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THE REGULARITY OF THE SPACE OF GERMS OF FRÉCHET VALUED HOLOMORPHIC FUNCTIONS AND THE MIXED HARTOG'S THEOREM

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Abstract

It is shown that $H(K, F)$ is regular for every reflexive Fréchet space F with the property (LB_∞) where K is a compact set of uniqueness in a Fréchet-Schwartz space E such that $E \in (\Omega)$. Using this result we give necessary and sufficient conditions for a Fréchet space F, under which every separately holomorphic function on $K \times F^*$ is holomorphic, where K is as above.

1. Introduction

The well known Hartogs theorem on the holomorphy of separately holomorphic functions in $\mathbf{C}^n$ was extended to the infinite dimensional case by several authors. In particular, this theorem is true for the classes of Fréchet spaces and of dual Fréchet-Schwartz spaces. However, the problem is more complicated in the mixed case. In the present paper we shall investigate the holomorphy of separately holomorphic functions in connection with their local Dirichlet representations and with the properties $(\Omega, \mathrm{LB}_\infty)$.

One of the keys, which is used to prove the above result is presented in the first part of this paper. In this section we consider regularity of the space $H(K, F)$, where K is a compact set of uniqueness in a Fréchet space with the property (Ω) and F is a reflexive Fréchet space having the property (LB_∞) (Theorem 3.1).

The remaining part will deal with the holomorphy and the local Dirichlet representation of separately holomorphic functions on $K \times F^*$, where K is a compact set of uniqueness in a Fréchet-Schwartz space with the property (Ω) and F is a nuclear Fréchet space with the property (LB_∞) (Theorem 4.4). To prove this theorem we use the result in the above section.

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2. Preliminaries

2.1. General notations

We shall use standard notations of the theory of locally convex spaces as presented in the book of Schaefer [11]. A locally convex space always is a complex vector space with a locally convex Hausdorff topology.

For a Fréchet space E we always assume that its locally convex structure is generated by an increasing system $\{\|.\|_k\}$ of semi-norms. Then we denote by E_k the completion of the canonically normed space $E/\operatorname{Ker}\|.\|_k$ and $\omega_k : E \to E_k$ denotes the canonical map and U_k denotes the set $\{x \in E : \|x\|_k < 1\}$.

If B is an absolutely convex subset of E we define a norm $\|.\|_B^*$ on E^*, the strong dual space of E, with values in $[0, +\infty]$ by

$$\|u\|_B^* = \sup\{|u(x)|, x \in B\}.$$

Obviously $\|.\|_B^*$ is the gauge functional of B^0. Instead of $\|.\|_{U_k}^*$ we write $\|.\|_k^*$.

For locally convex spaces E and F we denote by $L(E, F)$ the space of all continuous linear mappings, while $LB(E, F)$ denotes the set of all $A \in L(E, F)$ for which there exists a zero neighbourhood U in E such that $A(U)$ is bounded.

2.2. The space of Köthe sequences

If $A = (a_{j,k})_{(j,k)\in\mathbb{N}^2}$ is a Köthe matrix satisfying the condition in Pietsch [10] then we denote by $\lambda(A)$ the sequence space

$$\lambda(A) = \left\{x \in \mathbb{C}^{\mathbb{N}} : \|x\|_k = \sum_{j=1}^{\infty} |x_j| a_{j,k} < \infty \text{ for all } k \in \mathbb{N}\right\}.$$

Obviously $\lambda(A)$ is a Fréchet space under the natural locally convex topology induced by the semi-norm system $\{\|.\|_k\}$.

Let $\alpha = (\alpha_n)_{n\in\mathbb{N}}$ be an increasing unbounded sequence of positive real numbers such that $\lim_{n\to\infty} \alpha_n = +\infty$ and let $0 < r_k \uparrow R$ where $0 < R \le +\infty$.

By putting $a_{j,k} = r_k^{\alpha_j}$, we define the power series space

$$\Lambda_R(\alpha) = \lambda(A) = \left\{x \in \mathbb{C}^{\mathbb{N}} : \|x\|_k = \sum_{j=1}^{\infty} |x_j| r_k^{\alpha_j} < \infty \text{ for all } k \in \mathbb{N}\right\}.$$

$\Lambda_R(\alpha)$ is called a power series space. In the case $R = 1$ (resp. $R = +\infty$) $\Lambda_R(\alpha)$ is called the power series space of finite (resp. infinite) type.

2.3. Some linear topological invariants

Let E be a Fréchet space with a fundamental system of semi-norms $\{\|.\|_k\}_{k\in\mathbb{N}}$. We say that E has the properties

$$(\Omega)\text{: if } \forall\, p\ \exists\, q\ \forall\, k\ \exists\, d>0, C>0 \text{ such that } \|.\|_q^{*1+d} \leq C\|.\|_k^{*}\|.\|_p^{*d}$$

$$(\mathrm{DN})\text{: if } \exists\, p\ \exists\, d>0\ \forall\, q\ \exists\, k, C>0 \text{ such that } \|.\|_q^{1+d} \leq C\|.\|_k\|.\|_p^{d}$$

and (LB_∞) if

$$\forall \varrho_N \uparrow \infty, \exists p\, \forall q\, \exists k(q) \geq q, C(q)>0, \forall x \in E\, \exists m \text{ with } q \leq m \leq k(q)$$

such that

$$\|x\|_q^{1+\varrho_m} \leq C(q)\|x\|_m\|x\|_p^{\varrho_m}.$$

In [13] Vogt has proved that $E \in (\mathrm{LB}_\infty)$ if and only if

$$L(\Lambda_\infty(\alpha), E) = \mathrm{LB}(\Lambda_\infty(\alpha), E)$$

for some exponent sequence $\alpha = (\alpha_n)_{n\in\mathbb{N}}$.

Note that the following relation holds

$$(\mathrm{LB}_\infty) \Rightarrow (\mathrm{DN}).$$

The above properties have been introduced and investigated by Vogt [13], [14], [15]. Hereafter, to be brief, whenever E has the property (Ω) (resp. (DN), (LB_∞)) we write $E \in (\Omega)$ (resp. $E \in (\mathrm{DN})$, $E \in (\mathrm{LB}_\infty)$).

2.4. The space of germs of holomorphic functions on a compact set

Let E and F be locally convex spaces and let $D \subset E$ be open, $D \neq \emptyset$. By $H(D,F)$ we denote the vector space of all holomorphic functions on D with values in F. $H(D,F)$ equipped with the compact-open topology.

A seminorm ϱ on $H(D,F)$ is said to be τ_ω-continuous if there exists a compact set K in D and a continuous seminorm α on F such that, for every neighbourhood V of K in D, there exists $C(V)>0$ such that

$$\varrho(f) \leq C(V)\sup_{z\in V}\alpha(f(z)), \qquad \forall\, f \in H(D,F).$$

For each compact set K in E we associate $H(K,F)$, the space of germs of F-valued holomorphic functions on K, equipped with the inductive topology

$$H(K,F) := \lim_{K\subset U}\mathrm{ind}(H(U,F); \tau_\omega),$$

where U is taken over all neighbourhoods of K. It is known [9] that

$$H(K) \cong \liminf_{K \subset U} H^{\infty}(U),$$

where $H^{\infty}(U)$ is the Banach space of all holomorphic functions which are bounded on U.

For details concerning holomorphic maps on locally convex spaces we refer to the book of Dineen [3].

2.5. Separately holomorphic functions on a compact set

Let K be a compact set in a locally convex space E and V an open set in a locally convex space F. For a function $f : K \times V \to \mathbb{C}$ we put

$$f_x(z) = f(x, z) \quad \text{for} \quad z \in V,$$
$$f^z(x) = f(x, z) \quad \text{for} \quad x \in K.$$

The function f is called separately holomorphic if $f_x : V \to \mathbb{C}$ and $f^z : K \to \mathbb{C}$ are holomorphic for all $x \in K$ and $z \in V$, respectively. Here a function on K is said to be holomorphic if it can be extended holomorphically to a neighbourhood of K in E.

3. The regularity of the space of germs of F-valued holomorphic functions

We first recall that the space $H(K, F)$ of germs of F-valued holomorphic functions on a compact set K is regular if every bounded subset in

$$H(K, F) := \liminf_{K \subset U}(H(U, F); \tau_\omega),$$

is contained and bounded in some $[H(U, F); \tau_\omega]$.

The problem of the regularity of the space $H(K)$ was investigated by several authors. Chae [1] proved that $H(K)$ is regular for every compact subset K of a Banach space E. When E is a metrizable locally convex space, $H(K)$ is represented as an inductive limit of a sequence of (DF)-spaces. Using a theorem of Grothendieck on bounded subsets in an inductive limit of a sequence of (DF)-spaces, Mujica [9] generalized the result of Chae. In [16] Vogt gave a general characterization for the regularity of the inductive limit of a sequence of Fréchet spaces.

In this section we prove the following.

THEOREM 3.1. *Let F be a reflexive Fréchet space with $F \in (\mathrm{LB}_\infty)$ and E a Fréchet-Schwartz space with $E \in (\Omega)$. Then $H(K, F)$ is regular for all compact sets of uniqueness K in E.*

Here, we recall that a compact set K in a Fréchet space is called a set of uniqueness if for all $f \in H(K)$ and $f|_K = 0$ then $f = 0$ on a neighbourhood of K in E.

For proving this theorem we need the following

PROPOSITION 3.2 ([6]). *Let E be a Fréchet-Schwartz space. Then $E \in (\Omega)$ if and only if $[H(K)]^* \in (\Omega)$ for all compact sets K in E.*

LEMMA 3.3. *Let F be a Fréchet space with the property* (LB_∞) *and B a Banach space. Then the space $L(B, F)$ of all continuous linear maps from B into F has the property* (LB_∞).

PROOF. Given a sequence of positive numbers $\{\varrho_N\}$, choose $k \geq 1$ such that the property (LB_∞) satisfied. Then for all $f \in L(B, F)$ we have

$$\begin{aligned}\|f\|_q^{1+\varrho_m} &= \sup\{\|f(x)\|_q^{1+\varrho_m} : \|x\| \leq 1\}\\ &\leq C \max_{q \leq m \leq k(q)} \sup\{\|f(x)\|_m \|f(x)\|_p^{\varrho_m} : \|x\| \leq 1\}\\ &= C \max_{q \leq m \leq k(q)} \|f\|_m \|f\|_p^{\varrho_m}.\end{aligned}$$

Thus $L(B, F) \in (\mathrm{LB}_\infty)$.

LEMMA 3.4. *Let F be a Fréchet space with $F \in (\mathrm{LB}_\infty)$ and K a compact set in a Fréchet-Schwartz space E with $E \in (\Omega)$. Then*

$$L([H(K, B)]^*, F) = \mathrm{LB}([H(K, B)]^*, F)$$

for every Banach space B.

PROOF. Let a continuous linear map $T : [H(K, B)]^* \to F$ be given. Because $[H(K, B)]^* \cong [H(K)]^* \widehat{\otimes}_\pi B^*$, T induces a continuous linear map

$$\widehat{T} : [H(K)]^* \to L(B^*, F).$$

From [14] and Proposition 3.2 there exists an index set I such that $[H(K)]^*$ is isomorphic to a quotient space of

$$\Lambda_\infty(\alpha, \ell^1(I)) = \left\{(x_n) \subset \ell^1(I) : \sum_{n=1}^{\infty} \|x_n\| n^k < +\infty, \forall k \geq 1\right\}$$

where $\alpha = (\log(n+1))_{n\geq 1}$.

From $\sup_{n\geq 1} \frac{\log(n+1)}{\log n} < \infty$ and Lemma 3.3, repeating the proof of Theorem 3.2 of Vogt [13] we deduce that every continuous linear map from

$\Lambda_\infty(\alpha, \ell^1(I))$ and, hence, from $[H(K)]^*$ to $L(B^*, F)$ is bounded on a neighbourhood of $0 \in [H(K)]^*$. Thus $\widehat{T}$ is bounded on a zero neighbourhood U in $[H(K)]^*$. This yields that T is bounded on $\overline{\text{conv}}(U \otimes V)$, a zero neighbourhood in $[H(K)]^* \widehat{\otimes}_\pi B^*$, where V is the unit ball in B^*.

We mention the following without proof.

LEMMA 3.5 ([5]). *Let f be a bounded function from an open set D in a locally convex space E into the Banach space $\ell^\infty(I)$ and let the coordinate functions f_α be holomorphic. Then f is holomorphic.*

Now we can prove Theorem 3.1.

PROOF OF THEOREM 3.1. Given a bounded family $\{f_\alpha\}_{\alpha \in I}$ in $H(K, F)$, consider the linear map

$$h : F^*_{\text{bor}} \to H(K, \ell^\infty(I))$$

defined by

$$h(u) = (u \circ f_\alpha)_{\alpha \in I}$$

where F^*_{bor} is the bornological space associated to F^*. Lemma 3.5 and the uniqueness of K imply that $h(u)$ is correctly defined.

1) We first check that h is bounded.

Indeed, let B be a bounded set in F^*. Take $k \geq 1$ such that B is contained and bounded in F_k^*, where F_k is the Banach space associated to $\|.\|_k$. Let $\omega_k : F \to F_k$ be the canonical map. By the regularity of $H(K, F_k)$ [12] and the boundedness of $\{\omega_k f_\alpha\}_{\alpha \in I}$, we can find a neighbourhood V of K in E such that $\{\omega_k f_\alpha\}_{\alpha \in I}$ is contained and bounded in $H^\infty(V, F_k)$, the Banach space of F_k-valued bounded holomorphic functions on V. Hence $h(B)$ is contained and bounded in $H^\infty(V, \ell^\infty(I))$. Therefore h is continuous.

2) By Lemma 3.4, the map $h^* : [H(K, \ell^\infty(I))]^* \to [F^*_{\text{bor}}]^* \cong F$ is of type (LB). It follows that h^{**}, and hence, h is also of type (LB). Thus, we can find a neighbourhood W of zero in F^*_{bor} for which there exists, for every $u \in F^*$, a function $\widehat{h}(u)$ in $H^\infty(V, \ell^\infty(I))$ such that

i) $\widehat{h}(u) = h(u)$ on a neighbourhood of K in V,

ii) $\{\widehat{h}(u)\}_{u \in W}$ is bounded in $H^\infty(V, \ell^\infty(I))$.

Now, for each $\alpha \in I$, we define a holomorphic function $g_\alpha : V \to [F^*_{\text{bor}}]^* \cong F$ by

$$g_\alpha(z)(u) = u \circ f_\alpha(z) \qquad \text{for} \quad z \in V, u \in F^*_{\text{bor}}.$$

By (ii), $\{g_\alpha\}_{\alpha \in I}$ is bounded in $H^\infty(V, F)$. Thus, $\{f_\alpha\}_{\alpha \in I}$ is contained and bounded in $[H^\infty(V, F); \tau_\omega]$.

The theorem is completely proved.

4. Representation of separately holomorphic functions

First we introduce the notion of local Dirichlet representations. Let E be a locally convex space and D an open subset in E. A function $f : D \to \mathbb{C}$ is said to have a local Dirichlet representation on D if for every $x_0 \in D$ there exist a neighbourhood U of x_0 and sequences $(\xi_k) \subset \mathbb{C}$, $(u_k) \subset E^*$, such that

$$f(x) = \sum_{k\geq 1} \xi_k \exp u_k(x) \qquad \text{for} \quad x \in U$$

and

$$\sum_{k\geq 1} |\xi_k| \exp \|u_k\|_K^* < \infty \qquad \text{for every compact set } K \subset U.$$

The global Dirichlet representation of entire functions was investigated in [2].

We begin this section by presenting auxiliary lemmas, which are useful for proving Theorem 4.4, the main result of the paper.

LEMMA 4.1. *Let E be a Fréchet-Schwartz space with $E \in (\Omega)$. Then E has the following property:*

(Ω^∞) $\forall p \, \exists q \, \forall k \, \exists D(k) > 0, \forall u \in E^*$ *such that* $\|u\|_q^{*1+k} \leq D(k)\|u\|_k^* \|u\|_p^{*k}$.

PROOF. First we prove that

$$\Lambda_\infty(\alpha, \ell^1(I)) = \left\{ (x_n) \subset \ell^1(I) : \sum_{n=1}^{\infty} \|x_n\| n^k < +\infty, \forall k \geq 1 \right\}$$

where $\alpha = (\log(n+1))_{n\geq 1}$ and I is an index set, has the property (Ω^∞).

We recall a result of Meise and Vogt [8]: Let $\lambda(A)$ be given, where $A = (a_{i,j})$ is a Köthe matrix. Then for $d > 0, D > 0$ and $p, q, m \geq 1$ the following conditions are equivalent

1) $\|.\|_q^{*1+d} \leq D\|.\|_m^* \|.\|_p^{*d}$

2) $a_{n,m} . a_{n,p}^d \leq D a_{n,q}^{1+d}, \ \forall n \geq 1$.

On $\Lambda_\infty(\alpha, \ell^1(I))$ with $a_{n,j} = n^j$, we use the norm-system $\{\|.\|_j\}_{j\geq 1}$ given by

$$\|(x_n)\|_j = \sum_{n\geq 1} \|x_n\| n^j.$$

For each $p \geq 1$, choose $q = 2p$. Then for all $m \geq 1$ we have

$$m \leq mp < m(q-p) + q \implies m(1+p) < (m+1)q.$$

It implies that

$$a_{n,m}.a_{n,p}^{m} = n^{m(1+p)} < n^{(m+1)q} = a_{n,q}^{1+m}.$$

By Meise and Vogt [8] $\Lambda_\infty(\alpha, \ell^1(I))$ has the property (Ω^∞).

Now, by the same arguments as in the proof of Lemma 4.4 of Vogt [15], we can show that the property (Ω^∞) is equivalent to the following: $\forall p\ \exists q\ \forall m$ $\exists D(m) > 0$ such that

$$U_q \subset r^m U_m + \frac{D(m)}{r} U_p \qquad \text{for all} \quad r > 0.$$

From this, it is easy to see that the property (Ω^∞) is inherited by quotient spaces.

On the other hand, since $E \in (\Omega)$, by Vogt [14] and Proposition 3.2, there exists an index set I such that $[H(K)]^*$, where K is a compact set in E, is isomorphic to a quotient space of

$$\Lambda_\infty(\alpha, \ell^1(I)) = \left\{(x_n) \subset \ell^1(I) : \sum_{n=1}^{\infty} \|x_n\| n^k < +\infty, \forall k \geq 1\right\}$$

where $\alpha = (\log(n+1))_{n\geq 1}$. Hence E has the property (Ω^∞). The lemma is proved.

PROPOSITION 4.2. *Let E be a Fréchet space with $E \in (\Omega)$ and F a nuclear Fréchet space with $F \in (\mathrm{LB}_\infty)$. Then every E^*-valued holomorphic function on F^* is locally bounded.*

PROOF. Since $F \in (\mathrm{LB}_\infty)$, by Vogt [13] $F \in (\mathrm{DN})$. According to Vogt [15] F is isomorphic to a subspace of s, the space of rapidly decreasing sequences. Hence without loss of generality we may assume that F has an absolute basis $\{e_j\}$.

Let $\{e_j^*\}$ be a sequence of coefficient functionals associated to $\{e_j\}$.

Let $f : F^* \to E^*$ be a holomorphic function and $u_o \in F^*$. It is enough to consider the case $u_o = 0$.

By Lemma 4.1, $\forall \alpha\ \exists \beta\ \forall \gamma\ \exists D(\gamma) > 0, \forall u \in E^*$

$$\|u\|_\beta^{*1+\gamma} \leq D(\gamma) \|u\|_\gamma^* \|u\|_\alpha^{*\gamma}.$$

For each $q \in \mathbb{N}$, take $\alpha = \alpha(q)$ (we may assume that $\alpha(q) \geq q$), $\lambda_q > 0$ such that

$$M(\lambda_q, q, \alpha(q)) = \sup\left\{\|f(u)\|_{\alpha(q)}^* : \|u\|_q^* < \lambda_q\right\} < +\infty$$

and
$$\frac{\lambda_q e_j^*}{\|e_j^*\|_q^*} \in U_q^o \qquad \forall j \geq 1.$$

For $\{d_n\} = \{\alpha(n)\} \uparrow +\infty$, since $F \in (\mathrm{LB}_\infty)$, $\exists p\ \forall q\ \exists k(q) \geq q, C(q) > 0$, $\forall e_j \in F, \exists m$ with $q \leq m \leq k(q)$ such that

$$\|e_j\|_q^{1+\alpha(m)} \leq C(q)\|e_j\|_m \|e_j\|_p^{\alpha(m)}. \tag{4.1}$$

Observe that
$$\|e_j^*\|_k^* = \frac{1}{\|e_j\|_k}, \qquad \forall\, k \geq 1.$$

Thus, we obtain

$$\|e_j^*\|_q^{*1+\alpha(m)} \geq C(q)^{-1}\|e_j^*\|_m^* \|e_j^*\|_p^{*\alpha(m)}. \tag{4.2}$$

Put $\alpha_1 = \alpha(p)$. From Lemma 4.1 we get

$$\exists\, \beta\ \forall \gamma\ \exists\, D(\gamma) > 0, \forall\, u \in E^*$$

$$\|u\|_\beta^{*1+\gamma} \leq D(\gamma)\|u\|_\gamma^* \|u\|_{\alpha_1}^{*\gamma}. \tag{4.3}$$

We shall show that there exists a neighbourhood V of $0 \in F^*$ such that f maps it holomorphically into E_β^*. Write the Taylor expansion of f at $0 \in F^*$.

$$f(u) = \sum_{n \geq 0} P_n f(u)$$

where
$$P_n f(u) = \frac{1}{2\pi i} \int_{|\lambda|=1} \frac{f(\lambda u)}{\lambda^{n+1}}\, d\lambda, \qquad \text{for} \quad u \in F^*.$$

Put
$$A^m = \left\{\omega \in F^* : \|\omega\|_q^{*1+\alpha(m)} \geq C(q)^{-1}\|\omega\|_m^* \|\omega\|_p^{*\alpha(m)}\right\}$$
$$J^m = \{j \in \mathbf{N} : e_j^* \in A^m\}.$$

By virtue of the properties (LB_∞) with $q = p+1$ and for $q \le m \le k(q)$, $\gamma = \alpha(m)$ and from (4.2), (4.3) we have

$$\sum_{n\ge 0} \|P_n f(u)\|_\beta^*$$
$$\le \sum_{n\ge 0}\sum_{j_1,\dots,j_n\ge 1} \frac{\|\widehat{P_n f}(e_{j_1}^*,\dots,e_{j_n}^*)\|_\beta^*}{\|e_{j_1}^*\|_q^*\dots\|e_{j_n}^*\|_q^*}|u(e_{j_1})|\|e_{j_1}^*\|_q^*\dots|u(e_{j_n})|\|e_{j_n}^*\|_q^*$$
$$\le \sum_{n\ge 0}\sum_{q\le m\le k(q)}\sum_{j_1,\dots,j_n\in J^m} \frac{\|\widehat{P_n f}(e_{j_1}^*,\dots,e_{j_n}^*)\|_\beta^*}{\|e_{j_1}^*\|_q^*\dots\|e_{j_n}^*\|_q^*}|u(e_{j_1})|\|e_{j_1}^*\|_q^*\dots|u(e_{j_n})|\|e_{j_n}^*\|_q^*$$
$$\le \sum_{n\ge 0}\sum_{q\le m\le k(q)} D(\alpha(m))^{\frac{1}{1+\alpha(m)}}C(q)^{\frac{n}{1+\alpha(m)}}$$
$$\times \sum_{j_1,\dots,j_n\in J^m}\left[\frac{\|\widehat{P_n f}(e_{j_1}^*,\dots,e_{j_n}^*)\|_{\alpha(m)}^*}{\|e_{j_1}^*\|_m^*\dots\|e_{j_n}^*\|_m^*}\right]^{\frac{1}{1+\alpha(m)}}\left[\frac{\|\widehat{P_n f}(e_{j_1}^*,\dots,e_{j_n}^*)\|_{\alpha_1}^*}{\|e_{j_1}^*\|_p^*\dots\|e_{j_n}^*\|_p^*}\right]^{\frac{\alpha(m)}{1+\alpha(m)}}$$
$$\times |u(e_{j_1})|\|e_{j_1}^*\|_q^*\dots|u(e_{j_n})|\|e_{j_n}^*\|_q^*$$
$$\le \sum_{n\ge 0}\sum_{q\le m\le k(q)} D(\alpha(m))^{\frac{1}{1+\alpha(m)}}C(q)^{\frac{n}{1+\alpha(m)}}\frac{1}{\lambda_m^{\frac{n}{1+\alpha(m)}}\lambda_p^{\frac{n\alpha(m)}{1+\alpha(m)}}}$$
$$\times \sum_{j_1,\dots,j_n\in J^m}\left\|\widehat{P_n f}\left(\frac{\lambda_m e_{j_1}^*}{\|e_{j_1}^*\|_m^*},\dots,\frac{\lambda_m e_{j_n}^*}{\|e_{j_n}^*\|_m^*}\right)\right\|_{\alpha(m)}^{\frac{1}{1+\alpha(m)}}$$
$$\times \left\|\widehat{P_n f}\left(\frac{\lambda_p e_{j_1}^*}{\|e_{j_1}^*\|_p^*},\dots,\frac{\lambda_p e_{j_n}^*}{\|e_{j_n}^*\|_p^*}\right)\right\|_{\alpha_1}^{\frac{\alpha(m)}{1+\alpha(m)}}|u(e_{j_1})|\|e_{j_1}^*\|_q^*\dots|u(e_{j_n})|\|e_{j_n}^*\|_q^*,$$

where $\widehat{P_n f}$ is the symmetric n-linear form associated to $P_n f$. Putting

$$\widetilde{D}(q) = \max_{q\le m\le k(q)}\left\{D(\alpha(m))^{\frac{1}{1+\alpha(m)}}\right\}$$
$$\widetilde{M}(q) = \max_{q\le m\le k(q)}\left\{M(\lambda_m, m, \alpha(m))^{\frac{1}{1+\alpha(m)}}.M(\lambda_p, p, \alpha(p))^{\frac{\alpha(m)}{1+\alpha(m)}}\right\}$$

we have

$$\sum_{n\geq 0}\|P_n f(u)\|_\beta^* \leq \widetilde{D}(q)\sum_{n\geq 0}\sum_{q\leq m\leq k(q)} \frac{C(q)^{\frac{n}{1+\alpha(m)}}}{\lambda_m^{\frac{n}{1+\alpha(m)}}\lambda_p^{\frac{n\alpha(m)}{1+\alpha(m)}}}$$

$$\times M(\lambda_m, m, \alpha(m))^{\frac{1}{1+\alpha(m)}} M(\lambda_p, p, \alpha(p))^{\frac{\alpha(m)}{1+\alpha(m)}} \frac{n^n}{n!}$$

$$\times \sum_{j_1,\dots,j_n\in J^m} |u(e_{j_1})|\|e_{j_1}^*\|_q^* \dots |u(e_{j_n})|\|e_{j_n}^*\|_q^*$$

$$\leq \widetilde{D}(q)\widetilde{M}(q)\sum_{n\geq 0}\sum_{q\leq m\leq k(q)} \frac{C(q)^{\frac{n}{1+\alpha(m)}}}{\lambda_m^{\frac{n}{1+\alpha(m)}}\lambda_p^{\frac{n\alpha(m)}{1+\alpha(m)}}} \frac{n^n}{n!}\|u\|_q^{*n}$$

$$= \widetilde{D}(q)\widetilde{M}(q)\sum_{n\geq 0}\sum_{q\leq m\leq k(q)} \left[\frac{C(q)^{\frac{1}{1+\alpha(m)}}}{\lambda_m^{\frac{1}{1+\alpha(m)}}\lambda_p^{\frac{\alpha(m)}{1+\alpha(m)}}}\right]^n \frac{n^n}{n!}\|u\|_q^{*n}$$

$$< \widetilde{D}(q)\widetilde{M}(q)\sum_{n\geq 0}\sum_{q\leq m\leq k(q)} \left[\frac{\delta C(q)^{\frac{1}{1+\alpha(m)}}}{\lambda_m^{\frac{1}{1+\alpha(m)}}\lambda_p^{\frac{\alpha(m)}{1+\alpha(m)}}}\right]^n \frac{n^n}{n!} < \infty$$

for $\|u\|_q^* < \delta$, where

$$0 < \delta < \min_{q\leq m\leq k(q)} \frac{1}{e}\left\{\frac{\lambda_m^{\frac{1}{1+\alpha(m)}}\lambda_p^{\frac{\alpha(m)}{1+\alpha(m)}}}{C(q)^{\frac{1}{1+\alpha(m)}}}\right\}.$$

Therefore there exists a neighbourhood $V_q = V_{p+1}$ of $0 \in F_{p+1}^*$ such that f maps holomorphically V_{p+1} into E_β^* and $f(V_{p+1})$ is bounded in E_β^*. Since F is Schwartz, we may assume that V_{p+1} is relatively compact in U_{p+2}^o. By applying the above argument to each holomorphic map

$$g_u(v) = f(u+v) \qquad \text{for } u \in V_{p+1},\ v \text{ sufficiently near } 0 \in F^*$$

we can choose a neighbourhood V_{p+2} of $0 \in F_{p+2}^*$ such that f maps holomorphically $V_{p+1} + V_{p+2}$ into E_β^* and $f(V_{p+1} + V_{p+2})$ is bounded in E_β^*.

Continuing this process we get a sequence of neighbourhoods $\{V_{p+k}\}$ of $0 \in F_{p+k}^*$, $k \geq 1$ satisfying

$$f(V_{p+1} + \dots + V_{p+k}) \subset E_\beta^*$$

and f is holomorphic, bounded on $V_{p+1} + \dots + V_{p+k}$, for $k \geq 1$. Put $V = \sum_{k=1}^\infty V_{p+k}$. Then V is a neighbourhood of $0 \in F^*$ and $f(V) \subset E_\beta^*$. Since

$f|_{\sum_{j=1}^{k} V_{p+j}}$ is holomorphic for all $k \geq 1$, it follows that f is holomorphic on V. Thus $f : V \to E_\beta^*$ is holomorphic and bounded.

The proposition is proved.

Note that this resut is also true for the case where F is a Fréchet-Schwartz space having an absolute basis.

LEMMA 4.3. *Let $T : X \to Y$ be a nuclear linear map between Banach spaces and f a holomorphic function on an open set $D \subset Y$. Then $f \circ T$ admits a local Dirichlet representation on $\widehat{D} = T^{-1}(D)$.*

PROOF. Let $x_0 \in \widehat{D}$. We may assume that $0 \in \widehat{D}$ and $x_0 = 0$. In view of [5], there exist two sequences of complex numbers $\{\xi_k\}$ and $\{\alpha_k\}$ satisfying

$$z = \sum_{k\geq 1} \xi_k \exp \alpha_k z \qquad \text{for} \quad z \in \mathbb{C},$$

and

$$C_r = \sum_{k\geq 1} |\xi_k| \exp(|\alpha_k| r) < \infty \qquad \text{for} \quad r \geq 0.$$

We write

$$Tx = \sum_{j\geq 1} h_j(x) e_j$$

with

$$\{h_j\} \subset X^*, \{e_j\} \subset Y \qquad \text{and} \qquad b = \sum_{j\geq 1} \|h_j\| \|e_j\| < \infty.$$

Let $a > 0$ be sufficiently small such that

$$\sum_{n\geq 0} \frac{C_1^n a^n b^n n^n}{n!} < \infty$$

and

$$\frac{a e_j}{\|e_j\|} \in V \qquad \text{for} \quad j \geq 1,$$

where V is a neighbourhood of $0 \in Y$ satisfying

$$\|f\|_V := \sup\{|f(y)| : y \in V\} < \infty.$$

We now write the Taylor expansion of f at $0 \in V$

$$f(y) = \sum_{n\geq 0} P_n f(y) \qquad \text{for} \quad y \in V.$$

Hence we get

$$
\begin{aligned}
&f(T(a^2x))\\
&\quad= \sum_{n\ge 0} P_n f\biggl(\sum_{j\ge 1} h_j(a^2x)e_j\biggr)\\
&\quad= \sum_{n\ge 0} a^{2n} \sum_{j_1,\dots,j_n\ge 1} \|h_{j_1}\|\cdots\|h_{j_n}\|\widehat{P_n f}(e_{j_1},\dots,e_{j_n})\|\frac{h_{j_1}(x)}{\|h_{j_1}\|}\cdots\frac{h_{j_n}(x)}{\|h_{j_n}\|}\\
&\quad= \sum_{n\ge 0} a^{2n} \sum_{j_1,\dots,j_n\ge 1} \|h_{j_1}\|\cdots\|h_{j_n}\|\widehat{P_n f}(e_{j_1},\dots,e_{j_n})\\
&\qquad\times\biggl(\sum_{k\ge 1}\xi_k\exp\alpha_k\frac{h_{j_1}(x)}{\|h_{j_1}\|}\biggr)\cdots\biggl(\sum_{k\ge 1}\xi_k\exp\alpha_k\frac{h_{j_n}(x)}{\|h_{j_n}\|}\biggr)\\
&\quad= \sum_{n\ge 0} a^{2n} \sum_{\substack{j_1,\dots,j_n\ge 1\\ k_1,\dots,k_n\ge 1}} \|h_{j_1}\|\cdots\|h_{j_n}\|\xi_{k_1}\cdots\xi_{k_n}\\
&\qquad\times\widehat{P_n f}(e_{j_1},\dots,e_{j_n})\exp\biggl[\alpha_{k_1}\frac{h_{j_1}(x)}{\|h_{j_1}\|}+\cdots+\alpha_{k_n}\frac{h_{j_n}(x)}{\|h_{j_n}\|}\biggr].
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
&\sum_{n\ge 0} a^{2n} \sum_{\substack{j_1,\dots,j_n\ge 1\\ k_1,\dots,k_n\ge 1}} \|h_{j_1}\|\cdots\|h_{j_n}\|\ |\xi_{k_1}|\cdots|\xi_{k_n}|\\
&\qquad\times\widehat{P_n f}(e_{j_1},\dots,e_{j_n})\exp[|\alpha_{k_1}|+\cdots+|\alpha_{k_n}|]\\
&\le \sum_{n\ge 0} a^{2n} \sum_{j_1,\dots,j_n\ge 1} \|h_{j_1}\|\|e_{j_1}\|\cdots\|h_{j_n}\|\|e_{j_n}\|\\
&\qquad\times\biggl|\widehat{P_n f}\biggl(a\frac{e_{j_1}}{\|e_{j_1}\|},\dots,a\frac{e_{j_n}}{\|e_{j_n}\|}\biggr)\biggr|\biggl(\sum_{k\ge 1}|\xi_k|\exp|\alpha_k|\biggr)^n\\
&\le \|f\|_V\sum_{n\ge 0}\frac{a^n n^n}{n!}\biggl(\sum_{j\ge 1}\|h_j\|\|e_j\|\biggr)^n C_1^n\\
&= \|f\|_V\sum_{n\ge 0}\frac{a^n b^n C_1^n n^n}{n!} < \infty \qquad \text{for} \quad \|x\| < 1.
\end{aligned}
$$

Hence

$$
\begin{aligned}
(f \circ T)(x) &= f(Tx) = f(T(a^2 x/a^2)) \\
&= \sum_{n\geq 0} a^{2n} \sum_{\substack{j_1,\dots,j_n\geq 1\\ k_1,\dots,k_n\geq 1}} \|h_{j_1}\| \cdots \|h_{j_n}\| \xi_{k_1} \cdots \xi_{k_n} \\
&\quad \times \widehat{P_n f}(e_{j_1}, \dots, e_{j_n}) \exp\left[\alpha_{k_1} \frac{h_{j_1}(x)}{a^2\|h_{j_1}\|} + \cdots + \alpha_{k_n} \frac{h_{j_n}(x)}{a^2\|h_{j_n}\|}\right] \\
&= \sum_{n\geq 0} a^{2n} \sum_{\substack{j_1,\dots,j_n\geq 1\\ k_1,\dots,k_n\geq 1}} \|h_{j_1}\| \cdots \|h_{j_n}\| \xi_{k_1} \cdots \xi_{k_n} \\
&\quad \times \widehat{P_n f}(e_{j_1}, \dots, e_{j_n}) \exp\left\langle x, \alpha_{k_1} \frac{h_{j_1}}{a^2\|h_{j_1}\|} + \cdots + \alpha_{k_n} \frac{h_{j_n}}{a^2\|h_{j_n}\|}\right\rangle
\end{aligned}
$$

for $\|x\| < a^2$. The lemma is proved.

REMARK. In the case of entire functions, Lemma 4.3 was proved in [4].

We now present the main result of the paper.

THEOREM 4.4. *Let F be a nuclear Fréchet space and K be a compact set of uniqueness in a nuclear Fréchet space E with $E \in (\Omega)$. Then the following conditions are equivalent:*

a) $F \in (\mathrm{LB}_\infty)$

b) *Every separately holomorphic function on $K \times F^*$ is holomorphic*

c) *Every separately holomorphic function as in b) has a local Dirichlet representation.*

PROOF. a) *implies* b). Let a separately holomorphic function $f : K \times F^* \to \mathbb{C}$ be given. We shall prove that f is holomorphic at every point $(x_0, u_0) \in K \times V$, where V is a neighbourhood of u_0 in F^*. Without loss of generality we may assume that $0 \in K \times V$ and $x_0 = 0, u_0 = 0$.

Consider the holomorphic function

$$f_{F^*} : F^* \to H(K)$$

induced by f. From Propositions 3.2 and 4.2 it follows that there exists a neighbourhood W of $0 \in F^*$ such that $f_{F^*}(W)$ is bounded in $H(K)$.

Since $H(K)$ is regular (Theorem 3.1) there exists a neighbourhood U of K in E such that $f_{F^*}(W)$ is contained and bounded in $H^\infty(U)$. It follows that f is bounded on $W \times U$ and hence f is holomorphic at $0 \in K \times F^*$.

b) *implies* c). Let f be a separately holomorphic function on $K \times F^*$. Fix $(x_0, u_0) \in K \times F^*$. We may assume that $x_0 = 0$, $u_0 = 0$. The hypothesis implies that f is bounded on a balanced convex neighbourhood W of $0 \in E \times F^*$. Therefore we may consider f as a holomorphic function on a neighbourhood D of $0 \in (E \times F^*)_W$, the Banach space associated to W. Since $E \times F^*$ is nuclear, there exists a balanced convex neighbourhood U of $0 \in E \times F^*$ such that the canonical map $T : (E \times F^*)_U \to (E \times F^*)_W$ is nuclear. From Lemma 4.3 we deduce that $f \circ T$ and hence, f admits a Dirichlet representation at $0 \in E \times F^*$.

c) *implies* a). By [13] it suffices to prove that every continuous linear map $T : F^* \to E^*$, where $E = H(\mathbb{C})$ is the space of holomorphic functions on $\mathbb{C}$, is compact.

Consider the separately continuous bilinear associated map $f : E \times F^* \to \mathbb{C}$. Let K be a compact set of uniqueness in the nuclear Fréchet space E. We may assume that K is balanced, convex and $0 \in K \times F^*$. By the hypothesis we can find a balanced neighbourhood $U \times W$ of $0 \in E \times F^*$ and $\{\sigma_j\} \subset E^*$; $\{z_j\} \subset F^{**} = F$ satisfying

$$T(u)(x) = f|_{U \times W}(x, u) = \sum_{j \geq 1} \xi_j \exp[\langle x, \sigma_j \rangle + \langle u, z_j \rangle] \quad \text{for } x \in U, u \in W$$

and

$$\sum_{j \geq 1} |\xi_j| \exp[\|\sigma_j\|_B^* + \|z_j\|_L^*] < \infty$$

for every compact set $B \subset U$, $L \subset W$.

Since T is linear in $x \in E$, we imply that

$$T(u)(x) = \sum_{j \geq 1} \xi_j \sigma_j(x) \exp u(z_j) \quad \text{for } x \in E, u \in W$$

and

$$\sum_{j \geq 1} |\xi_j| \|\sigma_j\|_B^* \exp \|z_j\|_L^* < \infty$$

for every compact set $B \subset U$, $L \subset W$.

Since E is Fréchet, we can choose $p \geq 1$ such that

$$\sum_{j \geq 1} |\xi_j| \|\sigma_j\|_p^* < \infty.$$

Moreover, since $E \in (\Omega)$ we can find $q \geq 1$ for all $k \geq 1$ there exists $d > 0$ verifying

$$\|\cdot\|_q^{*1+d} \leq \|\cdot\|_k^* \|\cdot\|_p^{*d}.$$

Consider a compact set K and a $k \geq 1$ such that $U_k \subset K \subset U$. Thus for every compact set $L \subset \frac{1}{1+d} W$ we have

$$\begin{aligned}
&\sum_{j\geq 1} |\xi_j| \|\sigma_j\|_q^* \exp \|z_j\|_L^* \\
&\qquad \leq \sum_{j\geq 1} |\xi_j| \|\sigma_j\|_K^{*1/(1+d)} \|\sigma_j\|_p^{*d/(1+d)} \exp \|z_j\|_L^* \\
&\qquad = \sum_{j\geq 1} |\xi_j|^{1/(1+d)} |\xi_j|^{d/(1+d)} \|\sigma_j\|_K^{*1/(1+d)} \|\sigma_j\|_p^{*d/(1+d)} \exp \|z_j\|_L^* \\
&\qquad \leq \left(\sum_{j\geq 1} |\xi_j| \|\sigma_j\|_K^* \exp \|z_j\|_{(1+d)L} \right)^{1/(1+d)} \left(\sum_{j\geq 1} |\xi_j| \|\sigma\|_p^* \right)^{d/(1+d)} < \infty.
\end{aligned}$$

Hence T maps continuously F^* into F_q^*. This implies the compactness of T.

The theorem is completely proved.

REMARK. In the case $(\overline{\overline{\Omega}}, \mathrm{DN})$, the mixed Hartogs Theorem is proved by Nguyen Van Khue and Nguyen Ha Thanh [7] for separately holomorphic functions on an open set $E \times D$ in $E \times F^*$.

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