x\rangle}\, dx \\
&= c_A^{-1} c\, e^{\frac{1}{2}\overline{\langle Kz,z\rangle}} \int h(x) e^{-\langle x,Hx\rangle/2} e^{\langle z,Hx\rangle}\, dx \\
&= c_A^{-2} c\, e^{\frac{1}{2}\overline{\langle Kz,z\rangle}} e^{\frac{1}{2}\langle z,H\bar{z}\rangle} \int h(x) e^{-(\langle z,H\bar{z}\rangle - \langle z,Hx\rangle - \langle x,H\bar{z}\rangle + \langle x,Hx\rangle)/2}\, dx \\
&= c_A^{-2} c\, e^{\frac{1}{2}\overline{\langle Kz,z\rangle}} e^{\frac{1}{2}\langle z,H\bar{z}\rangle} \int h(x) e^{-\langle z-x,H(\bar{z}-\bar{x})\rangle/2}\, dx
\end{aligned}$$

for $\langle z, Hx\rangle = \langle \overline{Hx}, \bar{z}\rangle = \langle Hx, \bar{z}\rangle = \langle x, H\bar{z}\rangle$ and $\langle z, Hx\rangle = \langle z, H\bar{x}\rangle$. Thus we finally arrive at

$$R^*h(z) = c_A^{-2} c\; e^{\frac{1}{2}\langle z,H\bar{z}+Kz\rangle} e^{-\frac{1}{2}\langle x,H\bar{x}\rangle} * h(z). \tag{4.1}$$

Let $P : V_{\mathsf{R}} \to V_{\mathsf{R}}$ be positive. Define $\phi_P(x) = \sqrt{\det(P)}(2\pi)^{-n/2} e^{-\|\sqrt{P}x\|^2/2}$. For $t > 0$, let $P(t) = \frac{1}{t}P$.

LEMMA 4.1. *Let the notation be as above. Then $0 < t \mapsto \phi_{P(t)}$ is a convolution semigroup, i.e., $\phi_{P(t+s)} = \phi_{P(t)} * \phi_{P(s)}$.*

PROOF. This follows by change of parameters $y = \sqrt{P}x$ from the fact that $\phi_{\mathrm{Id}(t)}(x) = (2\pi t)^{-n/2} e^{-x^2/2t}$ is a convolution semigroup.

Define a unitary operator W on $L^2(V_{\mathsf{R}})$ by

$$Wf(x) = e^{i\,\mathrm{Im}\langle x,Kx\rangle} f(x) = e^{i\,\mathrm{Im}\langle x,Ax\rangle} f(x). \tag{4.2}$$

We see $W = I$ if $AV_{\mathsf{R}} \subseteq V_{\mathsf{R}}$ which is equivalent to the translation operators $T(x)$ being unitary.

LEMMA 4.2. *Let h be in the domain of definition of R^*. Then*

$$RR^*h = W(\phi_H * h).$$

PROOF. We notice first that $c_A^{-2}c^2 = (2\pi)^{-n/2}\sqrt{\det H}$ by Lemma 3.7. From (4.1) we then have

$$\begin{aligned}
RR^*h(x) &= c\, e^{-\frac{1}{2}\langle Ax,x\rangle} R^*h(x)\\
&= c_A^{-2}c^2\, e^{-\frac{1}{2}\langle Ax,x\rangle} e^{\frac{1}{2}\langle x,H\bar{x}+Kx\rangle} e^{-\frac{1}{2}\langle y,H\bar{y}\rangle} * h(x)\\
&= (2\pi)^{-n/2}\sqrt{\det(H)}\, e^{-\frac{1}{2}\langle Ax,x\rangle} e^{\frac{1}{2}\langle x,Ax\rangle} e^{-\frac{1}{2}\langle y,H\bar{y}\rangle} * h(x)\\
&= (2\pi)^{-n/2}\sqrt{\det(H)}\, e^{i\,\mathrm{Im}\langle x,Ax\rangle} \int e^{-\frac{1}{2}\langle y,Hy\rangle} h(x-y)\,dy.\\
&= (2\pi)^{-n/2}\sqrt{\det(H)} e^{i\,\mathrm{Im}\langle x,Ax\rangle} \int e^{-\frac{\|\sqrt{H}y\|^2}{2}} h(x-y)\,dy\\
&= W(\phi_H * h)(x).
\end{aligned}$$

In the last step, we used the fact that $\phi_H * h \in L^2(V_{\mathsf{R}})$, which follows from ϕ_H being in $L^1(V_{\mathsf{R}})$ and $h \in L^2(V_{\mathsf{R}})$.

Arguing as in the classical case, we see that R and R^* are everywhere defined and continuous.

Lemma 4.1 and Lemma 4.2 leads to the following corollary:

COROLLARY 4.3. *Suppose* $AV_{\mathsf{R}} \subseteq V_{\mathsf{R}}$. *Then*

$$|R^*|h(x) = \phi_{H(1/2)} * h(x) = \frac{\sqrt{\det(H)}}{\pi^{n/2}} \int_{V_{\mathsf{R}}} e^{-\|\sqrt{H}y\|^2} h(x-y)\,dy.$$

THEOREM 4.4 (The Segal-Bargmann Transform). *Suppose A leaves V_{R} invariant. Then the operator $U_A : L^2(V_{\mathsf{R}}) \to \mathsf{F}_A$ defined by*

$$U_A f(z) = \left(\frac{2}{\pi}\right)^{n/4} \frac{(\det H)^{3/4}}{(\det_{\mathsf{R}} A)^{1/4}} e^{\frac{1}{2}(\langle Hz,\bar{z}\rangle + \langle z,Kz\rangle)} \int e^{(H(z-y))\cdot(z-y)} f(y)\,dy$$

is a unitary isomorphism. We call the map U_A the generalized Segal-Bargmann transform.

PROOF. By polar decomposition, we can write $R^* = U\,|R^*|$ where $U : L^2(V_{\mathsf{R}}) \to \mathsf{F}_A$ is a unitary isomorphism. Taking adjoints gives $|R^*|\,U^* = R$. Hence $RU = |R^*|$. Thus

$$\begin{aligned}
c\,m(x)Uh(x) &= RUh(x) = (|R^*|\,h)(x)\\
&= \frac{\sqrt{\det(H)}}{\pi^{n/2}} \int_{V_{\mathsf{R}}} e^{-\|\sqrt{H}y\|^2} h(x-y)\,dy.
\end{aligned}$$

Since $m(x) = e^{-\frac{1}{2}(\langle x,Hx\rangle+\langle x,Kx\rangle)}$, we have using Lemma 3.7:

$$Uf(x) = \left(\frac{2}{\pi}\right)^{n/4} \frac{(\det H)^{3/4}}{(\det_{\mathsf{R}} A)^{1/4}} e^{\frac{1}{2}(\langle x,Hx\rangle+\langle x,Kx\rangle)} \int e^{(x-y,H(x-y))} f(y)\, dy.$$

Holomorphicity of Uf now implies $Uf = U_A f$.

5. The Gaussian Formulation

In infinite dimensions there is no useful notion of Lebesgue measure but Gaussian measure does make sense. So, with a view to extension to infinite dimensions, we will recast our generalized Segal-Bargmann transform using Gaussian measure instead of Lebesgue measure as the background measure on V_{R}. Of course, we have already defined the Fock space F_A using Gaussian measure.

As before, V is a finite-dimensional complex vector space with Hermitian inner-product $\langle\cdot,\cdot\rangle$, and $A : V \to V$ is a *real*-linear map which is *symmetric, positive-definite* with respect to the real inner-product $(\cdot,\cdot) = \mathrm{Re}\langle\cdot,\cdot\rangle$, i.e. $(Az, z) > 0$ for all $z \in V$ except $z = 0$. We assume, furthermore, that there is a real subspace V_{R} for which $V = V_{\mathsf{R}} + iV_{\mathsf{R}}$, the inner-product $\langle\cdot,\cdot\rangle$ is real-valued on V_{R}, and $A(V_{\mathsf{R}}) \subset V_{\mathsf{R}}$. Denote the linear map $v \mapsto iv$ by J. As usual, A is the sum

$$A = H + K$$

where $H = (A - JAJ)/2$ is complex-linear on V and $K = (A + JAJ)/2$ is complex-conjugate-linear. The real subspaces V_{R} and JV_{R} are $(\cdot,\cdot)$-orthogonal because for any $x, y \in V_{\mathsf{R}}$ we have $(x, Jy) = \mathrm{Re}\langle x, Jy\rangle = -\mathrm{Re}(J\langle x, y\rangle)$, since $\langle x, y\rangle$ is real, by hypothesis. Since A preserves V_{R} and is symmetric, it also preserves the orthogonal complement JV_{R}. Thus A has the block diagonal form:

$$A = \begin{bmatrix} R & 0 \\ 0 & T \end{bmatrix} = d(R, T)$$

Here, and henceforth, we use the notation $d(X, Y)$ to mean the real-linear map $V \to V$ given by $a \mapsto Xa$ and $Ja \mapsto JYa$ for all $a \in V_{\mathsf{R}}$, where X, Y are real-linear operators on V_{R}. Note that $d(X, Y)$ is complex-linear if and only if $X = Y$ and is complex-conjugate-linear if and only if $Y = -X$. The operator $d(X, X)$ is the unique complex-linear map $V \to V$ which restricts to X on V_{R}, and we denote it:

$$X_V = \begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix}$$

The hypothesis that A is symmetric and positive-definite means that R and T are symmetric, positive definite on V_{R}. Consequently, the real-linear operator S on V_{R} given by

$$S = 2(R^{-1} + T^{-1})^{-1}$$

is also symmetric, positive-definite.

The operators H and K on V are given by

$$H = \frac{1}{2}(R_V + T_V), \qquad K = d\left(\frac{1}{2}(R-T), \frac{1}{2}(T-R)\right).$$

Using the conjugation map

$$\sigma : V \to V : a + ib \mapsto a - ib \qquad \text{for} \quad a, b \in V_{\mathsf{R}}$$

we can also write K as

$$K = \frac{1}{2}(R_V - T_V)\sigma \tag{5.1}$$

Now consider the holomorphic functions ρ_T and ρ_S on V given by

$$\rho_T(z) = \frac{(\det T)^{1/2}}{(2\pi)^{n/2}} e^{-\frac{1}{2}(T_V z)\cdot z} \qquad \rho_S(z) = \frac{(\det S)^{1/2}}{(2\pi)^{n/2}} e^{-\frac{1}{2}(S_V z)\cdot z}$$

where $n = \dim V_{\mathsf{R}}$. Restricted to V_{R}, these are density functions for Gaussian probability measures.

The Segal-Bargmann transform in this setting is given by the map

$$S_A : L^2(V_{\mathsf{R}}, \rho_S(x)dx) \to \mathsf{F}_A : f \mapsto S_A f$$

where

$$S_A f(z) = \int_{V_{\mathsf{R}}} f(x)\rho_T(z-x)\,dx = \int_{V_{\mathsf{R}}} f(x)c(x,z)\rho_S(x)\,dx \tag{5.2}$$

with $c(x,z)$ given, for $x \in V_{\mathsf{R}}$ and $z \in V$, by

$$c(x,z) = \frac{\rho_T(x-z)}{\rho_S(x)}.$$

It is possible to take (5.2) as the starting point, with $f \in L^2(V_{\mathsf{R}}, \rho_S(x)dx)$ and prove that: (i) $S_A f(z)$ is well-defined, (ii) $S_A f$ is in F_A, (iii) S_A is a unitary isomorphism onto F_A. However, we shall not work out everything in this approach since we have essentially proven all this in the preceding sections. Full details of a direct approach would be obtained by generalizing the procedure used in [13]. In the present discussion we shall work out only some of the properties of S_A.

LEMMA 5.1. *Let $w, z \in V$. Then:*

(i) *The function $x \mapsto c(x, z)$ belongs to $L^2(V_{\mathsf{R}}, \rho_S(x)dx)$, thereby ensuring that the integral (5.2) defining $S_A f(z)$ is well-defined;*

(ii) *The S_A-transform of $c(\cdot, w)$ is $K_A(\cdot, \bar{w})$:*

$$[S_A c(\cdot, w)](z) = K_A(z, \bar{w})$$

and so, in particular:

$$K_A(z, w) = \int_{V_{\mathsf{R}}} \frac{\rho_T(x - z)\rho_T(x - \bar{w})}{\rho_S(x)}\, dx$$

(iii) *The transform S_A preserves inner-products on the linear span of the functions $c(\cdot, w)$:*

$$\langle c(\cdot, w), c(\cdot, z)\rangle_{L^2(V_{\mathsf{R}}, \rho_S(x)dx)} = K_A(w, z) = \langle K_A(\cdot, \bar{w}), K_A(\cdot, \bar{z})\rangle_{\mathsf{F}_A}$$

PROOF. (i) is equivalent to finiteness of $\int_{V_{\mathsf{R}}} \frac{|\rho_T(x-z)|^2}{\rho_S(x)}\, dx$, which is equivalent to positivity of the operator $2T - S$. To see that $2T - S$ is positive observe that

$$2T - S = 2T[(R^{-1} + T^{-1}) - T^{-1}](R^{-1} + T^{-1})^{-1}$$

$$= 2TR^{-1}(R^{-1} + T^{-1})^{-1} = TR^{-1}S \tag{5.3}$$

$$= 2(T^{-1} + T^{-1}RT^{-1})^{-1} \tag{5.4}$$

and in this last line $T^{-1} > 0$ (since $T > 0$) and $(T^{-1}RT^{-1}x, x) = (RT^{-1}x, T^{-1}x) \geq 0$ by positivity of R. Thus $2T - S$ is positive, being twice the inverse of the positive operator $T^{-1} + T^{-1}RT^{-1}$.

(ii) Recall $v \cdot w$ for $v, w \in V$ is the symmetric complex bilinear pairing given by $v \cdot w = \langle v, \bar{w}\rangle$, and we write v^2 for $v \cdot v$. We shall denote the complex-linear operator T_V which restricts to T on V_{R} simply by T. It is readily checked that T continues to be symmetric in the sense that $Tv \cdot w = v \cdot Tw$ for all $v, w \in V$. We start with

$$\begin{aligned} a &\stackrel{\text{def}}{=} [S_A c(\cdot, w)](z) \\ &= \int_{V_{\mathsf{R}}} \frac{\rho_T(x - w)}{\rho_S(x)} \rho_T(z - x)\, dx \\ &= (2\pi)^{-n/2} \frac{\det T}{(\det S)^{1/2}} \int_{V_{\mathsf{R}}} e^{-\frac{1}{2}[T(x-w)\cdot(x-w) + T(x-z)\cdot(x-z) - Sx\cdot x]}\, dx \\ &= (2\pi)^{-n/2} \frac{\det T}{(\det S)^{1/2}} \int_{V_{\mathsf{R}}} e^{-\frac{1}{2}[(2T-S)x\cdot x - 2Tx\cdot(w+z) + Tw\cdot w + Tz\cdot z]}\, dx. \end{aligned}$$

Recall from the proof of (i) that $2T - S > 0$. For notational simplicity let $L = (2T - S)^{1/2}$ and $M = L^{-1}T$. Then

$$a = (2\pi)^{-n/2} \frac{\det T}{(\det S)^{1/2}} \int_{V_{\mathsf{R}}} e^{-\frac{1}{2}(Lx - M(w+z))^2}\, dx\, e^{-\frac{1}{2}[Tw\cdot w + Tz\cdot z - M(w+z)\cdot M(w+z)]}$$
$$= \frac{\det T}{(\det S)^{1/2}(\det L)} e^{-\frac{1}{2}[Tw\cdot w + Tz\cdot z - M(w+z)\cdot M(w+z)]}.$$

To simplify the last exponent observe that by (5.4) and (5.1) we have

$$\begin{aligned} Tw\cdot w - Mw\cdot Mw &= Tw\cdot w - Tw\cdot L^{-2}Tw \\ &= Tw\cdot w - Tw\cdot(2T - S)^{-1}Tw \\ &= Tw\cdot w - \frac{1}{2}Tw\cdot(T^{-1} + T^{-1}RT^{-1})Tw \\ &= Tw\cdot w - \frac{1}{2}Tw\cdot(w + T^{-1}Rw) \\ &= \frac{1}{2}(Tw\cdot w - Rw\cdot w) \\ &= -\langle K\bar{w}, \bar{w}\rangle. \end{aligned}$$

The same holds with z in place of w. For the "cross term" we have

$$\begin{aligned} Mw\cdot Mz &= Tw\cdot L^{-2}Tz \\ &= Tw\cdot(2T - S)^{-1}Tz \\ &= \frac{1}{2}Tw\cdot(T^{-1} + T^{-1}RT^{-1})Tz \\ &= \frac{1}{2}(Tw\cdot z + w\cdot Rz) \\ &= 2w\cdot Hz. \end{aligned}$$

Putting everything together gives

$$[S_A c(\cdot, w)](z) = \frac{\det T}{(\det S)^{1/2}(\det L)} e^{\frac{1}{2}\langle K\bar{w},\bar{w}\rangle} e^{\langle Hw,\bar{z}\rangle} e^{\frac{1}{2}\langle K\bar{z},\bar{z}\rangle}.$$

In Lemma 6.2 below we prove that

$$\frac{\det T}{(\det S)^{1/2}(\det L)} = \left(\frac{\det_{\mathsf{R}}(A)}{\det_{\mathsf{R}}(H)}\right)^{-1/2} = c_A^{-2}.$$

So

$$[S_A c(\cdot, w)](z) = K_A(w, \bar{z}).$$

For (iii), we have first:

$$\langle c(\cdot, w), c(\cdot, z)\rangle_{L^2(\rho_S(x)dx)} = [S_A c(\cdot, w)](\bar{z}) = K_A(\bar{z}, \bar{w}) = K_A(w, z).$$

The second equality in (iii) follows since K_A is a reproducing kernel.

6. The evaluation map and determinant relations

Recall

$$K_A(z, w) = c_A^{-2} e^{\frac{1}{2}\langle z, Kz\rangle + \frac{1}{2}\langle Kw, w\rangle + \langle Hz, w\rangle}$$

where

$$c_A^{-2} = \left(\frac{\det_V H}{\det_V A}\right)^2$$

is a reproducing kernel for F_A. Thus

$$f(w) = \langle f, K_A(\cdot, w)\rangle = \pi^{-n} \int_V f(z) K_A(w, z)\, |dz|$$

where $|dz| = dxdy$ signifies integration with respect to Lebesgue measure on the real inner-product space V. Thus we have

PROPOSITION 6.1. *For any $z \in V$, the evaluation map*

$$\delta_z : \mathsf{F}_A \to \mathsf{C} : f \mapsto f(z)$$

is a bounded linear functional with norm

$$\|\delta_z\| = K_A(z, z)^{1/2} = c_A^{-1} e^{(Az, z)}.$$

PROOF. Note

$$|\delta_z f| = |f(z)| = |\langle f, K_A(\cdot, z)\rangle| \leq \|f\|_{\mathsf{F}_A} K_A(z, z)^{1/2} \tag{6.1}$$

follows from the reproducing kernel property

$$\|K_A(\cdot, z)\|^2_{\mathsf{F}_A} = \left\langle K_A(\cdot, z), K_A(\cdot, z)\right\rangle_{\mathsf{F}_A} = K_A(z, z).$$

This last calculation also shows that the inequality in (6.1) is an equality if $f = K_A(\cdot, z)$ and thereby shows that $\|\delta_z\|$ is actually equal to $K_A(z, z)^{1/2}$. The latter is readily checked to be equal to $c_A^{-1} e^{(Az, z)}$.

We have already used the first of the following two facts about c_A.

LEMMA 6.2. *For the constant c_A we have*

$$c_A^{-2} = \left(\frac{\det_V H}{\det_V A}\right)^2 = \frac{\det T}{(\det S)^{1/2} \det L}$$

where, as before, $L = (2T - S)^{1/2}$ *and* $S = 2(R^{-1} + T^{-1})^{-1}$.

PROOF. Recall from (5.3) that $2T - S = TR^{-1}S$. Note also that

$$S^{-1} = \frac{1}{2}(R^{-1} + T^{-1}) = R^{-1}\frac{R+T}{2}T^{-1} = R^{-1}(H|V_{\mathsf{R}})T^{-1}$$

So

$$\begin{aligned}
&\left(\frac{\det_V A}{\det_V H}\right)^{1/2} \frac{\det T}{(\det S)^{1/2} \det L} \\
&\quad = \frac{(\det R)^{1/2}(\det T)^{1/2}}{\det S^{-1} \det R \det T} \frac{\det T}{(\det S)^{1/2} \det T^{1/2} \det R^{-1/2} \det S^{1/2}} \\
&\quad = 1
\end{aligned}$$

which implies the desired result.

Next we prove a determinant relation which implies $c_A \geq 1$. (This "determinant AM-GM inequality" could be obtained by reference to standard matrix inequalities, but we include a complete proof.)

LEMMA 6.3. *If R and T are positive definite $n \times n$ matrices (symmetric if real) then*

$$\sqrt{\det R \det T} \leq \det\left(\frac{R+T}{2}\right)$$

with equality if and only if $R = T$.

PROOF. Noting that $R^{-1/2}TR^{-1/2} \geq 0$ we have, with $D = (R^{-1/2}TR^{-1/2})^{1/4}$,

$$\begin{aligned}
\frac{\det R \det T}{\left(\det \frac{R+T}{2}\right)^2} &= \frac{\det R \det(R^{1/2}D^4R^{1/2})}{\left[\det R^{1/2}\left(\frac{1+D^4}{2}\right)R^{1/2}\right]^2} \\
&= \left[\det\left(\frac{D^2 + D^{-2}}{2}\right)\right]^{-2} \\
&= \left[\det\left\{I + \left(\frac{1}{\sqrt{2}}D - \frac{1}{\sqrt{2}}D^{-1}\right)^2\right\}\right]^{-2}.
\end{aligned}$$

Diagonalizing D, it is clear that this last term is less or equal to 1 with equality if and only if $D = D^{-1}$. This is equivalent to $D^4 = I$ which holds if and only if $R = T$.

As consequence, we have for c_A:

$$c_A = \left(\frac{\det_V A}{\det_V H}\right)^{1/4} = \left(\frac{\det R \det T}{\left(\det \frac{R+T}{2}\right)^2}\right)^{1/4} = \left(\frac{\sqrt{\det R \det T}}{\det \frac{R+T}{2}}\right)^{1/2}$$

and so

$$c_A^{-2} = \frac{\det \frac{R+T}{2}}{\sqrt{\det R \det T}} \geq 1 \tag{6.2}$$

with equality if and only if $R = T$.

In extending this theory to infinite dimensions, to retain a meaningful notion of evaluation $\delta_z : f \mapsto f(z)$, the constant c_A^{-1}, which appears in the norm $\|\delta_z\|$, must be finite. The expression for c_A^{-2} in (6.2) gives a more explicit condition on R and T for this finiteness to hold.

If $R = rI$ and $T = tI$ then, by (6.2), $c_A^{-1} = [(r+t)/(2\sqrt{rt})]^{n/2}$ which is bounded as $n \nearrow \infty$ if and only if $r = t$ (this was noted in [13]).

7. Remarks on extension to infinite dimensions

The Gaussian formulation permits extension to infinite dimensions with some conditions placed on A. Suppose then that V is an infinite-dimensional separable complex Hilbert space, V_{R} a real subspace on which the inner-product is real-valued and for which $V = V_{\mathsf{R}} + iV_{\mathsf{R}}$, and $A : V \to V$ a bounded symmetric, positive-definite real-linear operator carrying V_{R} into itself. The operators R, T, S, H and K are defined as before. Assume that R and T commute and that there is an orthonormal basis $e_1, e_2, \ldots$ of V_{R} consisting of simultaneous eigenvectors of R and T (greater generality may be possible but we discuss only this case). Let V_n be the complex linear span of $e_1, \ldots, e_n$, and $V_{n,\mathsf{R}}$ the real linear span of $e_1, \ldots, e_n$. Then A restricts to an operator A_n on V_n, and we have similarly restrictions H_n, K_n on V_n and R_n, T_n, S_n on $V_{n,\mathsf{R}}$. The unitary transform S_A may be obtained as a limit of the finite-dimensional transforms S_{A_n}.

The Gaussian kernels ρ_S and ρ_T do not make sense anymore, and nor does the coherent state c, but the Gaussian measures $d\gamma_S(x) = \rho_S(x)dx$ and μ_A do have meaningful analogs. There is a probability space V'_{R}, with a σ-algebra $\mathscr{F}$ on which there is a measure γ_S, and there is a linear map $V_{\mathsf{R}} \to L^2(V'_{\mathsf{R}}, \gamma_A) : x \mapsto G(x) = (x, \cdot)$, such that the σ-algebra $\mathscr{F}$ is generated by the random variables $G(x)$, and each $G(x)$ is real Gaussian with mean 0 and variance $(S^{-1}x, x)$. Similarly, there is probability space V', with a σ-algebra $\mathscr{F}_1$ on which there is a measure μ_A, and there is a real-linear map $V \to L^2(V', \mu_A) : z \mapsto G_1(z) = (z, \cdot)$, such that the σ-algebra $\mathscr{F}_1$ is generated by the random variables $G_1(z)$, and each $G_1(z)$ is (real) Gaussian with mean 0 and variance $\frac{1}{2}(A^{-1}z, z)$. Then for each $z \in V$, written as $z = a + ib$ with $a, b \in V_\mathsf{R}$, we have the complex-valued random variable on V' given by $\tilde{z} = G_1(a) + iG_1(b)$. Suppose g is a holomorphic function of n complex variables such that $\int_V |g(\tilde{e}_1, \ldots, \tilde{e}_n)|^2 \, d\mu_A < \infty$. Define F_A to be the closed linear span of all functions of the type $g(\tilde{e}_1, \ldots, \tilde{e}_n)$ in $L^2(\mu_A)$ for all $n \geq 1$. We may then define S_A of a function $f\big(G(e_1), \ldots, G(e_n)\big)$ to be $(S_{A_n} f)(\tilde{e}_1, \ldots, \tilde{e}_n)$, and then extend S_A by continuity to all of $L^2(\gamma_S)$. In writing $S_{A_n} f$ we have identified V_n with C^n and $V_{n,\mathsf{R}}$ with R^n using the basis $e_1, \ldots, e_n$.

A potentially significant application of the infinite-dimensional case would be to situations where V_R is a *path space* and A arises from a suitable *differential operator*. For the "classical case" where $R = T = tI$ for some $t > 0$, this leads to the Hall transform [6] for Lie groups as well as the path-space version on Lie groups considered in [8].

REFERENCES

1. Bargmann, V., *On a Hilbert space of analytic functions and an associated integral transform, Part I*, Comm. Pure. Appl. Math. 14 (1961), 187–214.
2. Driver, B., and Hall, B., *Yang-Mills theory and the Segal-Bargmann transform*, Comm. Math. Phys. 201 (1999), 249–290.
3. Fock, V., *Verallgemeinerung und Lösung der Diracschen statistischen Gleichung*, Z. für Phys. 49 (1928), 339–357.
4. Fock, V., *Konfigurationsraum und zweite Quantelung*, Z. für Phys 75 (1932), 622–647.
5. Folland, G. B., *Harmonic Analysis in Phase Space*, Ann. of Math. Stud. 122 (1989).
6. Hall, B. C., *The Segal-Bargmann "coherent state" transform for compact Lie groups*, J. Funct. Anal. 112 (1994), 103–151.
7. Hall, B. C., *The range of the heat operator*, Preprint, 2004.
8. Hall, B. C., and Sengupta, A. N., *The Segal-Bargmann transform for path spaces in groups*, J. Funct. Anal. 152 (1998).
9. Ólafsson, G., *Analytic continuation in representation theory and harmonic analysis*, in J. P. Bourguignon, T. Branson, and O. Hijazi (eds.), Seminares et Congr. Vol. 4 (2000), 201–233, The French Math. Soc.
10. Ólafsson, G., and Ørsted, B., *Generalizations of the Bargmann transform*, Lie theory and its applications in physics, Clausthal 1995 (H.-D. Doebner, V. K. Dobrev and J. Hilgert, eds.), World Scientific, River Edge, New Jersey, 1996.
11. Segal, I. E., *Foundations of the theory of dynamical systems of infinitely many degrees of freedom I*, Mat. Medd. Danske Vid. Selsk. 31 (1959), no. 12.
12. Segal, I. E., *Mathematical Problems of Relativistic Physics*, Amer. Math. Soc., 1963.

13. Sengupta, A. N., *The two-parameter Segal-Bargmann transform*, Preprint (1999) at http://www.mpim-bonn.mpg.de/html/preprints/preprints.html.

DEPARTMENT OF MATHEMATICS
LOUISIANA STATE UNIVERSITY
BATON ROUGE, LA 70803
USA
E-mail: fabec@math.lsu.edu, olafsson@math.lsu.edu,sengupta@math.lsu.edu
URL: www.math.lsu.edu/˜fabec, www.math.lsu.edu/˜olafsson, www.math.lsu.edu/˜sengupta