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FUNCTIONAL COMPOSITION IN $B_{p,k}$ SPACES AND APPLICATIONS

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Abstract

Let $f(x,z)$, $x \in \mathsf{R}^N$, $z \in \mathsf{C}^M$, be a smooth function in the sense that its Fourier transform has a good behaviour. We study the composition $f(x,u(x))$, where u is in a generalized Hörmander $B_{p,k}$ space in the sense of Björck [1]. As a consequence we obtain results of local solvability and hypoellipticity of semilinear equations of the type $P(D)u + f(x, Q_1(D)u, \dots, Q_M(D)u) = g$, with $g \in B_{p,k}$, and fully nonlinear elliptic equations.

0. Introduction

During the last years the attention of several authors has been directed to local solvability for semilinear equations

$$P(x,D)u + f(x,Du) = g(x),$$

where the linear part is assumed to be locally solvable and the nonlinear term contains derivatives of lower order with respect to $P(x,D)$ (see Gramchev and Rodino [9], Hounie and Santiago [11], Marcolongo and Oliaro [13], De Donno and Oliaro [4], etc.). The regularity of the data is usually prescribed according to the regularity of the coefficients of $P(x,D)$ (see, for example, Mascarello-Rodino [14]). The functional frame is given in this case by linear spaces (Sobolev, Hilbert or Banach) that, under suitable assumptions form an algebra, and then one can apply Functional Analysis results (Inverse Function Theorem, Fixed Point Theorem, etc.). The algebra property guarantees a possible analytical dependence of f with respect to u, and consequently, brings a priori estimates for the nonlinearity f. In this note we prove that the functional composition $f(x,u(x))$, where u is in a generalized Hörmander $B_{p,k}$ space in the sense of Björck [1] and f belongs to some $B_{p,\bar{k}}$ space with respect to all the variables (see Theorem 2.6 for the details), is also in $B_{p,k}$. We use the more modern framework of ultradifferentiable functions and ultradistributions

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as introduced by Braun, Meise and Taylor [3]. In particular, we need a strong weight ω as defined in Meise and Taylor [15] for our purposes. Moreover, we see that if the weight ω is not strong then the algebra property fails in general (see Theorem 2.2 and Example 2.4). We follow the lines of Bourdaud, Reissig and Sickel [2], that study composition in $B_{p,k}$ spaces in the simpler case $p = 2$ and $k(t) = e^{t^{1/s}}$; we recapture as a particular case the result proved in Ref. [2].

As an application, we investigate local solvability of the semilinear operator

$$F(u) = P(D)u + f(x, Q_1(D)u, \ldots, Q_M(D)u)$$

where $P, Q_1, \ldots, Q_M$ are linear partial differential operators with constant coefficients. The problem consists in finding a local solution u in a neighborhood Ω of a point x^0, for any f in a given class of data. We prove, continuing the works of Messina and Rodino [17] and Messina [16], that for every $g \in B_{p,k}$ in the sense of Björck, where $k(\xi) = e^{\omega(\xi)}$ and ω is a strong weight, the equation $F(u) = g$ admits (locally near a point) a solution $u \in B_{p,k\tilde{P}}$ (cf. Theorems 3.2 and 3.5). Two different hypotheses on the nonlinearity f will be considered: the first, to assume that $f(x^0, z) = 0$, for some $x^0 \in \mathsf{R}^N$ and all $z \in \mathsf{C}^M$ and that $\tilde{Q}_i(\xi) \leq C\tilde{P}(\xi)$ for all $\xi \in \mathsf{R}^N$ and each $i = 1, \ldots, M$ and some constant $C > 0$; the second, the essentially weaker hypothesis on the nonlinear term $f(x, 0) = 0$ for every $x \in \mathsf{R}^N$, and the stronger one on the differential operators with constant coefficients $\frac{\tilde{Q}_i(\xi)}{\tilde{P}(\xi)} \to 0$ as $|\xi| \to +\infty$ for each $i = 1, \ldots, M$. We observe that we extend the corresponding results of [17] and [16], that solve locally the equation $F(u) = g$ when g belongs to a classical Hörmander $B_{p,k}$ and $f(x, z)$ is assumed to be holomorphic in the z-variable (cf. Example 3.7).

As a further application of the composition result we then analyze semilinear operators with hypoelliptic linear part, extending a result proved in [8].

We end this introduction by giving some examples of operators to which our results apply. Here we just give some hints, referring to Section 3 for the details. Let us consider a fully nonlinear equation

$$f(x, D^\alpha u)_{|\alpha| \leq m} = g(x), \tag{1}$$

where the operator $f[u] = f(x, D^\alpha u)_{|\alpha| \leq m}$ is supposed to be elliptic and $f(x, 0) = 0$ for all x; given arbitrarily x^0 and $g \in B_{p,k}$ we can find (locally near x^0) a solution of (1) belonging to some $B_{p,h}$ space, where the nonlinear function $f(x, z)$ is supposed to be sufficiently regular.

The same kind of result holds for the following semilinear version of the Schrödinger equation for a free particle:

$$P(D)u + f(x, u, D_{x_1}u, \ldots, D_{x_{N-1}}u, P(D)u) = g, \tag{2}$$

where $P(D) = -D_{x_N} + \sum_{j=1}^{N-1} D_{x_j}^2$ and $f(x^0, z) = 0$: we can solve (2) locally near the point x^0 in the space $B_{p,k}$.

We consider finally the following simple example in three variables $x = (x_1, x_2, x_3)$:

$$\frac{\partial}{\partial x_1}u + i\frac{\partial}{\partial x_2}u + f(x, u, \partial_{x_1}u, \partial_{x_2}u) = g(x). \tag{3}$$

Under similar assumptions as in the previous cases we can prove the local solvability of (3) in $B_{p,k}$ spaces at any point x^0, assuming $f(x^0, z) = 0$ or $f(x, 0) = 0$; observe that the assumption that the nonlinear term f does not depend on $\partial_{x_3}u$ is essential, since it was proved by Lewy [12] that the linear operator

$$\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2} - 2i(x_1 + ix_2)\frac{\partial}{\partial x_3}$$

is not locally solvable at the origin.

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1. Notation and preliminaries

First we introduce the spaces of functions and ultradistributions and most of the notation that will be used in the sequel (see [3], [1]).

1.1. Weight functions

Let $\omega : [0, \infty[\to [0, \infty[$ be a continuous function which is increasing and satisfies $\omega(0) > 0$. We consider the following conditions on ω:

(α) $\omega(2t) \leq K(1 + \omega(t))$ for some $K \geq 1$ and for all t.

(β) $\int_1^\infty \frac{\omega(t)}{t^2}\, dt < \infty$.

(γ) $\log(t) = o(\omega(t))$ as t tends to ∞.

$(\gamma)'$ $\log(1 + t) = O(\omega(t))$ as $t \to \infty$,

(δ) $\varphi : t \to \omega(e^t)$ is convex.

(ε) there exists $C > 0$ with $\int_1^\infty \frac{\omega(yt)}{t^2}\, dt \leq C\omega(y) + C$ for all $y \geq 0$.

If ω satisfies (α), (β), (γ) and (δ) is called weight function (in the sense of Braun, Meise and Taylor [3]). A weight function that satisfies (ε) is said strong weight. Björck considers in [1] subadditive weights that satisfy (β), $(\gamma)'$ to develop a theory of ultradifferentiable functions and ultradistributions.

For a weight function ω we define $\tilde{\omega} : \mathbb{C}^N \to [0, \infty[$ by $\tilde{\omega}(z) = \omega(|z|)$ and again call this function ω, by abuse of notation. The Young conjugate of φ is defined by $\varphi^*(x) = \sup_{y>0}\{xy - \varphi(y)\}$.

1.2. Spaces of functions and ultradistributions

Let ω be a weight function and let Ω be an open set in R^N. We define the set of ω-ultradifferentiable functions of Beurling type as

$$\mathscr{E}_{(\omega)}(\Omega) := \{f \in C^\infty(\Omega) : \|f\|_{K,\lambda} < \infty, \text{for every } K \subset \Omega, \text{ and every } \lambda > 0\},$$

where

$$\|f\|_{K,\lambda} := \sup_{x\in K} \sup_{\alpha\in\mathsf{N}_0^N} \left|f^{(\alpha)}(x)\right| \exp\left(-\lambda\varphi^*\left(\frac{|\alpha|}{\lambda}\right)\right). \tag{4}$$

This space is endowed with its natural Fréchet topology. We put

$$\mathscr{D}_{(\omega)}(K) := \{f \in \mathscr{E}_{(\omega)}(\Omega) : \operatorname{supp} f \subset K\}$$

and

$$\mathscr{D}_{(\omega)}(\Omega) := \operatorname*{ind}_{j\to} \mathscr{D}_{(\omega)}(K_j),$$

where $(K_j)_{j\in\mathbf{N}}$ denotes a fundamental sequence of compact sets of Ω. The elements of $\mathscr{D}'_{(\omega)}(\Omega)$ are called ω-ultradistributions of Beurling type.

Following Björck [1], we define the spaces $B_{p,k}$ introduced by Hörmander in the case that $k \in \mathscr{K}_\omega$. We refer to Fieker [7] for a version of the theory which includes the case of non subadditive weights.

For a given weight function ω we define $\mathscr{K}_\omega$ as the set of all functions $k : \mathsf{R}^N \to [0, +\infty[$ such that,

$$k(\xi + \eta) \leq e^{\lambda\omega(|\xi|)}k(\eta), \qquad \eta, \xi \in \mathsf{R}^N$$

for some $\lambda > 0$.

DEFINITION 1.1. Let ω be a weight function, $k \in \mathscr{K}_\omega$ and $1 \leq p \leq \infty$. We denote by $B_{p,k}$ the completion of the normed space consisting of those $u \in \mathscr{E}'_{(\omega)}(\mathsf{R}^N)$ such that

$$\|u\|_{p,k} = \left((2\pi)^{-N} \int_{\mathsf{R}^N} |k(\xi)\widehat{u}(\xi)|^p \, d\xi\right)^{1/p} < \infty, \tag{5}$$

where $\|u\|_{\infty,k}$ denotes $\operatorname{ess\,sup} k(\xi)|\widehat{u}(\xi)|$; $\mathscr{E}'_{(\omega)}(\mathsf{R}^N)$ is the set of all the elements of $\mathscr{D}'_{(\omega)}(\mathsf{R}^N)$ having compact support.

If k, k_1 and k_2 belong to $\mathscr{K}_\omega$ and $s \in \mathsf{R}$, then $k_1 + k_2$, $k_1 \cdot k_2$, $\max(k_1, k_2)$, $\min(k_1, k_2)$ and k^s also belong to $\mathscr{K}_\omega$. If $P \neq 0$ is a polynomial in $\mathsf{C}[z]$ then $\widetilde{P} \in \mathscr{K}_\omega$, where

$$\widetilde{P}(\xi) := \left[\sum_\alpha |D^\alpha P(\xi)|^2\right]^{1/2}.$$

2. Composition in $B_{p,k}$ spaces

In this section we analyze the composition $F(x, \boldsymbol{u}(x))$, where both F and $\boldsymbol{u}(x)$ are in the $B_{p,k}$ spaces. Let ω be a fixed strong weight and $k(\xi) = e^{\omega(\xi)}$; we prove that the space $B_{p,k}$ is an algebra under certain additional condition on ω. Moreover, we observe that the elements of $B_{p,k}$ are actually functions, since ω satisfies (γ) (or $(\gamma)'$, if we assume that $M_\gamma \log(1+t) \le \omega$, for $M_\gamma > 0$ big enough) and, consequently, we give sense to this type of composition. As in Ref. [15], the function

$$\chi(x) = \int_1^\infty \frac{\omega(xt)}{t^2}\, dt = x \int_x^\infty \frac{\omega(t)}{t^2}\, dt$$

is an equivalent concave weight, satisfying $\chi'(x) \ge 0$: for some constant $K > 0$,

$$\omega(x) \le \chi(x) \le K\omega(x) + K \tag{6}$$

for all $x > 0$. Then, the function $k(\xi) = \exp(\chi(\xi))$ is in $\mathscr{K}_\omega$. In the following lemma we can suppose that ω satisfies the more relaxed condition $(\gamma)'$ instead of (γ) (see [15, 1.3]).

Lemma 2.1. *There exist constants* $0 < \mathscr{C} < 1$ *and* $0 < \delta < 1$ *such that*

$$\chi(x+y) - \chi(x) - \chi(y) \le -\mathscr{C} \min\{\chi(x), \chi(y)\} + \delta \tag{7}$$

for all $x, y > 0$.

Proof. We will see that

$$\chi(x+y) - \chi(x) - \chi(y) + \mathscr{C}\chi(x) \le \delta \tag{8}$$

if $0 < x < y$. We can rewrite the first term of (8) as

$$(x+y)\int_{x+y}^\infty \frac{\omega(t)}{t^2}\, dt - x\int_x^\infty \frac{\omega(t)}{t^2}\, dt - y\int_y^\infty \frac{\omega(t)}{t^2}\, dt + \mathscr{C}x\int_x^\infty \frac{\omega(t)}{t^2}\, dt$$

$$= (\mathscr{C} - 1)x\int_x^{x+y} \frac{\omega(t)}{t^2}\, dt - y\int_y^{x+y} \frac{\omega(t)}{t^2}\, dt + \mathscr{C}\, x\int_{x+y}^\infty \frac{\omega(t)}{t^2}\, dt. \tag{9}$$

The first integral in (9) is < 0 (we take $\mathscr{C} < 1$). We prove that the last two integrals sum a number $< \delta$, for some $0 < \delta < 1$. In fact,

$$y\int_y^{x+y} \frac{\omega(t)}{t^2}\, dt \ge y\omega(y)\int_y^{x+y} t^{-2}\, dt = \omega(y)\frac{x}{x+y},$$

and using $\chi(2y) \leq 2\chi(y)$ for each $y > 0$ (by the definition of χ) and (6), we have for the last integral,

$$\mathscr{C}x \int_{x+y}^{\infty} \frac{\omega(t)}{t^2}\,dt = \mathscr{C}\frac{x}{x+y}\chi(x+y) \leq \mathscr{C}\frac{x}{x+y}\chi(2y) \leq \mathscr{C}\frac{x}{x+y}2\chi(y)$$
$$\leq \frac{x}{x+y}\mathscr{C}(2K\omega(y)+2K) \leq \frac{x}{x+y}2\mathscr{C}K\omega(y)+2K\mathscr{C}$$

Then, taking $\mathscr{C} < 1/(2K)$, we finally obtain

$$-y\int_{y}^{x+y}\frac{\omega(t)}{t^2}\,dt + \mathscr{C}x\int_{x+y}^{\infty}\frac{\omega(t)}{t^2}\,dt \leq (2\mathscr{C}K-1)\frac{x}{x+y}\omega(y)+2\mathscr{C}K$$
$$\leq 2\mathscr{C}K.$$

Moreover, if the weight χ satisfies (8), then it's a strong weight, that is, it satisfies (ε). From now on, ω is supposed to satisfy (7). Proceeding as in the proof of [2, Theorem 2.1] we have

THEOREM 2.2. *Let us take $u, v \in B_{p,k}$, $1 \leq p \leq +\infty$, with $k(\xi) = e^{\omega(\xi)}$. Then $uv \in B_{p,k}$ and there exists a constant C_{alg} such that the following estimate holds:*

$$\|uv\|_{p,k} \leq C_{\text{alg}}\|u\|_{p,k}\|v\|_{p,k}. \tag{10}$$

Moreover, one can see that the constant $C_{\text{alg}} = C\|e^{-\mathscr{C}\omega(\xi)}\|_{L^{p'}}$ is finite, since ω satisfies (γ) (or $M_\gamma \log t \leq \omega(t)$ as $t \to \infty$ for a constant M_γ big enough). We observe that, at least for the case $p' \neq \infty$, we can write

$$C_{\text{alg}} = C\left(\int_{\mathsf{R}^N} e^{-\mathscr{C}p'\omega(\xi)}\,d\xi\right)^{\frac{1}{p'}} = C_1\left(\int_0^{+\infty} e^{-\mathscr{C}p'\omega(t)}t^{N-1}\,dt\right)^{\frac{1}{p'}} \tag{11}$$

where in the last integral we apply polar coordinates.

EXAMPLE 2.3. It is easy to see that the weights $\omega(t) = \exp\left[(\log(1+t))^\alpha\right]$, with $0 < \alpha < 1$, or $\omega(t) = t^\alpha(\log(1+t))^\beta$, with $0 < \alpha < 1$ and $\beta > 0$ are strong weights that satisfy condition (8).

As in Meise and Taylor [15, 3.11] we can construct more examples of weights in the following way: Let $(M_j)_{j\in\mathsf{N}_0}$ be a sequence of positive numbers which has the following properties:

(M1) $M_j^2 \leq M_{j-1}M_{j+1}$ for all $j \in \mathsf{N}$;

(M2) there exist $A, H > 1$ with $M_n \leq AH^n \min_{0\leq j\leq n} M_jM_{n-j}$ for all $j \in \mathsf{N}$;

(M3) there exists $A > 0$ with $\sum_{q=j+1}^{\infty} \frac{M_{q-1}}{M_q} \leq A\, j \frac{M_j}{M_{j+1}}$ for all $j \in \mathsf{N}$; and define $\omega_M : [0, +\infty[\to [0, +\infty[$ by

$$\omega_M(t) = \begin{cases} \sup_{j \in \mathsf{N}_0} \log \dfrac{|t|^j M_0}{M_j}, & \text{for } |t| > 0 \\ 0 & \text{for } t = 0. \end{cases}$$

Then there exists a strong weight $\kappa(t)$ which satisfies (8) and

$$\omega_M(t) \leq \kappa(t) \leq C\omega_M(t) + C \quad \text{for some } C > 0 \text{ and all } t > 0.$$

From the properties of ω_M and κ it follows that for each open set Ω of R^N we have

$$\mathscr{E}_{(\kappa)}(\Omega) = \mathscr{E}^{(M_j)}(\Omega) = \left\{ f \in C^\infty(\Omega) \;\middle|\; \sup_{\alpha \in \mathsf{N}_0^N} \sup_{x \in K} \frac{|f^{(\alpha)}(x)|}{h^{|\alpha|} M_{|\alpha|}} < \infty \right.$$
$$\left. \text{for each } h > 0 \text{ and each } K \subset \Omega \text{ compact} \right\}.$$

We refer to Examples 2.7 and 2.8 for more details.

In the following example, we will see that $B_{p,k}$ need not be an algebra when ω does not satisfy property (ε).

EXAMPLE 2.4. We take the weight $\omega(t) = t(\log(1+t))^{-\beta}$, $\beta > 0$. Then ω is not a strong weight. As in [2, Theorem 2.1], we choose $u \equiv v$ such that

$$\mathscr{F}u(\xi)e^{\omega(\xi)/2} = \frac{1}{(1+|\xi|)^\alpha}, \qquad \alpha > \frac{N}{2}.$$

We have $\max(|\eta|, |\xi - \eta|) \leq (L + 1/2)|\xi|$ in $B(\xi/2, L|\xi|)$, where $L > 0$ is a constant that we will fix later. Moreover, if $L < 1/2$, $\min(|\eta|, |\xi - \eta|) \geq (1/2 - L)|\xi|$. Then, for $|\xi|$ large enough, we get

$$\begin{aligned} &\omega(\xi) - \omega(\xi - \eta) - \omega(\eta) \\ &= |\xi|(\log(1+|\xi|))^{-\beta} - |\xi - \eta|(\log(1+|\xi - \eta|))^{-\beta} - |\eta|(\log(1+|\eta|))^{-\beta} \\ &\geq |\xi|(\log(1+|\xi|))^{-\beta} - 2(1/2 + L)|\xi|[\log((1/2 + L)(1+|\xi|))]^{-\beta} \\ &\geq -1/2. \end{aligned}$$

Now, proceeding in a similar way to the proof of [2, Theorem 2.1], we easily obtain that $\|u^2\|_{2,k} = \infty$, where $k(\xi) = e^{\omega(\xi)}$ and $\alpha \leq 3N/4$.

The following remark is essential to prove the composition result.

REMARK 2.5. Let us suppose, as in Theorem 2.2, that u and v belong to the space $B_{p,k}$, and moreover that $\operatorname{supp}\hat{u}$, $\operatorname{supp}\hat{v} \subset \{\xi \in \mathbb{R}^N : |\xi| \geq R\}$. Then repeating the same proof as in Theorem 2.2 we get the following more refined estimate:

$$\|uv\|_{p,k} \leq C(R)\|u\|_{p,k}\,\|v\|_{p,k},$$

where the constant $C(R)$ is of the form

$$C(R) = C\biggl(\int_R^{+\infty} e^{-\mathscr{C}p'\omega(t)}t^{N-1}\,dt\biggr)^{\frac{1}{p'}}. \tag{12}$$

Now we want to state the main result of this section. We consider a function $f(x,z)$, $x \in \mathbb{R}^N$, $z \in \mathbb{C}^M$, and we represent it as

$$G(x, \Re z, \Im z) = G(x,y), \quad x \in \mathbb{R}^N,\ y \in \mathbb{R}^{2M}. \tag{13}$$

THEOREM 2.6. *Let us fix* $p \in [1,+\infty]$ *and three weights* $k(t) = e^{\omega(t)}$, $\tilde{k}(t) = e^{\tilde{\omega}(t)}$ *and* $\overline{k}(t) = e^{\sigma(t)}$, *where:*

$$A\omega(t) \leq \tilde{\omega}(t) \qquad as \quad t \to +\infty, \tag{14}$$

for some suitable constant $A > 1$; *there exists* $\nu \in (0,\mathscr{C})$ *such that*

$$\omega\biggl\{t\omega^{-1}\biggl[\frac{1}{\mathscr{C}-\nu}\log\frac{t}{\omega(t)}\biggr]\biggr\} = o\bigl(\sigma(t)\bigr) \qquad as \quad t \to +\infty, \tag{15}$$

where $\mathscr{C}$ *is the constant of* (7). *We suppose that* $G(x,y)$ *is* $B_{p,\tilde{k}}$ *in the* x*-variable and* $B_{p,\overline{k}}$ *in the* y*-variable, more precisely:*

$$\biggl[(2\pi)^{-N-2M}\int_{\mathbb{R}^N}\int_{\mathbb{R}^{2M}}\bigl|\tilde{k}(\xi)\overline{k}(\eta)\widehat{G}(\xi,\eta)\bigr|^p\,d\eta\,d\xi\biggr]^{\frac{1}{p}} < \infty. \tag{16}$$

Let us fix a compact set $K \subset \mathbb{R}^N$. *Then:*

(i) *if* $\boldsymbol{u}(x) = (u_1(x),\dots,u_M(x))$ *with* $\operatorname{supp} u_j \subset K$ *and* $u_j \in B_{p,k}$ *for every* $j = 1,\dots,M$, *then* $f(x,\boldsymbol{u}(x)) \in B_{p,k}$; *more precisely, there exists a function* $\Psi_K : [0,+\infty)^M \to [0,+\infty)$ *such that:*
   a) Ψ_K *is bounded on bounded sets, i.e. for every* $B \subset [0,+\infty)^M$ *bounded in* $\mathbb{R}^M$ *there exists* C_B *such that* $\sup_{y\in B}|\Psi_K(y)| \leq C_B$;
   b) *we have*

$$\|f(x,\boldsymbol{u}(x))\|_{p,k} \leq \Psi_K(\|u_1\|_{p,k},\dots,\|u_M\|_{p,k}). \tag{17}$$

(ii) *Let us define* $B_{p,k,T} := \{u \in B_{p,k} : \|u\|_{p,k} \leq T\}$, $T > 0$; *then for every* $\boldsymbol{u}^{(1)}(x) = (u_1^{(1)}(x),\dots,u_M^{(1)}(x))$ *and* $\boldsymbol{u}^{(2)}(x) = (u_1^{(2)}(x),\dots,u_M^{(2)}(x))$

with the same properties as $\boldsymbol{u}(x)$ *in* (i) *and satisfying* $u_h^{(j)}(x) \in B_{p,k,T}$ *for* $j = 1, 2$ *and* $h = 1, \dots, M$, *there exists a constant* $C_{K,T}$ *depending on* K *and* T *such that*

$$\|f(x, \boldsymbol{u}^{(1)}(x)) - f(x, \boldsymbol{u}^{(2)}(x))\|_{p,k} \leq C_{K,T} \sum_{j=1}^{M} \|u_j^{(1)} - u_j^{(2)}\|_{p,k}. \tag{18}$$

Before giving the proof of Theorem 2.6 we want to analyze in some particular cases the meaning of the conditions (14) and (15).

EXAMPLE 2.7. Let us fix

$$\omega(t) = t^{1/s}, \qquad s > 1,$$

which corresponds to the Gevrey case. In this particular case the composition in $B_{p,k}$-spaces has been already studied, in the case $p = 2$, in [2], [5], proving results similar to the one of Theorem 2.6. Since $\omega^{-1}(r) = r^s$ we easily obtain that

$$\omega\left\{t\,\omega^{-1}\left[\frac{1}{\mathscr{C} - \nu} \log \frac{t}{\omega(t)}\right]\right\} = \left(1 - \frac{1}{s}\right) \frac{1}{\mathscr{C} - \nu} t^{1/s} \log t;$$

we can then rewrite (14)–(15) in the following way:

$$t^{1/s} \leq \tilde{A}\tilde{\omega}(t) \qquad \text{as} \quad t \to +\infty, \tag{14$'$}$$

$$t^{1/s} \log t = o\big(\sigma(t)\big) \qquad \text{as} \quad t \to +\infty. \tag{15$'$}$$

We notice in particular that conditions (14′)–(15′) are the same as in [2], [5], and so we recover known results in the Gevrey frame.

EXAMPLE 2.8. Let us analyze now the case

$$\omega(t) = \big(\log t\big)^{\beta}, \qquad \beta \geq 1,$$

for t large enough. This weight satisfies condition (7), since the function $x \mapsto (\log(x + y))^{\beta} - (\log(x))^{\beta}$ is decreasing (for a fixed y). Moreover, we have $\omega^{-1}(r) = e^{r^{1/\beta}}$; then

$$\omega\left\{t\,\omega^{-1}\left[\frac{1}{\mathscr{C} - \nu} \log \frac{t}{\omega(t)}\right]\right\} \sim \max\left\{1, \frac{1}{\mathscr{C} - \nu}\right\} \big(\log t\big)^{\beta} \qquad \text{as} \quad t \to +\infty,$$

which implies that the conditions (14)–(15) in this case become

$$\big(\log t\big)^{\beta} \leq \tilde{A}\,\tilde{\omega}(t) \qquad \text{as} \quad t \to +\infty, \tag{14$''$}$$

$$\big(\log t\big)^{\beta} = o\big(\sigma(t)\big) \qquad \text{as} \quad t \to +\infty. \tag{15$''$}$$

We can fix for example $\tilde{\omega}(t) = \sigma(t) := \vartheta(t)$ with $\left(\log t\right)^{\beta} = o\left(\vartheta(t)\right)$ as $t \to +\infty$.

From the concavity of the weight ω we obtain

LEMMA 2.9. (1) *Let* $\alpha > 0$*; for every* $t > 0$ *we have* $\omega(\alpha t) \leq \max\{1, \alpha\}\omega(t)$.
(2) *The function* $t \mapsto \log \frac{t}{\omega(t)}$ *is increasing, and it tends to infinity for* $t \to \infty$.

The fundamental tool to prove the composition result of Theorem 2.6 is the following technical proposition.

PROPOSITION 2.10. *Let us fix a compact set* K *as in Theorem 2.6. Then for every* $\nu \in (0, \mathscr{C})$ *we can find* $a_\nu > 0$ *(*$\mathscr{C}$ *is the constant of* (7)*), and moreover there exist positive constants* $M, C, C_1, b, C_K, \widetilde{C}_K$*, where* $M > 1$*, and* $C_K, \widetilde{C}_K$ *depend on* K*, such that for every real-valued function* $u \in B_{p,k}$*,* $p \in [1, +\infty]$*, with* $\operatorname{supp} u \subset K$ *the following estimates hold:*

$$\|e^{iu(x)} - 1\|_{p,k} \leq C_{K,M} \tag{19}$$

for $\|u\|_{p,k} \leq M$*, and*

$$\|e^{iu(x)} - 1\|_{p,k} \leq Ce^{a_\nu \omega(\|u\|_{p,k})}\left[C_1 + \widetilde{C}_K e^{b\omega\left\{\|u\|_{p,k}\omega^{-1}\left[\frac{1}{\mathscr{C}-\nu}\log\frac{\|u\|_{p,k}}{\omega(\|u\|_{p,k})}\right]\right\}}\right] \tag{20}$$

for $\|u\|_{p,k} > M$.

Estimates of the type of (19)–(20) have been proved in [2] in the simpler case when $p = 2$ and the weight function is of the kind $\omega(t) = t^{1/s}$ (which corresponds to Gevrey type spaces); we use the idea developed in [2], but we need more involved techniques. In particular, in [2] the fact that $\|u\|_{L^2} = \|\hat{u}\|_{L^2}$ is crucial at a certain point of the proof, but here we cannot use a similar tool, since we admit also $p \neq 2$, and this is the reason for the dependence of the result on the compact set K; on the other hand, this is enough for proving several results on local solvability (cf. Section 3), where we shall be allowed to definitely fix K, considering only compactly supported functions.

PROOF OF PROPOSITION 2.10. We define, for $R > 0$, the following set:

$$P_R = \{\xi \in \mathsf{R}^N : |\xi_j| \leq R \text{ for } j = 1, \dots, N\};$$

moreover, for $\boldsymbol{\epsilon} = (\epsilon_1, \dots, \epsilon_N)$ with $\epsilon_j \in \{0, 1\}$ we put

$$P_R(\boldsymbol{\epsilon}) = \{\xi \in \mathsf{R}^N : \operatorname{sgn} \xi_j = (-1)^{\epsilon_j}, j = 1, \dots, N\} \setminus P_R.$$

The proof consists of three steps: we shall analyze at first the case when the Fourier transform of u is supported in the sets that we have just defined, and then we shall consider a general function u.

First step. We start by considering $u \in B_{p,k}$ such that $\operatorname{supp} \hat{u} \subset P_R(\epsilon)$. Since $P_R(\epsilon) \subset \{\xi \in \mathsf{R}^N : |\xi| \geq R\}$ we can apply Remark 2.5, obtaining

$$\begin{aligned}\|e^{iu(x)} - 1\|_{p,k} \leq \sum_{h=1}^{\infty} \frac{C(R)^{h-1}\|u\|_{p,k}^h}{h!} &\leq \|u\|_{p,k} e^{C(R)\|u\|_{p,k}} \\ &\leq e^{c\omega(\|u\|_{p,k})} e^{C(R)\|u\|_{p,k}},\end{aligned} \tag{21}$$

as we can deduce from the property (γ) of the weight $\omega(\cdot)$.

Second step. Let us suppose now that $u \in B_{p,k}$ with $\operatorname{supp} \hat{u} \subset P_R$. For every $\ell \in \mathsf{N}$ we set

$$u_1(x) = \sum_{h=1}^{\ell} \frac{\big(iu(x)\big)^h}{h!}.$$

By the standard properties of the Fourier transform and of the convolution product we then have that $\operatorname{supp} \hat{u}_1(x) \subset P_{\ell R}$; since $\mathsf{R}^N = \big(\cup_\epsilon P_{\ell R}(\epsilon)\big) \cup P_{\ell R}$ we obtain:

$$\begin{aligned}\|e^{iu(x)} - 1\|_{p,k} &\leq \sum_\epsilon \Big\| e^{\omega(\xi)} \mathscr{F}_{x\to\xi}\Big(\sum_{h=\ell+1}^{\infty} \frac{\big(iu(x)\big)^h}{h!}\Big)\Big\|_{L^p(P_{\ell R}(\epsilon))} \\ &\quad + \big\| e^{\omega(\xi)} \mathscr{F}_{x\to\xi}\big(e^{iu(x)} - 1\big)\big\|_{L^p(P_{\ell R})} \\ &= \sum_\epsilon T_1^{(\epsilon)} + T_2,\end{aligned} \tag{22}$$

where $T_1^{(\epsilon)}$ and T_2 are the norms in $L^p(P_{\ell R}(\epsilon))$ and $L^p(P_{\ell R})$, respectively. We now analyze separately $T_1^{(\epsilon)}$ and T_2.

(i) By Theorem 2.2 we have that

$$T_1^{(\epsilon)} = \Big\| \sum_{h=\ell+1}^{\infty} \frac{\big(iu(x)\big)^h}{h!} \Big\|_{p,k} \leq \frac{1}{C_{\text{alg}}} \sum_{h=\ell+1}^{\infty} \frac{\big(C_{\text{alg}}\|u\|_{p,k}\big)^h}{h!};$$

then, taking

$$4C_{\text{alg}}\|u\|_{p,k} \leq \ell \leq 4C_{\text{alg}}\|u\|_{p,k} + 1, \tag{23}$$

we easily obtain by Stirling's formula that

$$T_1^{(\epsilon)} \leq C, \tag{24}$$

C being a constant independent of $\|u\|_{p,k}$.

(ii) Let us analyze now T_2. At first we observe that there exists a constant A such that for every $\xi \in P_{\ell R}$ we have $|\xi| \leq A\ell R$; since ω is increasing we then have

$$T_2 \leq e^{\omega(A\ell R)} \big\| \mathscr{F}_{x\to\xi}\big(e^{iu(x)} - 1\big)\big\|_{L^p(P_{\ell R})}. \tag{25}$$

Using that $P_{\ell R}$ is compact and $|e^{it} - 1| \leq C|t|$ for every $t \in \mathsf{R}$, we get from (25) and the well known mapping property of the Fourier transform $\mathscr{F} : L^1(\mathsf{R}^N) \to L^\infty(\mathsf{R}^N)$

$$\begin{aligned} T_2 &\leq e^{\omega(A\ell R)} \big|P_{\ell R}\big|^{1/p} \big\| \mathscr{F}_{x\to\xi}\big(e^{iu(x)} - 1\big)\big\|_{L^\infty(P_{\ell R})} \\ &\leq C e^{\omega(A\ell R)} \big|P_{\ell R}\big|^{1/p} \|u\|_{L^1(\mathsf{R}^N)}, \end{aligned} \tag{26}$$

$|P_{\ell R}|$ being the measure of the set $P_{\ell R}$. Now, writing $u(x) = (2\pi)^{-N} \int e^{ix\xi} \hat{u}(\xi)\, d\xi$ we have by a simple integration by parts that

$$\begin{aligned} \|u\|_{L^1(\mathsf{R}^N)} &= (2\pi)^{-N} \int \big(1 + |x|^2\big)^{-N} \left| \int e^{ix\xi} \big(1 + \Delta_\xi\big)^N \hat{u}(\xi)\, d\xi \right| dx \\ &\leq C \big\| \big(1 + \Delta_\xi\big)^N \hat{u}(\xi) \big\|_{L^1(\mathsf{R}^N)}, \end{aligned} \tag{27}$$

where Δ_ξ is the Laplacian in the ξ-variables. Since $\operatorname{supp} \hat{u} \subset P_R$, the last norm in (27) is in $L^1(P_R)$. Now P_R is compact, so for every $g \in L^p(P_R)$ we have by Hölder inequality that $\|g\|_{L^1(P_R)} \leq |P_R|^{1/p'} \|g\|_{L^p(P_R)}$, where $|P_R|$ is the measure of P_R and $\frac{1}{p} + \frac{1}{p'} = 1$; then by (27) we deduce that $\|u\|_{L^1(\mathsf{R}^N)} \leq C|P_R|^{1/p'} \big\| \mathscr{F}_{x\to\xi}\big((1 + |x|^2)^N u(x)\big)\big\|_{L^p(P_R)}$. We can now apply this last inequality in (26): since $e^{\omega(\xi)} \geq 1$ for every $\xi \in \mathsf{R}^N$, taking into account that for every $T > 0$ we have $|P_T| = (2T)^N$ we obtain

$$\begin{aligned} T_2 &\leq C_1 e^{\omega(A\ell R)} \big|P_{\ell R}\big|^{1/p} \big|P_R\big|^{1/p'} \big\| e^{\omega(\xi)} \mathscr{F}_{x\to\xi}\big((1 + |x|^2)^N u(x)\big)\big\|_{L^p(P_R)} \\ &= C_2 e^{\omega(A\ell R)} \ell^{N/p} R^N \big\| (1 + |x|^2)^N u(x) \big\|_{p,k}. \end{aligned} \tag{28}$$

We finally have from (22), (24) and (28) that

$$\|e^{iu(x)} - 1\|_{p,k} \leq C_1 + C_2 e^{\omega(A\ell R)} \ell^{N/p} R^N \big\| (1 + |x|^2)^N u(x) \big\|_{p,k}, \tag{29}$$

for every $u \in B_{p,k}$ with $\operatorname{supp} \hat{u} \subset P_R$, where ℓ is a positive integer satisfying (23).

Third step. We consider now a generic $u \in B_{p,k}$. Fix functions $\chi_R(\xi)$ and $\chi_\epsilon(\xi)$ with the following properties:

$\chi_R(\xi) \in C_0^\infty(\mathsf{R}^N)$, $0 \le \chi(\xi) \le 1$ for every $\xi \in \mathsf{R}^N$, $\chi_R(\xi) \equiv 1$ for every $\xi \in P_R$; moreover, we assume that $\operatorname{supp}\chi_R \subset P_{R+1}$ and χ_R is real-valued and even.

For every $\boldsymbol{\epsilon} = (\epsilon_1, \dots, \epsilon_N)$ with $\epsilon_j \in \{1, 0\}$, we set

$$\chi_{\boldsymbol{\epsilon}}(\xi) = \begin{cases} 1 - \chi_R(\xi) & \text{for } \xi \in P_R(\boldsymbol{\epsilon}) \\ 0 & \text{for } \xi \notin P_R(\boldsymbol{\epsilon}) \end{cases}$$

Observe that $\chi_R(\xi) + \sum_{\boldsymbol{\epsilon}} \chi_{\boldsymbol{\epsilon}}(\xi) = 1$ for every $\xi \in \mathsf{R}^N$; then we can write

$$u(x) = \mathscr{F}^{-1}_{\xi\to x}(\hat{u}(\xi)) = \sum_{\boldsymbol{\epsilon}} u_{\boldsymbol{\epsilon}}(x) + u_0(x), \tag{30}$$

where $u_{\boldsymbol{\epsilon}}(x) = \mathscr{F}^{-1}_{\xi\to x}\big[\chi_{\boldsymbol{\epsilon}}(\xi)\hat{u}(\xi)\big]$ and $u_0(x) = \mathscr{F}^{-1}_{\xi\to x}\big[\chi_R(\xi)\hat{u}(\xi)\big]$. Since $\chi_{\boldsymbol{\epsilon}}(\xi) \le 1$ for every $\xi \in \mathsf{R}^N$, we have that for each $\boldsymbol{\epsilon}$ the following estimate holds:

$$\begin{aligned} \|u_{\boldsymbol{\epsilon}}\|_{p,k} &= (2\pi)^{-N/p}\big\|e^{\omega(\xi)}\chi_{\boldsymbol{\epsilon}}(\xi)\hat{u}(\xi)\big\|_{L^p} \\ &\le (2\pi)^{-N/p}\big\|e^{\omega(\xi)}\hat{u}(\xi)\big\|_{L^p} = \|u\|_{p,k}; \end{aligned} \tag{31}$$

by definition, $\operatorname{supp}\hat{u}_{\boldsymbol{\epsilon}} \subset P_R(\boldsymbol{\epsilon})$, so we can apply the inequality (21), which gives, together with (31),

$$\|e^{iu_{\boldsymbol{\epsilon}}(x)} - 1\|_{p,k} \le e^{c\omega(\|u_{\boldsymbol{\epsilon}}\|_{p,k})}e^{C(R)\|u_{\boldsymbol{\epsilon}}\|_{p,k}} \le e^{c\omega(\|u\|_{p,k})+C(R)\|u\|_{p,k}}, \tag{32}$$

since ω is an increasing function.

Let us recall now the following identity, that was proved in [2, Lemma 4.6]: for every $a_1, \dots, a_m \in \mathsf{C}$ we have

$$a_1a_2\cdots a_m - 1 = \sum_{h=1}^{m} \sum_{\substack{\boldsymbol{j}=(j_1,\dots,j_h) \\ 0\le j_1<j_2<\dots<j_h\le m}} (a_{j_1} - 1)\cdots(a_{j_h} - 1). \tag{33}$$

We want to analyze the norm of the quantity $e^{iu(x)} - 1$; by (30) we can write $e^{iu(x)} - 1 = \big[\prod_{\boldsymbol{\epsilon}} e^{iu_{\boldsymbol{\epsilon}}(x)}\big]\cdot e^{iu_0(x)} - 1$; then by applying (33) in this last expression and by using the algebra property (10) we can estimate the norm $\|e^{iu(x)} - 1\|_{p,k}$ by a sum of terms whose factors are of the type $\|e^{iu_{\boldsymbol{\epsilon}}(x)} - 1\|_{p,k}$ or $\|e^{iu_0(x)} - 1\|_{p,k}$. Regarding $\|e^{iu_{\boldsymbol{\epsilon}}(x)} - 1\|_{p,k}$ we can apply (32), while for $\|e^{iu_0(x)} - 1\|_{p,k}$ the estimate (29) holds with $R+1$ in place of R, since $\operatorname{supp}\hat{u}_0 \subset P_{R+1}$. Since we have 2^N different choices of $\boldsymbol{\epsilon}$ we then obtain

$$\begin{aligned} \|e^{iu(x)} - 1\|_{p,k} &\le Ce^{2^Nc\omega(\|u\|_{p,k})+2^NC(R)\|u\|_{p,k}} \\ &\quad \times \Big(C_1 + C_2e^{\omega(A\ell(R+1))}\ell^{N/p}(R+1)^N\big\|(1+|x|^2)^Nu_0(x)\big\|_{p,k}\Big). \end{aligned} \tag{34}$$

Let us estimate now the term $\|(1+|x|^2)^N u_0(x)\|_{p,k}$. By definition, we have $u_0(x)=\mathscr{F}^{-1}_{\xi\to x}(\chi_R(\xi)\hat{u}(\xi))$; then, easy computations and the basic properties of the Fourier transform give us

$$\begin{aligned}(1+|x|^2)^N u_0(x) &= \sum_{|\alpha|\le 2N} c_\alpha x^\alpha \mathscr{F}^{-1}_{\xi\to x}\big(\chi_R(\xi)\hat{u}(\xi)\big)\\ &= \sum_{|\alpha|\le 2N}\sum_{\beta\le\alpha} c_{\alpha\beta}\mathscr{F}^{-1}_{\xi\to x}\big[(D^\beta\chi_R)(\xi)\big(\widehat{x^{\alpha-\beta}u}\big)(\xi)\big],\end{aligned}$$

where $c_{\alpha\beta}=\binom{\alpha}{\beta}(-1)^{|\beta|}c_\alpha$. Now we observe that we can find a constant B depending on N but not on R such that $\|D^\beta\chi_R\|_{L^\infty}\le B$ for every $|\beta|\le 2N$; so we obtain

(35)
$$\begin{aligned}\|(1+|x|^2)^N u_0(x)\|_{p,k} &\le C\sum_{|\alpha|\le 2N}\sum_{\beta\le\alpha}\big\|e^{\omega(\xi)}(D^\beta\chi_R)(\xi)\big(\widehat{x^{\alpha-\beta}u}\big)(\xi)\big\|_{L^p}\\ &\le CB\sum_{|\alpha|\le 2N}\sum_{\beta\le\alpha}\|x^{\alpha-\beta}u(x)\|_{p,k}.\end{aligned}$$

Since by hypothesis u is supported in K, we have that $x^{\alpha-\beta}u(x)\equiv\phi_K(x)x^{\alpha-\beta}u(x)$, where ϕ_K is a suitable (test) function such that $\phi_K(x)\equiv 1$ on K and $\phi_K(x)x^{\alpha-\beta}\in B_{p,k}$ for every $\beta\le\alpha$, $|\alpha|\le 2N$. We deduce from (35) and Theorem 2.2 that

(36)
$$\begin{aligned}&\|(1+|x|^2)^N u_0(x)\|_{p,k}\\ &\qquad\le C'\sum_{|\alpha|\le 2N}\sum_{\beta\le\alpha}C_{\mathrm{alg}}\|\phi_K(x)x^{\alpha-\beta}\|_{p,k}\|u\|_{p,k}=\overline{C}_K\|u\|_{p,k}.\end{aligned}$$

Let us observe moreover that we can find a constant D satisfying $R+1\le DR$ for every $R\ge 1$, so $\omega(A\ell(R+1))\le\omega(A'\ell R)$ with $A'=AD$, ω being an increasing function; we then deduce by (34) and (36) the following estimate:

(37)
$$\begin{aligned}&\|e^{iu(x)}-1\|_{p,k}\\ &\qquad\le Ce^{2^N c\omega(\|u\|_{p,k})+2^N C(R)\|u\|_{p,k}}\big(C_1+C_2'e^{\omega(A'\ell R)}\ell^{N/p}R^N\overline{C}_K\|u\|_{p,k}\big),\end{aligned}$$

for every $R\ge 1$, where ℓ satisfies (23). Let us analyze at first the case $\|u\|_{p,k}\le M$, M being a positive constant; since (37) holds for all $R\ge 1$ we can fix $R=1$, and moreover from (23) we get $\ell\le 4C_{\mathrm{alg}}M+1$, so (37) gives us (19).

Now let us consider the case $\|u\|_{p,k}>M$. At first we want to analyze the constant $C(R)$, cf. (12); since ω is increasing, for every $\nu\in(0,\mathscr{C})$ we can write

$$C(R) \le Ce^{-(\mathscr{C}-\nu)\omega(R)}\left(\int_0^{+\infty} e^{-\nu p'\omega(t)} t^{N-1}\,dt\right)^{\frac{1}{p'}} = C_\nu e^{-(\mathscr{C}-\nu)\omega(R)}. \tag{38}$$

We then get

$$R \le \omega^{-1}\left[\frac{1}{\mathscr{C}-\nu}\log\left(\frac{C_\nu}{C(R)}\right)\right]. \tag{39}$$

Since (37) holds for every $R \ge 1$, we still have the freedom to choose R; we fix R in such a way that

$$C(R) = C_\nu \frac{\omega(\|u\|_{p,k})}{\|u\|_{p,k}}, \tag{40}$$

and then by (39) we obtain

$$R \le \omega^{-1}\left[\frac{1}{\mathscr{C}-\nu}\log\left(\frac{\|u\|_{p,k}}{\omega(\|u\|_{p,k})}\right)\right]. \tag{41}$$

Since we want to estimate $\|e^{iu(x)}-1\|_{p,k}$ through (37) we now consider some of the terms appearing there, starting from $\omega(A'\ell R)$. Recall that we are supposing that $\|u\|_{p,k} > M$; then, taking $M \ge 1$ we have from (23) that $\ell \le (4C_{\text{alg}} + 1)\|u\|_{p,k}$. Then by (41) and Lemma 2.9 we get:

$$\omega(A'\ell R) \le A_1\omega\left\{\|u\|_{p,k}\omega^{-1}\left[\frac{1}{\mathscr{C}-\nu}\log\left(\frac{\|u\|_{p,k}}{\omega(\|u\|_{p,k})}\right)\right]\right\}, \tag{42}$$

for every $\nu \in (0,\mathscr{C})$, where $A_1 = \max\{1, A'(4C_{\text{alg}}+1)\}$ is independent of ν.

Let us consider now the term $\ell^{N/p}$ in (37). By Lemma 2.9 again we have, for $\|u\|_{p,k} > M$ with M sufficiently large, $\omega^{-1}\left(\frac{1}{\mathscr{C}-\nu}\log\frac{\|u\|_{p,k}}{\omega(\|u\|_{p,k})}\right) \ge 1$; so, using the condition $\ell \le (4C_{\text{alg}}+1)\|u\|_{p,k}$ and the fact that $\log t = o(\omega(t))$ as $t \to +\infty$ we easily obtain the following estimate for every $\|u\|_{p,k} > M$:

$$\begin{aligned} \ell^{\frac{N}{p}} &\le (4C_{\text{alg}}+1)^{\frac{N}{p}} e^{\frac{N}{p}b_1\omega(\|u\|_{p,k})} \\ &\le (4C_{\text{alg}}+1)^{\frac{N}{p}} e^{\frac{N}{p}b_1\omega\left\{\|u\|_{p,k}\omega^{-1}\left[\frac{1}{\mathscr{C}-\nu}\log\frac{\|u\|_{p,k}}{\omega(\|u\|_{p,k})}\right]\right\}}. \end{aligned} \tag{43}$$

Similarly, from (41) we get the following estimates:

$$\begin{aligned} R^N &\le e^{Nb_1\omega\left\{\|u\|_{p,k}\omega^{-1}\left[\frac{1}{\mathscr{C}-\nu}\log\frac{\|u\|_{p,k}}{\omega(\|u\|_{p,k})}\right]\right\}}, \\ \|u\|_{p,k} &\le e^{b_1\omega\left\{\|u\|_{p,k}\omega^{-1}\left[\frac{1}{\mathscr{C}-\nu}\log\frac{\|u\|_{p,k}}{\omega(\|u\|_{p,k})}\right]\right\}}. \end{aligned} \tag{44}$$

It then follows from (37), (40), (42), (43) and (44) that (20) is satisfied for every $\|u\|_{p,k} > M$, with $a_\nu = 2^N(c + C_\omega)$, $\widetilde{C}_K = C_2'(4C_{\text{alg}} + 1)^{\frac{N}{p}}\overline{C}_K$ and $b = A_1 + b_1\big(\frac{N}{p} + N + 1\big)$. The proof of Proposition 2.10 is then complete.

REMARK 2.11. We observe that the estimates (19)–(20) can be obviously unified in (20). On the other hand, considering the proof of Theorem 2.6 in the one-dimensional case, we always have $f(x, u(x)) \in B_{p,k}$ for a single $u \in B_{p,k}$. In fact, it is sufficient that the last expression in formula (26) be finite, but the Fourier transform of u in (26) has compact support, and therefore $u \in L^1(\mathsf{R}^N)$. In this case we can avoid the requirement that u has support in a fixed compact set K.

Now we give some lemmas that, together with Proposition 2.10, shall allow us to prove Theorem 2.6. Using that $\varphi_\omega : t \mapsto \omega(e^t)$ is convex we obtain

LEMMA 2.12. *The weight ω satisfies $\omega^{-1}(t+s) \le \omega^{-1}(t)\omega^{-1}(s)$, for every $t, s \in \mathsf{R}_+$.*

LEMMA 2.13. *Let $G(x, y)$ satisfy the hypotheses of Theorem 2.6, and let us denote $R(x) = G(x, 0)$. Then $R(x) \in B_{p,k}$.*

PROOF. First, we observe that

$$R(x) = (2\pi)^{-N-2M} \int_{\mathsf{R}^{N+2M}} e^{ix\xi}\widehat{G}(\xi, \eta)\, d\xi\, d\eta. \tag{45}$$

The statement follows then by applying Hölder's inequality, (14) and (16).

LEMMA 2.14. *Let $E(x)$ be a function belonging to the space $B_{p,k}$. The following estimate holds for every $\xi \in \mathsf{R}^N$:*

$$\|e^{ix\xi}E(x)\|_{p,k} \le Ce^{\omega(\xi)}\|E\|_{p,k},$$

where C is a positive constant independent of ξ, and we have set as usual $k(\xi) = e^{\omega(\xi)}$.

PROOF. By definition of the $B_{p,k}$-norm we have

$$\|e^{ix\xi}E(x)\|_{p,k}^p = (2\pi)^{-N} \int_{\mathsf{R}^N} e^{p\omega(\zeta)-p\omega(|\zeta|-|\xi|)}\, e^{p\omega(|\zeta|-|\xi|)}\big|\widehat{E}(\zeta - \xi)\big|^p\, d\zeta.$$

Since, by (7), $e^{\omega(\zeta)-\omega(|\zeta|-|\xi|)} \le e^{\omega(\xi)+\delta}$, we get immediately the conclusion.

Now we can pass to the proof of the composition result in the spaces $B_{p,k}$.

PROOF OF THEOREM 2.6. (i) We start by proving (17). We observe at first that we can write the function $G(x, y)$ in the following way:

$$G(x,y) = (2\pi)^{-N-2M} \int e^{ix\xi}\big(e^{iy\eta} - 1\big)\widehat{G}(\xi,\eta)\, d\xi\, d\eta + R(x), \tag{46}$$

where $R(x)$ is given by (45). Let us write

$$\boldsymbol{v}(x) = (\Re u_1(x), \ldots, \Re u_M(x), \Im u_1(x), \ldots, \Im u_M(x)); \tag{47}$$

since we have represented $f(x, z)$ as in (13), using Lemmas 2.13 and 2.14 we obtain

$$\begin{aligned} \|f(x,\boldsymbol{u}(x))\|_{p,k} &= \|G(x,\boldsymbol{v}(x))\|_{p,k} \\ &\le C \int e^{\omega(\xi)} \|e^{i\boldsymbol{v}(x)\eta} - 1\|_{p,k} |\widehat{G}(\xi,\eta)|\, d\xi\, d\eta + C_1, \end{aligned} \tag{48}$$

where $C_1 = \|R\|_{p,k}$ is a constant depending only on G. So we have just to estimate $\|e^{i\boldsymbol{v}(x)\eta} - 1\|_{p,k}$; by the formula (33) and Theorem 2.2 we get

$$\begin{aligned} &\|e^{i\boldsymbol{v}(x)\eta} - 1\|_{p,k} \\ &\qquad \le C \sum_{h=1}^{2M} \sum_{\substack{\boldsymbol{j}=(j_1,\ldots,j_h) \\ 0\le j_1<\ldots<j_h\le 2M}} \|e^{iv_{j_1}(x)\eta_{j_1}} - 1\|_{p,k} \cdots \|e^{iv_{j_h}(x)\eta_{j_h}} - 1\|_{p,k}. \end{aligned} \tag{49}$$

By using Proposition 2.10 and taking into account Remark 2.11 we obtain, for every $\ell = 1, \ldots, 2M$, the following estimate:

$$\begin{aligned} &\|e^{iv_\ell(x)\eta_\ell} - 1\|_{p,k} \\ &\qquad \le C e^{a_\nu \omega(|\eta_\ell| \|v_\ell\|_{p,k})} \bigg[C_1 + \widetilde{C}_K e^{b\omega\left\{|\eta_\ell| \|v_\ell\|_{p,k}\, \omega^{-1}\left[\frac{1}{\mathscr{C}-\nu} \log \frac{|\eta_\ell| \|v_\ell\|_{p,k}}{\omega(|\eta_\ell| \|v_\ell\|_{p,k})}\right]\right\}} \bigg]. \end{aligned} \tag{50}$$

Now we consider two cases.

If $\|v_\ell\|_{p,k} \le 1$, since ω is increasing and $|\eta_\ell| \le |\eta|$, we obtain from Lemma 2.9 that

$$\|e^{iv_\ell(x)\eta_\ell} - 1\|_{p,k} \le C e^{a_\nu \omega(\eta)} \bigg[C_1 + \widetilde{C}_K e^{b\omega\left\{|\eta| \omega^{-1}\left[\frac{1}{\mathscr{C}-\nu} \log \frac{|\eta|}{\omega(|\eta|)}\right]\right\}} \bigg]. \tag{51}$$

When $\|v_\ell\|_{p,k} > 1$, by Lemmas 2.9 and 2.12 we get:

$$\begin{aligned} &\omega\bigg\{ |\eta_\ell| \|v_\ell\|_{p,k} \omega^{-1}\bigg[\frac{1}{\mathscr{C}-\nu} \log \frac{|\eta_\ell| \|v_\ell\|_{p,k}}{\omega(|\eta_\ell| \|v_\ell\|_{p,k})} \bigg] \bigg\} \\ &\qquad \le \Theta_1(\|v_\ell\|_{p,k}) \omega\bigg\{ |\eta|\, \omega^{-1}\bigg[\frac{1}{\mathscr{C}-\nu} \log \frac{|\eta|}{\omega(|\eta|)} \bigg] \bigg\} \end{aligned} \tag{52}$$

where $\Theta_1(\|v_\ell\|_{p,k}) = \|v_\ell\|_{p,k} \max\left\{1, \omega^{-1}\left[\frac{1}{\mathscr{C}-\nu}\log(1+\|v_\ell\|_{p,k})\right]\right\}$. We then have by (50), (52) and Lemma 2.9 that

$$\|e^{iv_\ell(x)\eta_\ell}-1\|_{p,k} \le Ce^{\Theta_2(\|v_\ell\|_{p,k})\omega(|\eta|)}\left[C_1+\widetilde{C}_K e^{b\Theta_1(\|v_\ell\|_{p,k})\omega\left\{|\eta|\omega^{-1}\left[\frac{1}{\mathscr{C}-\nu}\log\frac{|\eta|}{\omega(|\eta|)}\right]\right\}}\right],$$

for every $\|v_\ell\|_{p,k} > 1$, where $\Theta_2(\|v_\ell\|_{p,k}) = a_\nu\|v_\ell\|_{p,k}$. Observe now that writing $\Re\, w = \frac{w+\overline{w}}{2}$ and $\Im\, w = i\,\frac{\overline{w}-w}{2}$ we easily obtain that for every $w \in B_{p,k}$

$$\|\Re w\|_{p,k} \le \|w\|_{p,k} \qquad \text{and} \qquad \|\Im w\|_{p,k} \le \|w\|_{p,k}; \tag{53}$$

then, since $v_\ell(x)$ is the real or imaginary part of some $u_{\tilde{\ell}}(x)$ and $\Theta_j(\cdot)$, $j = 1, 2$, is increasing, using (53) we have, for $\|v_\ell\|_{p,k} > 1$,

$$\|e^{iv_\ell(x)\eta_\ell}-1\|_{p,k} \le Ce^{\Theta_2(\|u_{\tilde{\ell}}\|_{p,k})\omega(|\eta|)}\left[C_1+\widetilde{C}_K e^{b\Theta_1(\|u_{\tilde{\ell}}\|_{p,k})\omega\left\{|\eta|\omega^{-1}\left[\frac{1}{\mathscr{C}-\nu}\log\frac{|\eta|}{\omega(|\eta|)}\right]\right\}}\right]. \tag{54}$$

Now we can complete the estimate of $\|e^{i\boldsymbol{v}(x)\eta}-1\|_{p,k}$ in (49) through (51) and (54), and then we can continue the estimate (48), getting that $\|f(x,\boldsymbol{u}(x))\|_{p,k}$ is estimated by a sum of integrals where the leading term in η (for $|\eta|$ large) is of the form

$$\int e^{\omega(\xi)}e^{\Theta\left(\|u_1\|_{p,k},\dots,\|u_M\|_{p,k}\right)\omega\left\{|\eta|\omega^{-1}\left[\frac{1}{\mathscr{C}-\nu}\log\frac{|\eta|}{\omega(|\eta|)}\right]\right\}}\left|\widehat{G}(\xi,\eta)\right|\,d\xi\,d\eta$$
$$\le \left\|e^{\tilde{\omega}(\xi)}e^{\sigma(\eta)}\widehat{G}(\xi,\eta)\right\|_{L^p(\mathsf{R}^{N+2M})}$$
$$\times\left\|e^{\omega(\xi)-\tilde{\omega}(\xi)}e^{\Theta\left(\|u_1\|_{p,k},\dots,\|u_M\|_{p,k}\right)\omega\left\{|\eta|\omega^{-1}\left[\frac{1}{\mathscr{C}-\nu}\log\frac{|\eta|}{\omega(|\eta|)}\right]\right\}-\sigma(\eta)}\right\|_{L^{p'}(\mathsf{R}^{N+2M})}$$

as we can deduce by Hölder inequality. In this last expression the norm in L^p is finite by the hypothesis (16); the one in $L^{p'}$ is finite, too, by the conditions (14) and (15), since for $c_1, c_2 > 0$ fixed, $\|e^{-c_1\omega(\xi)}\|_{L^{p'}(\mathsf{R}^N)}$ and $\|e^{-c_2\sigma(\eta)}\|_{L^{p'}(\mathsf{R}^{2M})}$ are finite, as we can deduce by the property (γ) of $\omega(\xi)$ and $\sigma(\eta)$. We then have that $f(x,\boldsymbol{u}(x)) \in B_{p,k}$ and (17) is satisfied; the boundedness of Ψ_K on bounded sets follows from the fact that the functions Θ_1, Θ_2 (and then also Θ in the last estimate) have such a property.

(ii) We want now to prove (18). From the Cavalieri-Lagrange formula and Theorem 2.2 we get

$$\|f(x,\boldsymbol{u}^{(1)}(x)) - f(x,\boldsymbol{u}^{(2)}(x))\|_{p,k} \le \sum_{j=1}^{2M} C_{\mathrm{alg}}\|v_j^{(1)}(x)-v_j^{(2)}(x)\|_{p,k} \tag{55}$$

$$\times \int_0^1 \big\|(\partial_{y_j} G)\big(x, \boldsymbol{v}^{(2)}(x) + t\big(\boldsymbol{v}^{(1)}(x) - \boldsymbol{v}^{(2)}(x)\big)\big)\big\|_{p,k}\, dt,$$

with notation as in (47). From (53) we then have

$$\|v_j^{(1)}(x) - v_j^{(2)}(x)\|_{p,k} \le \|u_{h_j}^{(1)}(x) - u_{h_j}^{(2)}(x)\|_{p,k} \tag{56}$$

for some h_j depending on j. Let us analyze now the function $(\partial_{y_j} G)(x, y)$. Setting $\sigma_1(t) = \sigma(t) - \log(1+t)$, $k_1(t) = e^{\sigma_1(t)}$ and recalling that $\overline{k}(t) = e^{\sigma(t)}$ we have

$$\int \big|\tilde{k}(\xi) k_1(\eta) (\widehat{\partial_{y_j} G})(\xi, \eta)\big|^p \, d\eta\, d\xi \le \int \tilde{k}(\xi)\overline{k}(\eta)\big|\widehat{G}(\xi, \eta)\big|^p \, d\eta\, d\xi,$$

that is finite by hypothesis (16). Now the property (γ) of the weight $\omega(\xi)$ ensures us that

$$\frac{\log(1+t)}{\omega\big\{t\,\omega^{-1}\big[\frac{1}{\mathscr{C}-\nu}\log\frac{t}{\omega(t)}\big]\big\}} \to 0 \qquad \text{as} \quad t \to +\infty,$$

and so we have that σ_1 satisfies

$$\omega\left\{t\,\omega^{-1}\left[\frac{1}{\mathscr{C}-\nu}\log\frac{t}{\omega(t)}\right]\right\} = o\big(\sigma_1(t)\big) \qquad \text{as} \quad t \to +\infty;$$

then for every $j = 1, \dots, 2M$ we can apply the point (i) of Theorem 2.6 to the function $(\partial_{y_j} G)(x, y)$, with σ_1 in place of σ, obtaining that

$$\big\|(\partial_{y_j} G)\big(x, \boldsymbol{v}^{(2)}(x) + t\big(\boldsymbol{v}^{(1)}(x) - \boldsymbol{v}^{(2)}(x)\big)\big)\big\|_{p,k} \le C_{K,T}^{(j)}, \tag{57}$$

for every $t \in [0, 1]$ and $j = 1, \dots, 2M$, since the function Ψ_K is bounded on bounded sets and by hypothesis $\big\|u_h^{(2)} + t\big(u_h^{(1)} - u_h^{(2)}\big)\big\|_{p,k} \le 3T$ for every $h = 1, \dots, M$. Then (18) follows from (55), (56) and (57).

REMARK 2.15. Observe that the proof above does not depend on the property (γ) on the weight ω. In fact, it is sufficient to take some constant $M_\gamma > 0$ such that $M_\gamma \log(1+t) \le \omega(t)$ as t tends to infinity, to have that the integrals in formulas (11), (12), and (38) are finite. In this case, we recover the classical functions $k \in \mathscr{K}_\omega$ of polynomial growth defined by Hörmander, and we extend the work of Messina and Rodino [17]. Compare, for example, the condition (2.1) in [17] to obtain that $B_{p,k}$ is an algebra. Here, the function

$$K(\xi, \eta) = \frac{e^{\omega(\xi)}}{e^{\omega(\xi-\eta)} e^{\omega(\eta)}} \le e^{-\mathscr{C}\min\{\omega(\xi-\eta), \omega(\eta)\} + \delta},$$

and it is sufficient to take $M_\gamma > 0$ big enough to have that such condition holds.

3. Applications to local solvability

Our aim is to give a local solution near a point x^0 for the following semilinear operator

$$F(u) = P(D)u + f(x, Q_1(D)u, Q_2(D)u, \ldots, Q_M(D)u), \tag{58}$$

where $P(D)$ and $Q_i(D)$, $i = 1, \ldots, M$ are linear partial differential operators with constant coefficients. As mentioned in the introduction, we will study two types of hypothesis on the nonlinear term f. First, we assume that there exists a point $x^0 \in \mathbb{R}^N$ such that $f(x^0, z) = 0$, for every $z \in \mathbb{C}^M$. Here, we need that $P(D)$ is stronger than $Q_i(D)$ for all $1 \le i \le M$ in the classical sense of Hörmander [10] and Trèves [19].

We recall some known results regarding $B_{p,k}$ spaces. We observe that if $h \in C^\infty$ in a neighborhood of $x^0 \in \mathbb{R}^N$ and $h(x^0) = 0$, we can write

$$h(x) = \sum_{j=1}^{N} (x_j - x_j^0) \int_0^1 \partial_j h(x^0 + t(x - x^0))\, dt, \tag{59}$$

and we have

Lemma 3.1. *Let $\psi \in C_0^\infty$ and $\psi_\varepsilon(x) = \psi(\frac{x-x^0}{\varepsilon})$. Then, for each $j = 1, 2, \ldots, N$,*

$$\|(x_j - x_j^0)\psi_\varepsilon(x)\|_{1,1} = \varepsilon \|x_j \psi(x)\|_{1,1}.$$

Another important property is the following (see [1, Theorem 2.2.7]): given a test function $\phi \in \mathscr{D}_{(\omega)}(\mathbb{R}^N)$, and $k \in \mathscr{K}_\omega$, there exists a constant $C > 0$ such that

$$\|\phi\, u\|_{p,k} \le C\|\phi\|_{1,1} \cdot \|u\|_{p,k} \le C'\|u\|_{p,k} \tag{60}$$

for all $u \in B_{p,k}$. The first result of local solvability is the following.

Theorem 3.2. *Let $g \in B_{p,k}$, with $k(\xi) = e^{\omega(\xi)}$, and consider the operator F defined by (58). We suppose that there exists a point $x^0 \in \mathbb{R}^N$ such that $f(x^0, z) = 0$ for all $z \in \mathbb{C}^M$ and $\tilde{Q}_i(\xi) \le C\tilde{P}(\xi)$ for all $\xi \in \mathbb{R}^N$ and $1 \le i \le M$ and some constant $C > 0$. We also assume that f satisfies (16) of Theorem 2.6. Then one can find a constant $\varepsilon_0(P, Q_1, \ldots, Q_M) > 0$ and $u^0 \in B_{p,k\tilde{P}}$ such that*

$$F(u^0)(x) = g(x) \tag{61}$$

when $\|x - x^0\| < \varepsilon_0$.

PROOF. It is well-known that there exists a fundamental solution $E \in B^{\mathrm{loc}}_{\infty,\tilde P}$ of the linear term $P(D)$ of the semilinear operator F. Set $E* =: L$. By Björck [1, Theorem 2.3.8], L is well defined from $B_{p,k} \cap \mathscr{E}'_{(\omega)}(\mathsf{R}^N)$ to $B_{p,k\tilde P}$.

Choose $\psi \in \mathscr{D}_{(\omega)}(\mathsf{R}^N)$, $\psi \equiv 1$ in $B_1(0)$ and $\varphi \in \mathscr{D}_{(\omega)}(\mathsf{R}^N)$, $\varphi \equiv 1$ in $B_1(x^0)$. Define $\psi_\varepsilon(x) = \psi(\frac{x-x^0}{\varepsilon})$ and consider the new operator

$$\tilde F(v) = g - \psi_\varepsilon \varphi f(x, Q_1(D)(\varphi L\varphi v), \dots, Q_M(D)(\varphi L\varphi v)).$$

Observe now that, if $v \in B_{p,k}$, then $\varphi v \in B_{p,k} \cap \mathscr{E}'_{(\omega)}(\mathsf{R}^N)$. Therefore $L\varphi v \in B^{\mathrm{loc}}_{p,k\tilde P}$ and $\varphi L\varphi \in B_{p,k\tilde P}$, then

$$Q_i(D)(\varphi L\varphi v) \in B_{p,k\tilde P/\tilde Q_i} \subset B_{p,k}$$

for $i = 1, 2, \dots, M$. We can apply now the result of composition from Theorem 2.6 to obtain $\tilde F(v) \in B_{p,k}$.

We will prove that, fixed $T \geq 2\|g\|_{p,k}$, there exists $\varepsilon_0 > 0$ such that the corresponding operator

$$\tilde F_0(v) = g - \psi_{\varepsilon_0} \varphi f(x, Q_1(D)(\varphi L\varphi v), \dots, Q_M(D)(\varphi L\varphi v))$$

is defined from $B_{p,k,T} = \{u \in B_{p,k} : \|u\|_{p,k} \leq T\}$ into itself and it is a contraction.

Let $v \in B_{p,k,T}$ and $u := \varphi L\varphi v$, as in the proof of [17, Theorem 3.1], $\|u\|_{p,k\tilde P} \leq CT$ for some constant $C > 0$. Set now $s := (s_1, \dots, s_M)$, where $s_i = Q_i(D)u$, then s_i is compactly supported for all $i = 1, 2, \dots, M$, and

$$\|s_i\|_{p,k} = \|Q_i(D)u\|_{p,k} \leq C_i \|u\|_{p,k\tilde P/\tilde Q_i} \leq C_i' \|u\|_{p,k\tilde P} \leq C_i'' T,$$

that is, $s_i \in B_{p,k}$. Now, we use that $f(x^0, z) = 0$ for all $z \in \mathsf{C}^M$. By (59), it follows that

$$f(x, z) = \sum_{j=1}^N (x_j - x_j^0) \int_0^1 \partial_{x_j} f(x^0 + t(x - x^0), z)\, dt,$$

and from Lemma 3.1 and (60),

$$\begin{aligned}
&\|\psi_\varepsilon \varphi f(x,s)\|_{p,k}\\
&\quad\le C\sum_{j=1}^{N}\|\varphi(x)\int_0^1 \partial_{x_j} f(x^0+t(x-x^0),s)\,dt\|_{p,k}\cdot\|(x_j-x_j^0)\psi_\varepsilon(x)\|_{1,1}\\
&\quad= C\varepsilon\sum_{j=1}^{N}\|\varphi(x)\int_0^1 \partial_{x_j} f(x^0+t(x-x^0),s)\,dt\|_{p,k}\cdot\|x_j\psi(x)\|_{1,1}\\
&\quad\le \tilde{C}\varepsilon\|\varphi\|_{1,1}\sum_{j=1}^{N}\int_0^1 \|\partial_{x_j} f(x^0+t(x-x^0),s)\|_{p,k}\cdot\|x_j\psi(x)\|_{1,1}\,dt.
\end{aligned}$$

Now, proceeding in a similar way to the proof of (ii) of Theorem 2.6 it is easy to see that $\partial_{x_j} f(x^0+t(x-x^0),z)$ (or better, the corresponding G with real variables) satisfies the hypotheses of Theorem 2.6 for $\tilde{\omega}_1=\tilde{\omega}-\log(1+t)$. We can conclude from the inequalities above that

$$\|\psi_\varepsilon \varphi f(x,s)\|_{p,k}\le \varepsilon C_1.$$

Then, choosing ε sufficiently small, $\|\tilde{F}(v)\|_{p,k}\le \varepsilon C_1+\|g\|_{p,k}\le T$.

We now prove that $\tilde{F}: B_{p,k,T}\to B_{p,k,T}$ is a contraction. Since the function $\partial_{x_j} f(x^0+t(x-x^0),z)$ satisfies Theorem 2.6, we can use (ii) of this result. Using the notation $h(x,s)=f(x,s^1)-f(x,s^2)$, observing that $h(x^0,s)=0$ and arguing as before

$$\begin{aligned}
&\|\tilde{F}(v^1)-\tilde{F}(v^2)\|_{p,k}=\|\varphi\psi_\varepsilon h(x,s)\|_{p,k}\\
&\quad\le \varepsilon C\|\varphi\|_{1,1}\sum_{i=1}^{N}\int_0^1 \|\partial_{x_j} h(x^0+t(x-x^0),s)\|_{p,k}\cdot\|x_j\psi(x)\|_{1,1}\,dt\\
&\quad\le \varepsilon C C_{\operatorname{supp}\varphi,T}\sum_{j=1}^{M}\|s_j^{(1)}-s_j^{(2)}\|_{p,k}
\end{aligned}$$

where $C_{\operatorname{supp}\varphi,T}$ is the constant that appears in (18). Then the operator $\tilde{F}: B_{p,k,T}\to B_{p,k,T}$ is a contraction choosing $\varepsilon=\varepsilon_0$ sufficiently small and, therefore, there exists a fixed point v^0 for the corresponding operator $\tilde{F}_0$. As in [17, Theorem 3.1], we conclude that the equation (61) admits a local solution $u^0\in B_{p,k\tilde{P}}$ for $\|x-x_0\|<\varepsilon_0$.

EXAMPLE 3.3. As in [17], we can consider the Schrödinger operator

$$P(D) = -D_{x_N} + \sum_{j=1}^{N-1} D_{x_j}^2,$$

and the linear partial differential operators $Q_j(D)u = D_{x_j}u$, for $j = 1, \dots, N-1$, $Q_N(D)u = P(D)u$, $Q_{N+1}(D)u = u$.

Let $H \in \mathsf{C}[z]$ be a polynomial defined in C^{2N+1}. We consider that the first N variables are real, and $H(x^0, z) = 0$ for some point $x^0 \in \mathsf{R}^N$ and all $z \in \mathsf{C}^{N+1}$. Then, by Theorem 3.2, the equation $P(D)u + H(x, D)u = g$ is locally solvable for $g \in B_{p,k}$ in a neighborhood of x^0. To see this, we write $H(x, z) = G(x, y)$, where $y \in \mathsf{R}^{2N+2}$, and we multiply G by a suitable cut-off function: As in the proof of Theorem 3.2, we fix $T \geq 2\|g\|_{p,k}$. Since we work with functions $u \in B_{p,k\tilde{P}}$ with support in a fixed compact set K, and $\|Q_i(D)u\|_{p,k} \leq CT$, for $1 \leq i \leq N+1$, we also have

$$\begin{aligned}(62)\quad \|Q_i(D)u\|_{L^\infty} &\leq C_1\|\widehat{Q_i(D)}u\|_{L^1} \\ &\leq C_2\|e^{\omega(\xi)}\widehat{Q_i(D)}u(\xi)\|_{L^p} \leq C_3T =: \tilde{T}.\end{aligned}$$

Then, it is enough to consider $H(x, z)$ on the set $B_\delta(x^0) \times P(0; \tilde{T}, \dots, \tilde{T})$, for some $\delta > 0$, where $P(0; \tilde{T}, \dots, \tilde{T})$ is a poly-disc in C^{N+1}. Therefore, we can multiply G by a suitable cut-off function in such a way that the product satisfies (16) of Theorem 2.6.

EXAMPLE 3.4. Let us consider the following nonlinear equation:

$$(63)\qquad F(u) = \sum_{|\alpha|\leq m} c_\alpha D^\alpha u + f(x, D^\alpha u)_{|\alpha|\leq m} = g(x),$$

where $c_\alpha \in \mathsf{C}$, the operator $P(D) = \sum_{|\alpha|\leq m} c_\alpha D^\alpha$ is elliptic, there exists an x^0 such that $f(x^0, z) = 0$ for every $z \in \mathsf{C}^M$, with $M = \#\{\alpha \in \mathsf{N}_0^N : |\alpha| \leq m\}$, and $f(x, z)$ (eventually multiplied by a suitable cut-off function, cf. Example 3.3) satisfies the hypotheses of Theorem 2.6. We observe that, writing $Q_\alpha(D) = D^\alpha$, the ellipticity of $P(D)$ implies that there exists a positive constant C such that $\tilde{Q}_\alpha(\xi) \leq C\tilde{P}(\xi)$ for every α with $|\alpha| \leq m$. Then we can apply Theorem 3.2 to the equation (63), obtaining that for every $g \in B_{p,k}$ there exists (locally near x^0) a solution $u(x)$ of (63) belonging to $B_{p,k\tilde{P}}$.

We consider now the same operator F but with a weaker hypothesis on the nonlinearity f. We will need in this case that $P(D)$ is infinitely stronger that $Q_i(D)$, for all $1 \leq i \leq M$, in the sense of Hörmander [10]. We obtain the following extension of [16, Theorem 11].

THEOREM 3.5. *Let $g \in B_{p,k}$, with $k(\xi) = e^{\omega(\xi)}$, and consider the operator F defined by (58). We suppose that $f(x,0) = 0$ for all $x \in \mathsf{R}^N$ and that $\frac{\tilde{Q}_i(\xi)}{\tilde{P}(\xi)} \to 0$ as $|\xi| \to +\infty$ for all $1 \leq i \leq M$. We also assume that f satisfies the hypothesis of Theorem 2.6. Then, for every $x^0 \in \mathsf{R}^N$, one can find $\varepsilon_0 > 0$ and $u^0 \in B_{p,k\tilde{P}}$ such that*

$$F(u^0)(x) = g(x)$$

when $\|x - x^0\| < \varepsilon_0$.

As a further application of the algebra result proved in Section 2 we prove now the following theorem concerning nonlinear elliptic equations. For such equations the local solvability is well-known in the frame of Sobolev spaces and analytic nonlinearities (a proof can be found in [18]); we give here a more general result in $B_{p,k}$ spaces.

THEOREM 3.6. *Let us consider the following equation:*

$$f(x, \partial^\alpha u)_{|\alpha| \leq m} = g(x), \tag{64}$$

where:

(i) *$f(x,z)$ is of the form*

$$f(x,z) = f_1(x, \Re z) + i f_2(x, \Im z), \tag{65}$$

where f_1 and f_2 are real-valued (or alternatively pure imaginary valued) functions;

(ii) *$F_1(v) := f_1(x, \partial^\alpha v)_{|\alpha| \leq m}$ and $F_2(w) := f_2(x, \partial^\alpha w)_{|\alpha| \leq m}$ are elliptic;*

(iii) *the functions $f_1(x,y)$ and $f_2(x,y)$ satisfy the hypotheses of Theorem 2.6.*

(iv) *$f(x,0) = 0$ for every $x \in \mathsf{R}^N$.*

Then for every $g \in B_{p,k,\delta}$ with δ sufficiently small we can find (locally near x^0) a solution $u \in B_{p,k\tilde{P}}$ of the equation (64).

PROOF. We have already observed that a function u belongs to $B_{p,k}$ if and only if $\Re u$ and $\Im u$ belong to $B_{p,k}$, cf. (53). Then, because of the particular form of the nonlinearity $f(x,z)$, we have that the equation (64) is equivalent to

$$f_1\big(x, \partial^\alpha(\Re u)\big)_{|\alpha| \leq m} = \Re\, g(x), \qquad f_2\big(x, \partial^\alpha(\Im u)\big)_{|\alpha| \leq m} = \Im g(x) \tag{66}$$

(an analogous consideration holds for the case of pure imaginary valued f_1 and f_2). We analyze only the first equation (the same procedure applies also to the second one). Let us write for simplicity w instead of $\Re u$ and $h(x)$ instead of $\Re g(x)$; we then have to find a solution of the equation

$$F_1(w) := f_1(x, \partial^\alpha w)_{|\alpha|\le m} = h(x), \tag{67}$$

where both w and h are now real-valued. We consider the linearization of $F(u)$, defined in the following way:

$$F_1'(0)(x, D) := \sum_{|\alpha|\ge 0} i^{|\alpha|} \frac{\partial f_1}{\partial w_\alpha}(x, 0) D^\alpha = \sum_{|\alpha|\ge 0} \frac{\partial f_1}{\partial w_\alpha}(x, 0)\partial^\alpha, \tag{68}$$

where w_α indicates the variable corresponding to $\partial^\alpha w$ in (67); we fix arbitrarily $x^0 \in \mathsf{R}^N$ and we write $P_1(D) := F_1'(0)(x^0, D)$. We write

$$F_1(w) = P_1(D)w + Q_1(x, D)w + G_1(w),$$

where $P_1(D) = F_1'(0)(x^0, D)$, $Q_1(x, D) = F_1'(0)(x, D) - P_1(D)$ and $G_1(w) = F_1(w) - F_1'(0)(x, D)w$. Observe that $G_1'(0)(x, D) = 0$. Now, if E_1 is a fundamental solution of $P_1(D)$ and $L_1 := E_1 *$ we consider, similarly to [18], the following equation:

$$R_1(v) = v + K_1 v + \psi_\epsilon G_1\big(\varphi\Re(L_1\varphi v)\big) = h, \tag{69}$$

where $K_1 v := \psi_\epsilon Q_1(x, D)\big(\varphi\Re(L_1\varphi v)\big)$ and the real-valued functions φ, ψ_ϵ have the same meaning as in the proof of Theorem 3.2. Observe that $R_1(v) = v + \psi_\epsilon F_1\big(\varphi\Re(L_1\varphi v)\big) - \psi_\epsilon P_1(D)\big(\varphi\Re(L_1\varphi v)\big)$. We have already proved that $\varphi L_1\varphi v \in B_{p,k\tilde{P}}$, which implies that $\varphi\Re(L_1\varphi v) = \Re(\varphi L_1\varphi v) \in B_{p,k\tilde{P}}$, cf. (53), and so $P_1(D)\big(\varphi\Re(L_1\varphi v)\big) \in B_{p,k}$. Moreover, for every $|\alpha| \le m$, we have $\partial^\alpha\big(\varphi\Re(L_1\varphi v)\big) \in B_{p,k}$ and $\|\partial^\alpha\big(\varphi\Re(L_1\varphi v)\big)\|_{p,k} \le C\|v\|_{p,k}$; so we obtain from Theorem 2.6

$$\|F_1\big(\varphi\Re(L_1\varphi v)\big)\|_{p,k} \le \tilde{\Psi}_{\operatorname{supp}\varphi}\big(\|v\|_{p,k}\big).$$

We then have

$$R_1 : B_{p,k} \to B_{p,k}.$$

Now proceeding in the same way as in the proof of Theorem 3.2 we get

$$\|K_1 v\|_{p,k} \le \epsilon C\|v\|_{p,k},$$

for a fixed positive constant C. Then we get $\|K_1\|_{\mathscr{L}(B_{p,k}, B_{p,k})} \to 0$ as $\epsilon \to 0$, which implies that shrinking ϵ we have that $R_1'(0)(x, D) = I + K_1 : B_{p,k} \to B_{p,k}$ is invertible. We can then apply the Inverse Function Theorem in the Banach space $B_{p,k}$ to get a solution $v \in B_{p,k}$ of (69), and then a (local and real-valued) solution $w = \Re(L_1 \varphi v) \in B_{p,k\tilde{P}}$ of the first equation in (66); in the same way we can treat the second equation in (66), and so we have found a local solution of (64).

In the next example, we show that the hypotheses in Theorem 2.6 include the well-known analytic case.

EXAMPLE 3.7. Now, we consider a different condition on $f(x,z)$ to have another application of Theorem 2.6. Let σ be a weight function; we denote by $\mathscr{E}_{(\sigma)}(\mathsf{R}^N_x, H(\mathsf{C}^M))$ (compare with [17, Def. 2.1]) the set of those functions $f(x,z) = \sum_{|\alpha|\geq 0} a_\alpha(x) z^\alpha$, $x \in \mathsf{R}^N$, $z \in \mathsf{C}^M$, such that $a_\alpha(x) \in \mathscr{E}_{(\sigma)}(\mathsf{R}^N)$ and, if for each compact set $K \subset \mathsf{R}^N$ and every $n \in \mathsf{N}$ we denote by $C_{n,\alpha} = \| a_\alpha \|_{K,n}$, being $\| \cdot \|_{K,n}$ the seminorm defined in (4), then the function $\sum_{|\alpha|\geq 0} C_{n,\alpha} z^\alpha$ is entire for each $n \in \mathsf{N}$ and $K \subset\subset \mathsf{R}^N$.

We write $f(x,z) = G(x,y)$ with $y \in \mathsf{R}^{2M}$, i.e., $G(x,y) = \sum_{|\alpha|\geq 0} a_\alpha(x) (y_1 + iy_2)^\alpha$ with $y = (y_1, y_2)$, and $y_1, y_2 \in \mathsf{R}^M$. We fix a compact set $\overline{K} \subset \mathsf{R}^{N+2M}$. Then, $|y| \leq M_0$ for $(x,y) \in \overline{K}$ and some constant $M_0 > 0$. We put $\delta = (\delta_1, \ldots, \delta_{2M}) \in \mathsf{N}_0^{2M}$ and $\tilde{\delta} = (\delta_1 + \delta_{M+1}, \delta_2 + \delta_{M+2}, \ldots, \delta_M + \delta_{2M}) \in \mathsf{N}_0^M$. For $\tilde{\delta} \leq \alpha$, we have

$$
\begin{aligned}
&\big| D_x^\gamma D_y^\delta \big(a_\alpha(x) (y_1 + iy_2)^\alpha \big) \big| \\
&\quad \leq C_{n,\alpha} \exp\left(n\varphi_\sigma^* \left(\frac{|\gamma|}{n} \right) \right) \tilde{\delta}! \binom{\alpha}{\tilde{\delta}} M_0^{|\alpha| - |\tilde{\delta}|} \\
&\quad \leq C_{n,\alpha} \exp\left(n\varphi_\sigma^* \left(\frac{|\gamma|}{n} \right) \right) \tilde{\delta}! (2M_0)^{|\alpha|} \\
&\quad \leq C_{n,\alpha} D_n \exp\left(n\varphi_\sigma^* \left(\frac{|\gamma|}{n} \right) \right) (2M_0)^{|\alpha|} \exp\left(n\varphi_\sigma^* \left(\frac{|\tilde{\delta}|}{n} \right) \right),
\end{aligned}
$$

since $\tilde{\delta}! \leq D_n \exp\big(n\varphi_\sigma^*\big(\frac{|\tilde{\delta}|}{n}\big)\big) \leq D_n \exp\big(n\varphi_\sigma^*\big(\frac{|\delta|}{n}\big)\big)$ for some constant $D_n > 0$ and all multi-indexes δ. We finally obtain

$$
\big| D_x^\gamma D_y^\delta G(x,y) \big| \leq C_n \exp\left(n\varphi_\sigma^* \left(\frac{|\gamma + \delta|}{n} \right) \right),
$$

where $C_n = D_n \sum_{|\alpha|\geq 0} C_{n,\alpha} (2M_0)^{|\alpha|}$ is a constant that only depends on $\overline{K}$ and $n \in \mathsf{N}$. We have proved that $G(x,y) \in \mathscr{E}_{(\sigma)}(\mathsf{R}^{N+2M})$. If necessary we can modify G multiplying by a cut-off function and take a suitable weight σ to have (16) of Theorem 2.6.

REMARK 3.8. Let $\Omega \subset \mathsf{R}^N$ be an open set, and define

$$
B_{p,k}^{\mathrm{loc}}(\Omega) = \{ u \in \mathscr{D}'_{(\omega)}(\Omega) : \text{for every } \varphi \in \mathscr{D}_{(\omega)}(\Omega),\ u\varphi \in B_{p,k} \},
$$

where as usual $k(\cdot) = e^{\omega(\cdot)}$; we consider a function $f(x, z)$ satisfying the hypotheses of Theorem 2.6. We then have that for every $\boldsymbol{u}(x) = (u_1(x), \ldots, u_M(x))$ with $u_j \in B^{\mathrm{loc}}_{p,k}(\Omega)$ for all $j = 1, \ldots, M$,

$$f(x, \boldsymbol{u}(x)) \in B^{\mathrm{loc}}_{p,k}(\Omega). \tag{70}$$

In fact, let us fix arbitrarily $\varphi \in \mathscr{D}_{(\omega)}(\Omega)$, and choose a real-valued function $\psi \in \mathscr{D}_{(\omega)}(\Omega)$, $\psi \equiv 1$ on $\operatorname{supp}\varphi$. From Theorem 2.6 we then obtain

$$\varphi(x) f(x, \boldsymbol{u}(x)) = \varphi(x) f(x, \psi(x)\boldsymbol{u}(x)) \in B_{p,k},$$

and this gives (70).

Let us fix now an increasing function $h : [0, +\infty) \to [0, +\infty)$ satisfying

$$0 \le h(t) \le c(1 + |t|)^d \qquad \text{and} \qquad \frac{h(t+s)}{h(t)h(s)} \le C \tag{71}$$

for suitable $d, c, C > 0$ and every $t, s \in [0, +\infty)$. Taking an ω satisfying (γ), by the same arguments as in the proof of Theorem 3.2 we have that for every fixed integer n the weight $\omega(t) + \log(h(t)^n)$ satisfies (14) (with a different constant $A_1 > 1$), and so we obtain as before that, for $\boldsymbol{u} = (u_1, \ldots, u_M)$ with $u_j \in B^{\mathrm{loc}}_{p,kh^n}(\Omega)$, $f(x, \boldsymbol{u}(x)) \in B^{\mathrm{loc}}_{p,kh^n}(\Omega)$.

Now, by specifying the hypothesis $\frac{\tilde{Q}_i(\xi)}{\tilde{P}(\xi)} \to 0$ as $|\xi| \to +\infty$ in Theorem 3.5, we obtain the following extension of [8, Theorem 3.2], giving a result about regularity of the solutions of semilinear equations with hypoelliptic linear part.

THEOREM 3.9. *Let us consider the equation*

$$P(D)u = f(x, Q_1(D)u, \ldots, Q_M(D)u), \tag{72}$$

where $P(D)$ is a hypoelliptic operator, i.e. there exist positive constants C, ρ such that

$$|\partial^\alpha P(\xi)| \le C|P(\xi)|\,|\xi|^{-\rho|\alpha|}$$

for every $\alpha \in \mathsf{Z}^N_+$ and $|\xi| \ge C$; we suppose moreover that there exists a function $h : [0, +\infty) \to [0, +\infty)$ satisfying (71) and such that

$$\frac{\tilde{P}(\xi)}{\tilde{Q}_i(\xi)} > h(\xi) \qquad \textit{for every} \quad i = 1, \ldots, M; \tag{73}$$

let the nonlinearity f satisfy the hypotheses of Theorem 2.6, and $f(x, 0) = 0$. Let $u \in B^{\mathrm{loc}}_{p,k\tilde{P}}(\Omega)$ be a solution of (72). Then, for every positive integer n,

$$u \in B^{\mathrm{loc}}_{p,k\tilde{P}h^n}(\Omega).$$

PROOF. Since $u \in B^{\mathrm{loc}}_{p,k\tilde{P}}(\Omega)$ we have that for every $i = 1, \dots, M$ the condition (73) implies $Q_i(D)u \in B^{\mathrm{loc}}_{p,k\frac{\tilde{P}}{\tilde{Q}_i}}(\Omega) \hookrightarrow B^{\mathrm{loc}}_{p,kh}(\Omega)$, and so from Remark 3.8 we have

$$P(D)u = f(x, Q_1(D)u, \dots, Q_M(D)u) \in B^{\mathrm{loc}}_{p,kh}(\Omega);$$

then the hypoellipticity of P implies that $u \in B^{\mathrm{loc}}_{p,k\tilde{P}h}(\Omega)$, cf. [10, Theorem 11.1.8]. The conclusion follows by induction on n.

REFERENCES

1. Björck, G., *Linear partial differential operators and generalized distributions*, Ark. Mat. 6 (1966), 351–407.
2. Bourdaud, G., Reissig, M., and Sickel, W., *Hyperbolic equations, function spaces with exponential weights and Nemytskij operators*, Ann. Mat. Pura Appl. 182, 4 (2003), 409–455.
3. Braun, R. W., Meise, R., and Taylor, B. A., *Ultradifferentiable functions and Fourier analysis*, Results Math. 17 (1990), 206–237.
4. De Donno, G., Oliaro, A., *Local solvability and hypoellipticity for semilinear anisotropic partial differential equations*, Trans. Amer. Math. Soc. 355, 8 (2003), 3405–3432.
5. De Donno, G., Oliaro, A., *Hypoellipticity and local solvability of anisotropic PDEs with Gevrey nonlinearity*, Boll. Un. Mat. Ital. B (to appear).
6. Fernández, C., Galbis, A., and Jornet, D., *ω-hypoelliptic differential operators of constant strength*, J. Math. Anal. Appl. 297 (2004), 561–576.
7. Fieker, C., *P-Konvexität und ω-Hypoelliptizität für partielle Differentialoperatoren mit konstanten Koeffizienten*, Diplomarbeit, Heinrich-Heine-Universität, Düsseldorf (1993).
8. Garello, G., *Continuity for Pseudodifferential Operators with Symbols in Weighted Function Spaces*, Proceedings Workshop "Partial Differential Operators", Torino, 8–10 May, 2000, L. Rodino ed., 63–74.
9. Gramchev, T., Rodino, L., *Gevrey solvability for semilinear partial differential equations with multiple characteristics*, Boll. Un. Mat. Ital. B (8) 2 (1999), 65–120.
10. Hörmander, L., *The analysis of Linear Partial Differential Operators*, Springer-Verlag, Berlin, I–II, 1983–85.
11. Hounie, J., Santiago, P., *On the local solvability of semilinear equations*, Comm. Partial Differential Equations 20 (1995), 1777–1789.
12. Lewy, H., *An example of a smooth linear partial differential equation without solution*, Ann. of Math. (2) 66 (1957), 155–158.
13. Marcolongo, P., Oliaro, A., *Local solvability for semilinear anisotropic partial differential equations*, Ann. Mat. Pura Appl. (4) 179 (2001), 229–262.
14. Mascarello, M., Rodino, L., *Partial Differential Equations with Multiple Characteristics*, Math. Top. 13 (1997).
15. Meise, R., Taylor, B. A., *Whitney's extension theorem for ultradifferentiable functions of Beurling type*, Ark. Mat. 26 (1988), 265–287.
16. Messina, F., *Local solvability for semilinear partial differential equations of constant strength*, Rend. Sem. Mat. Univ. Politec. Torino 61 (2003), no. 1, 33–54.
17. Messina, F., Rodino, L., *Local solvability for nonlinear partial differential equations*, Nonlinear Anal. TMA 47 (2001), 2917–2927.
18. Oliaro, A., and Rodino, L., *Solvability for Semilinear PDE with Multiple Characteristics*, Banach Center Publ. 60 (2003), 295–303.

19. Trèves, F., *Linear Partial Differential Equations with Constant Coefficients*, Gordon and Breach Science Publishers, New York, 1966.

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