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THE COMPLEMENT OF A D-TREE IS COHEN-MACAULAY

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Abstract

In this work we obtain the result that the complement of a d-tree is a Cohen-Macaulay graph. To do this we use a theorem by Fröberg that estabilishes a condition for a Stanley-Reisner ring of a simplicial complex to be Cohen-Macaulay and a useful lemma to pass from a Stanley-Reisner ideal of a simplicial complex to an edge ideal of a graph.

1. Introduction

We can associate to a graph G with n vertices the quotient ring $\mathsf{K}[v]/I(G)$, where $v = (v_1, \dots, v_n)$ and $I(G)$ is the ideal generated by all the squarefree monomials $v_i \cdot v_j$ such that $\{v_i, v_j\}$ is an edge of G. The graph G is said to be *Cohen-Macaulay* (C-M for short) if $\mathsf{K}[v]/I(G)$ is Cohen-Macaulay. C-M graphs were studied in several works, see for example [8], where one can find constructions of C-M graphs and properties about bipartite C-M graphs. These last were characterized in [6].

A complete classification of C-M graphs does not exist. However, together with the cited result [6], Herzog, Hibi and Zheng found out that a chordal graph is C-M if and only if it is unmixed (see [7]). In particular this is true for trees and forests.

There is a generalization of such result to so called simplicial trees introduced by Sara Faridi. She found a criterion for Cohen-Macaulay simplicial trees in [1]. In particular she showed that all grafted trees (trees obtained by adding a leaf for every facet) are Cohen-Macaulay. For a simplicial tree to be grafted and to be unmixed it is equivalent.

A similar criterion to build Cohen-Macaulay graphs is in [3], where it is shown that every graph obtained by adding a leaf to every vertex is C-M.

There are other works on the properties of monomial ideals to be sequentially Cohen-Macaulay. In [2] it is proved that the facet ideal of a simplicial tree is sequentially C-M, by using the Alexander dual of a simplicial tree. A

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similar method is used by Francisco and Van Tuyl in the paper [4], that complements the results in [2] and [7]. In particular they showed that a chordal graph is sequentially Cohen-Macaulay, via the componentwise linearity of the Alexander dual of the edge ideal.

In this article we prove that if G is a d-tree, then the complement (i.e. the graph whose edges are all the non-edges of G) is Cohen-Macaulay. This enables us to get lots of examples of CM-graphs. To do this we use a theorem by Fröberg ([5]) which gives a condition for the Stanley-Reisner ring of a simplicial complex with a 2-linear resolution to be Cohen-Macaulay.

The Stanley-Reisner ring is a quotient ring associated to a simplicial complex. More precisely, if Δ is a simplicial complex on the vertex set $V = \{v_1, \dots, v_n\}$ and I_Δ is the ideal generated by the non-faces of Δ, the Stanley-Reisner ring is $\mathsf{K}[\Delta] = \mathsf{K}[v_1, \dots, v_n]/I_\Delta$.

In the next section we recall some concepts from graph theory and commutative algebra that we use. In particular we define the concept of d-tree, and we recall what an m-linear resolution of a ring is.

2. The background

In this section we recall all the definitions and properties we use throughout the paper. In particular we treat the edge ideal, the concepts dealing with simplicial complexes we need for the paper and the definition of d-tree. Moreover we recall basic notions on resolutions.

DEFINITION 2.1. The *complementary graph* of $G = (V, E)$ is the graph $\bar{G}$ with the vertices of V and edges all the pairs $\{v_i, v_j\}$ such that $i \neq j$ and $\{v_i, v_j\} \notin E$.

DEFINITION 2.2. Let $G = (V, E)$ be a graph, with $V = \{v_1, \dots, v_n\}$. The *edge ideal* $I(G)$ is the ideal in the polynomial ring in the n variables $v_1, \dots, v_n$ over a field K, $\mathsf{K}[v] = \mathsf{K}[v_1, \dots, v_n]$, generated by those squarefree quadratic monomials $v_i \cdot v_j$ such that $\{v_i, v_j\}$ is an edge of G.

The graph G is called *Cohen-Macaulay* over K if the quotient ring $\mathsf{K}[v]/I(G)$ is Cohen-Macaulay.

DEFINITION 2.3. The *q-skeleton* of a simplicial complex Δ is the simplicial complex Δ^q consisting of all the p-simplices of Δ with $p \leq q$.

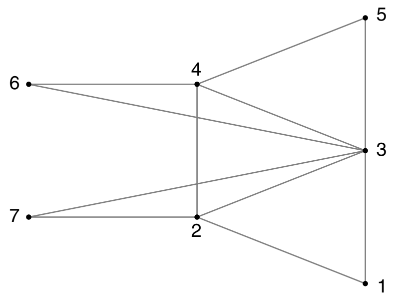
Let $G = (V, E)$ be a graph. $\Delta(G)$ is the simplicial complex whose faces are subsets F of V such that every pair of elements in F is an edge in E.

DEFINITION 2.4. We can define a *d-tree* inductively in the following way:

- a complete graph with $d + 1$ vertices is a d-tree;

- if G is a d-tree and we add to G a vertex v, together with the edges connecting v with a sub-complete graph with d vertices of G, then $\{v\} \cup G$ is a d-tree.

Note that a 1-tree is a usual tree, and a 2-tree consists of triangles where each triangle is attached to the remaining part of the graph in an edge. (See the following picture)



2.1. Basic notions on resolutions

Let K be a field and $R = \mathsf{K}[x] = \mathsf{K}[x_1, \ldots, x_n]$. A sequence $\mathscr{F}_\bullet$ of maps Φ_i of free R-modules

$$\mathscr{F}_\bullet 0 \longleftarrow F_0 \overset{\Phi_1}{\longleftarrow} F_1 \overset{\Phi_2}{\longleftarrow} F_2 \cdots F_{l-1} \overset{\Phi_l}{\longleftarrow} F_l \longleftarrow 0$$

is a *complex* if $\Phi_i \circ \Phi_{i+1} = 0$ for all i.

A complex $\mathscr{F}_\bullet$ is a *free resolution* of a module M over R if $\mathrm{Ker}(\Phi_i) = \mathrm{Im}(\Phi_{i+1})$ for all $i > 0$[1] and $M = F_0/\,\mathrm{Im}(\Phi_1)$.

A resolution is *pure* if for all i all the nonzero entries of Φ_i are homogeneous of the same degree. In such a resolution every F_i can be written in the form $R(-c_i)^{\beta_i}$, the sequence of numbers $c_1, c_2 - c_1, \ldots, c_l - c_{l-1}$ is called *degree type* and the β_i are called the *Betti numbers*.

Finally a resolution is *m-linear* if its degree type is $m, 1, \ldots, 1$.

3. The result

Before stating our main result, we recall a theorem we use

THEOREM 3.1 (see [5]). *Let Δ be a simplicial complex. Then the following are equivalent:*

[1] Trivially it is always $\mathrm{Im}(\Phi_{i+1}) \subseteq \mathrm{Ker}(\Phi_i)$.

(i) *The Stanley-Reisner ring* $\mathsf{K}[\Delta]$ *is Cohen-Macaulay of Krull dimension* $d+1$ *and has 2-linear resolution.*

(ii) *The 1-skeleton* $G(\Delta)$ *of* Δ *is a* d*-tree and* $\Delta = \Delta(G(\Delta))$.

The next lemma describes the connection between the Stanley-Reisner ring associated to a graph, and the edge ideal of the complementary graph.

LEMMA 3.2. *The Stanley-Reisner ideal of the simplicial complex* $\Delta(G)$*,* $I_{\Delta(G)}$*, is equal to the edge ideal of the complement of* G*,* $I(\bar{G})$*.*

PROOF. The Stanley-Reisner ideal of a simplicial complex is the ideal generated by all its non-faces, i.e. $I_{\Delta(G)} = (\{v_{i_1} \cdots v_{i_r}$ s.t. $i_1 < \cdots < i_r$, $\{v_{i_1}, \ldots, v_{i_r}\} \notin \Delta(G)\})$, now if $\{v_{i_1}, \ldots, v_{i_r}\} \notin \Delta(G)$ it means that there exists at least one edge, call it $\{v_{i_1}, v_{i_2}\}$, that does not belong to the edges of G, i.e. $\{v_{i_1}, v_{i_2}\}$ belongs to the edges of $\bar{G}$. And so $I_{\Delta(G)} = I(\bar{G})$.

Now we are ready to prove our main theorem.

THEOREM 3.3. *The complement of a* d*-tree is Cohen-Macaulay.*

PROOF. The Stanley-Reisner ring of Δ is the quotient $\mathsf{K}[\Delta] = \mathsf{K}[v_1, \ldots, v_n]/I_{\Delta(G)}$, where $v_1, \ldots, v_n$ are the variables representing the n vertices of Δ. By using the definition of Cohen-Macaulay for a graph, $\bar{G}$ is Cohen-Macaulay if the quotient $\mathsf{K}[v_1, \ldots, v_n]/I(\bar{G})$ is Cohen-Macaulay and the previous theorem assures that this is equivalent to the condition that the 1-skeleton G is a d-tree and $\Delta = \Delta(G(\Delta))$; this last is satisfied whenever G is a d-tree and so the theorem is proved.

4. Some nice examples

In this section we will give some examples of classes of graphs on which we can use our main theorem.

We start by considering the complements of trees with diameter ≤ 3.

4.1. A complete graph is Cohen-Macaulay

Consider a tree $T2$ of diameter 2, i.e. a tree whose shortest path between any two vertices is at most 2, so there is a vertex of $T2$, call it v, that is joined with all the other vertices and these last are the only edges in $T2$, as shown in Figure 1.

Then in $\overline{T2}$ all the vertices but v are joined with each other and v becomes isolated. This means that $\overline{T2}$ has two connected components: a complete graph and an isolated vertex. See Figure 2.

Since $\overline{T2}$ is the complement of a 1-tree, the main theorem implies that this graph is Cohen-Macaulay. It is known that a graph is Cohen-Macaulay if

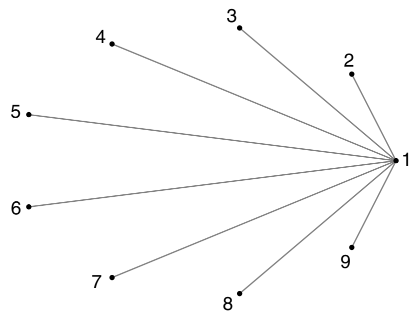


FIGURE 1. $T2$

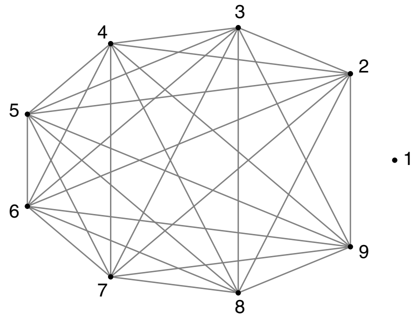


FIGURE 2. $\overline{T2}$

and only if its connected components are Cohen-Macaulay (see [8]) and an isolated vertex is trivially Cohen-Macaulay. Therefore the construction of $\overline{T2}$ and application of the main theorem shows how to recover, in a simple way, the known fact that a complete graph is Cohen-Macaulay.

4.2. A quasi-complete graph is Cohen-Macaulay

Consider a tree $T3$ of diameter 3, i.e. a tree whose shortest path between any two vertices is at most 3. Such a tree looks like that in Figure 3.

There are two vertices, call them v and w (1 and 2 in the picture), joined together and every other vertex is joined either with v or with w. Call V the set of vertices joined with v (3 and 4 in the picture) and let $|V| = k$, and W the set of vertices joined with w (5, 6 and 7 in the picture), with $|W| = h$, (so $n = k + h + 2$ if n is the total number of vertices in $T3$).

Consider the complement of such a tree, $\overline{T3}$. We call a graph *quasi-complete* if it is built in the following way: there is a complete subgraph with $k + h$ vertices, k of its vertices are joined with w and the other h are joined with v. Such a graph looks like the one drawn in Figure 4 and it is Cohen-Macaulay, by the main theorem.

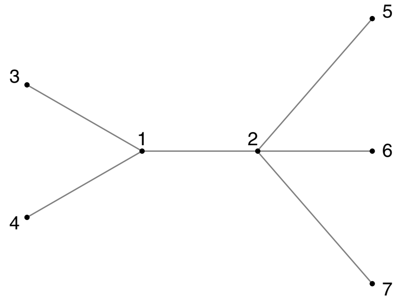


FIGURE 3. $T3$

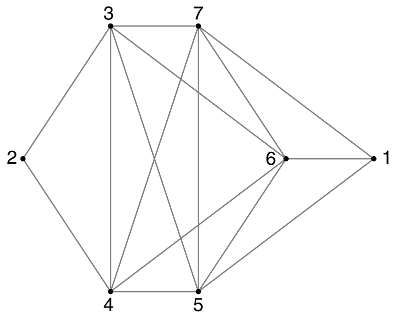
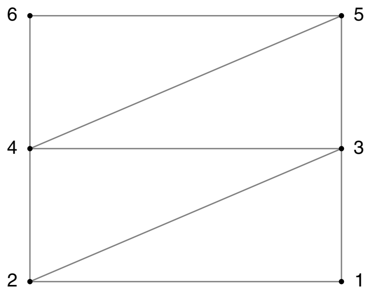


FIGURE 4. $\overline{T3}$

4.3. d-paths

We define a *d-path* as a graph on the vertex set $\{v_1, \ldots, v_n\}$ which is the union of the complete graphs $\{v_1, \ldots, v_{d+1}\}, \{v_2, \ldots, v_{d+2}\}, \ldots, \{v_{n-d}, \ldots, v_n\}$.

Thus a 1-path is a usual path with edges $(v_1, v_2), (v_2, v_3), \ldots, (v_{n-1}, v_n)$, and a 2-path looks like the following picture:

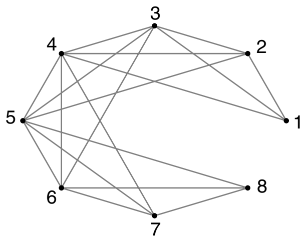


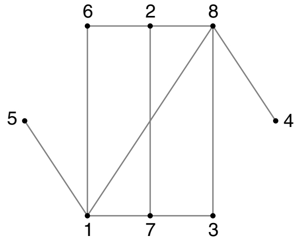
Of course a d-path is a particular d-tree.

The complement of a d-path on $\{v_1, \dots, v_n\}$ has edges (v_i, v_j) for all i, j with $j - i > d$. Such a graph is Cohen Macaulay, by the main theorem.

Here we show as an example the complement of the 4-path with 8 vertices.

This last is the union of the complete graphs $\{v_1, \dots, v_4\}$, $\{v_2, \dots, v_5\}$, $\{v_3, \dots, v_6\}$, $\{v_4, \dots, v_7\}$, $\{v_5, \dots, v_8\}$ and looks like the following picture (to the left)





The complement of the graph in the example looks like the picture to the right, and is C-M.

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