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S-REGULARITY AND THE CORONA FACTORIZATION PROPERTY

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Abstract

Stability is an important and fundamental property of C^*-algebras. Given a short exact sequence of C^*-algebras $0 \longrightarrow B \longrightarrow E \longrightarrow A \longrightarrow 0$ where the ends are stable, the middle algebra may or may not be stable. We say that the first algebra, B, is S-regular if every extension of B by a stable algebra A has a stable extension algebra, E. Rørdam has given a sufficient condition for S-regularity. We define a new condition, weaker than Rørdam's, which we call the *corona factorization property*, and we show that the corona factorization property implies S-regularity. The corona factorization property originated in a study of the Kasparov $KK^1(A, B)$ group of extensions, however, we obtain our results without explicit reference to KK-theory.

Our main result is that for a separable stable C^*-algebra B the first two of the following properties (which we define later) are equivalent, and both imply the third. With additional hypotheses on the C^*-algebra, all three properties are equivalent.

(1) B has the corona factorization property.

(2) Stability is a stable property for full hereditary subalgebras of B.

(3) B is S-regular.

We also show that extensions of separable stable C^*-algebras with the corona factorization property give extension algebras with the corona factorization property, extending the results of [9].

1. Introduction

Stability is a fundamental property of C^*-algebras, and in general is quite well-behaved. It was therefore surprising when Rørdam gave an example showing that stability does not have the two-out-of-three property. Specifically, Rørdam constructed an example of a short exact sequence

$$0 \longrightarrow B \longrightarrow C \longrightarrow A \longrightarrow 0$$

of C^*-algebras in which B and A are stable but C is not [16].

This motivates the following definition:

Definition 1.1. A C^*-algebra B is *S-regular* if in any short exact sequence of the form

$$0 \longrightarrow B \longrightarrow C \longrightarrow A \longrightarrow 0,$$

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C is stable when A is.

A S-regular C^*-algebra is necessarily stable, since an ideal in a stable algebra is stable. The letter "S" in *S-regularity* is there to differentiate between our concept and Rørdam's, and could reasonably be taken to stand for either stability or for short exact sequences.

Rørdam [18] proved the following:

THEOREM 1.2. *Let B be a separable C^*-algebra. If full hereditary subalgebras of B are stable whenever they have no nonzero unital quotients and no nonzero bounded traces, then B is S-regular.*

Rørdam uses the term *regular* for algebras satisfying the hypotheses of theorem 1.2.

Some situations in which Rørdam's theorem is applicable are:

(1) Simple, separable, exact C^*-algebras, with real rank zero, stable rank one, and weakly unperforated ordered K_0 group [18].
(2) Simple, separable, purely infinite C^*-algebras.
(3) Type I C^*-algebras with finite decomposition rank [12].

The purpose of this note is to give a simple algebraic sufficient condition for S-regularity. This condition is called the *corona factorization property*, and originated in a study of KK-theory [3], [10], [13].

DEFINITION 1.3. A C^*-algebra B is said to have the *corona factorization property* if every norm-full projection P in the stable multiplier algebra $\mathcal{M}(B \otimes \mathcal{K})$ is Murray-von Neumann equivalent to $1_{\mathcal{M}(B\otimes\mathcal{K})}$.

We remark that in a separable stable C^*-algebra, the corona factorization property has several equivalent forms. The following is obtained by combining the results of [10] and [13], see also [12]:

THEOREM 1.4. *In a stable separable C^*-algebra, the corona factorization property is equivalent to any, hence all, of the following:*

(1) *Every positive element a of the corona such that $C^*(a)$ has zero intersection with proper ideals of the corona is properly infinite.*
(2) *Every positive element a of the multipliers such that $C^*(a)$ has zero intersection with proper ideals of the multipliers is properly infinite.*
(3) *For every norm-full projection P in the corona algebra $\mathcal{M}(B)/B$, there is an $x \in \mathcal{M}(B)/B$ such that $xPx^* = 1_{\mathcal{M}(B)/B}$.*
(4) *Every norm-full projection in $\mathcal{M}(B)/B$ is properly infinite.*
(5) *If c is a norm-full positive element of $\mathcal{M}(B)$, then $\overline{cBc}$ contains a stable subalgebra that is full in B.*

(6) *Every norm-full extension of B is nuclearly absorbing.*

(7) *Every norm-full trivial extension of B is nuclearly absorbing.*

It follows from property v of the above list that regularity in Rørdam's sense will in most cases imply the corona factorization property. In this note, we study the relationships between the corona factorization property and S-regularity. Our first result is the following:

THEOREM 1.5. *Let B be a separable stable C^*-algebra. If B has the corona factorization property then B is S-regular*

Hence, the corona factorization property rules out the type of counterexample constructed by Rørdam in [17].

For separable, stable, simple C^*-algebras of minimal real and stable ranks, the converse of the above theorem will be shown to hold. It is an interesting question whether weaker hypotheses would still allow the converse to be shown. Also, is there a counterexample to the converse holding in general? We feel that there will be for sufficiently unusual algebras, but do not currently have a counterexample.

The ideas and techniques in this note are strongly influenced by Rørdam's fundamental papers on stability [18], [17], [19], [16].

2. The S-regularity property

We now prove our first result, theorem 1.5. Thus, we consider short exact sequences of separable C^*-algebras,

$$0 \longrightarrow B \longrightarrow C \longrightarrow A \longrightarrow 0,$$

where the canonical ideal B is stable and has the corona factorization property. We are to show that C is stable if and only if A is, and the difficulties lie in proving the "if" direction. We now assemble some lemmas and results needed for our proof.

First, the following proposition, which is due to Rørdam [18, prop. 6.8]:

PROPOSITION 2.1. *Let A be a stable, closed, two-sided $*$-ideal of a separable C^*-algebra B. Suppose that B/A is stable, or zero. Then the following are equivalent:*

(1) *B is stable.*

(2) *For each positive contraction $b \in B$, the hereditary subalgebra $(1 - b)A(1 - b)$ of A is stable.*

In particular, this proposition has an interesting special case (cf. [5, cor. 4.3]):

COROLLARY 2.2. *Let B be a stable and separable C^*-algebra. For each positive contraction $b \in B$, the hereditary subalgebra $\overline{(1-b)B(1-b)}$ is stable.*

We also need the following proposition [8], [13], see also [4, 1.3.17]:

PROPOSITION 2.3. *Let B be a stable σ-unital C^*-algebra. Let ℓ be a nonzero positive element of $\mathscr{M}(B)$. There is a projection P in $\mathscr{M}(B)$ such that the hereditary C^*-algebra $\overline{\ell B \ell}$ is $*$-isomorphic to PBP.*

Moreover, if ℓ is a full element of $\mathscr{M}(B)$, then P is a full projection in $\mathscr{M}(B)$.

The above isomorphism is given by an unitary equivalence in an appropriate Hilbert module.

THEOREM 2.4. *Let A and C be separable C^*-algebras such that $A \otimes \mathscr{K}$ has the corona factorization property. Suppose that there is an extension of C^*-algebras of the form*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0.$$

Then B is stable if and only if A and C are stable.

PROOF. The "only if" direction follows from the fact that ideals and quotients of a stable C^*-algebra are always stable.

We now prove the converse direction, using proposition 2.1. Thus, we are to show that $\overline{(1-b)A(1-b)}$ is stable for any positive contraction b of B. In proving this, it is clear that we may as well replace B by its image in $\mathscr{M}(A)$, since we are only concerned with properties of the action of B on A. We shall prove that the given element $1-b$ is full in $\mathscr{M}(A)$, for then by proposition 2.3, the algebra $\overline{(1-b)A(1-b)}$ is isomorphic to a hereditary subalgebra generated by a full multiplier projection, and such a subalgebra is stable by the corona factorization property.

We now show that $1-b$ is full, using a method from [3, § 16]. Suppose, to the contrary, that $1-b$ is not full in $\mathscr{M}(A)$. Then there is a nonzero $*$-homomorphism π on $\mathscr{M}(A)$, such that $\pi(1-b) = 0$. Restricting π to $\tilde{B}$, the unitization of B in $\mathscr{M}(A)$, we see that $\pi(1) = \pi(b)$, implying that $\pi(B)$ is actually unital (and nonzero). But then, if $J := \ker \pi|_B$, we have the short exact sequence

$$0 \longrightarrow \frac{A}{J \cap A} \longrightarrow \frac{B}{J} \longrightarrow \frac{C}{(J/A)} \longrightarrow 0$$

giving a unital algebra as an extension of stable (or zero) algebras. One sees that $\frac{B}{J}$ cannot in fact be unital[1].

[1] This argument is similar to lemma 6.6 in [18].

We will later show that under appropriate conditions, there is actually a converse to the above result.

3. Short exact seqences of C^*-algebras with the corona factorization property

It would be interesting to know whether S-regularity has a two-of-three property: in particular, if the C^*-algebras A and C in

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

are S-regular, does the C^*-algebra B have this property? We do not know if this is the case or not, but we can show that the possibly stronger corona factorization property *is* preserved when forming extension algebras. This generalizes an earlier result in [9], where we had to restrict the class of exact sequences considered.

THEOREM 3.1. *Suppose that we are given a short exact sequence of separable C^*-algebras*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0.$$

Then:

(1) *If A and C are stable C^*-algebras with the corona factorization property, then B is a stable C^*-algebra with the corona factorization property.*

(2) *If B is a stable C^*-algebra with the corona factorization property, then C is a stable C^*-algebra with the corona factorization property.*

REMARK 3.2. Statement ii holds, with the same proof, if B is just σ-unital. In fact, the only place in this article where we apply separability is in the use of proposition 2.1

PROOF. First, we prove statement i. Stability of B follows from theorem 2.4. It is sufficient to prove that if P is a full projection in $\mathcal{M}(B)$, then PBP is stable. By [20, Proposition 2.2.16], the given maps in the short exact sequence induce a surjective unital map $r : \mathcal{M}(B) \longrightarrow \mathcal{M}(C)$ and a unital map $i : \mathcal{M}(B) \longrightarrow \mathcal{M}(A)$ that is not necessarily injective – in fact, this holds even if the algebras given are not separable. A unital $*$-homorphism will map a full projection to a full projection [2], so we see that the projection P will map to a full projection in $\mathcal{M}(A)$ and in $\mathcal{M}(C)$. Thus, the first and last algebras in the short exact sequence

$$0 \longrightarrow i(P)A\,i(P) \longrightarrow PBP \longrightarrow r(P)C\,r(P) \longrightarrow 0$$

are stable by the corona factorization propery of A and of C.

[2] If $m \in \mathcal{M}(B)$ is full, and under some nonzero homomorphism h, the image $h(m)$ were not full, then $h(m)$ would be in the kernel of some unital homomorphism, and composing the two homomorphisms, we would see that m is in the kernel of a nonzero homomorphism out of $\mathcal{M}(B)$.

By theorem 2.4, the algebra A is S-regular. Recall that by Brown's theorem [2], there is an isomorphism of $i(P)Ai(P)$ and A, and it follows from the definition of S-regularity that S-regularity is preserved by isomorphism. Thus, $i(P)Ai(P)$ is a S-regular C^*-algebra, and hence PBP is stable as claimed.

Next, we prove statement ii. Since B is stable, C is stable. Now suppose that P is a norm-full projection in $\mathcal{M}(C)$. Let b be a norm-full positive element of $\mathcal{M}(B)$ such that $r(b) = P$. By proposition 2.3, let Q be a norm-full projection in $\mathcal{M}(B)$ such that QBQ is $*$-isomorphic to $\overline{bBb}$. Since $\mathcal{M}(B)$ has the corona factorization property, Q is Murray-von Neumann equivalent to the unit of $\mathcal{M}(B)$, and QBQ is a stable C^*-algebra. Hence, $\overline{bBb}$ is a stable C^*-algebra, implying that $PCP \cong r(\overline{bBb})$ is a stable, full, hereditary subalgebra of C, and since P is arbitrary, it follows that C has the corona factorization property.

We note that the corona factorization property does not pass to ideals. For example, let X be the countably infinite Cartesian product of spheres. Let $C := C(X) \otimes \mathcal{K}$. Since C is stable, there exists a sequence $\{S_i\}_{i=1}^{\infty}$ of partial isometries in $\mathcal{M}(C)$ that generate a copy of O_∞. Let C_1 be the unital subalgebra of $\mathcal{M}(C)$, generated by C and $\{S_i\}_{i=1}^{\infty}$. Let $B := C_1 \otimes \mathcal{K}$, and let J be the ideal of B given by $J := C \otimes \mathcal{K}$. Then B has the corona factorization property, but in fact $J \cong C(X) \otimes \mathcal{K}$ does not have the corona factorization property [13], [12].

4. When stability is a stable property

In this section, we relate the corona factorization property to an interesting property concerning the stability of full hereditary subalgebras of C^*-algebras.

DEFINITION 4.1. A C^*-algebra B_0 is said to be *asymptotically stable* if $M_n(B_0)$ is stable for some positive integer n.

Stable algebras are asymptotically stable. One might hope that the converse is true, but Rørdam has constructed an example of a simple, separable, σ_P-unital AH-algebra B_0, with stable rank one, such that $M_2(B_0)$ is stable but B_0 is not stable (see [18]). In the presence of the corona factorization property, this phenomena cannot occur:

THEOREM 4.2. *Suppose that B is a σ-unital stable C^*-algebra. Then the following are equivalent:*

(1) *B has the corona factorization property.*
(2) *A full hereditary subalgebra D of B is stable if and only if $M_n(D)$ is stable for some positive integer n.*

As a corollary to theorem 4.2, we get the following result:

COROLLARY 4.3. *Suppose that B_0 is a σ-unital C^*-algebra such that $B_0 \otimes \mathscr{K}$ has the corona factorization property. Then B_0 is stable if and only if $M_n(B_0)$ is stable for some positive integer n.*

Hence, in the presence of the corona factorization property, stability is a stable property.

The above result, together with the abovementioned example of Rørdam in [18], will give us an example of a simple, separable, stable, σ_P-unital nuclear C^*-algebra without the corona factorization property:

COROLLARY 4.4. *There exits a simple, separable, σ_P-unital AH-algebra B_0, with stable rank one, such that $B_0 \otimes \mathscr{K}$ does not have the corona factorization property.*

We will need a result due to Brown ([1, th. 4.23]) in order to prove the main result of this section:

THEOREM 4.5. *Let B be a separable stable C^*-algebra, and let P be a projection in $\mathscr{M}(B)$. Then PBP is a stable full hereditary subalgebra of B if and only if P is Murray-von Neumann equivalent to $1_{\mathscr{M}(B)}$.*

REMARK 4.6. This theorem implies that if p is a full projection in B, then $p \otimes 1 \in \mathscr{M}(B \otimes \mathscr{K})$ is equivalent to 1.

PROOF OF THEOREM 4.2. First, we prove that i implies ii. Suppose that B has the corona factorization property, D is a full, hereditary subalgebra of B and $n \geq 1$ is a positive integer, such that $M_n(D)$ is a stable C^*-algebra.

Now since D is a full hereditary subalgebra of B, by Brown's stable isomorphism theorem [1], the stabilization $D \otimes \mathscr{K}$ is $*$-isomorphic to B, and therefore $D \otimes \mathscr{K}$ has the corona factorization property. Let P be the projection in $\mathscr{M}(D \otimes \mathscr{K})$ given by $P := 1 \otimes e_{11}$, where 1 is the unit of $\mathscr{M}(D)$ and e_{11} is a rank one projection in $\mathscr{K}$. Clearly $P(D \otimes \mathscr{K})P$ is a full hereditary subalgebra of $D \otimes \mathscr{K}$, and is $*$-isomorphic to D. Since $D \otimes \mathscr{K}$ is stable, let $v_1, v_2, \ldots, v_n$ be isometries with pairwise orthogonal ranges which generate a unital copy of the Cuntz algebra O_n in $\mathscr{M}(D \otimes \mathscr{K})$. The stable C^*-algebra $M_n(D)$ is $*$-isomorphic to $Q(D \otimes \mathscr{K})Q$, where Q is the projection in $\mathscr{M}(D \otimes \mathscr{K})$ given by $Q := v_1 P v_1^* + v_2 P v_2^* + \cdots + v_n P v_n^*$. By Theorem 4.5, the projection Q is Murray-von Neumann equivalent to the unit of $\mathscr{M}(D \otimes \mathscr{K})$, so in particular P is a norm-full projection in $\mathscr{M}(D \otimes \mathscr{K})$. By the corona factorization property, the subalgebra $P(D \otimes \mathscr{K})P$ is therefore stable, implying that D is stable.

Next, we prove that ii implies i. Suppose that B satisfies ii. Suppose that P is a norm-full projection in $\mathscr{M}(B)$, so that $\overline{\mathscr{M}(B)P\mathscr{M}(B)} = \mathscr{M}(B)$. Hence, $\overline{BPB} = B$, and PBP is a full, hereditary subalgebra of B. Since P is norm-full in $\mathscr{M}(B)$, there is a positive integer n such that $\bigoplus_1^n P$ is equivalent to

$1_{\mathscr{M}(B)}$. This means that $M_n(PBP)$ is (isomorphic to) a stable C^*-algebra, and therefore, by the hypothesis PBP is stable. Thus, by theorem 4.5, the projection P is Murray-von Neumann equivalent to the unit of $\mathscr{M}(B)$.

5. A converse for C^*-algebras with cancellation

To prove a converse to theorem 1.5, we need a preliminary result. We show that that if (G, G_+) is a partially ordered group with the Riesz decomposition property, then (G, G_+) has "asymptotic two-unperforation."

LEMMA 5.1. *Let (G, G_+) be a partially ordered group with the Riesz decomposition property. Suppose that x, y are elements of G_+ such that $2x \leq 2y$ in G_+. Then, for every positive integer $n \geq 1$, there exist $x_1, x_2 \in G_+$ such that*

(1) $x = x_1 + x_2$,

(2) $nx_2 \leq x_1$, *and*

(3) $x_1 \leq y$.

PROOF. We prove the result by induction on the integer n.

Initial step ($n = 1$): Since $2x \leq 2y$, we have $x \leq 2y$. Hence, let $z \in G_+$ such that $x + z = y + y$. Since (G, G_+) has the Riesz decomposition property, let $\{u_{i,j}\}_{1 \leq i,j \leq 2}$ be a subset of G_+ such that $x = u_{1,1} + u_{1,2}$ and $y = u_{1,1} + u_{2,1} = u_{1,2} + u_{2,2}$. From the second equation, and by another application of the Riesz decomposition property, we have a subset $\{w_{i,j}\}_{1 \leq i,j \leq 2}$ of G_+ such that

$$
\begin{aligned}
u_{1,1} &= w_{1,1} + w_{1,2} \\
u_{2,1} &= w_{2,1} + w_{2,2} \\
u_{1,2} &= w_{1,1} + w_{2,1} \\
u_{2,2} &= w_{1,2} + w_{2,2}
\end{aligned}
$$

Hence, $x = 2w_{1,1} + w_{1,2} + w_{2,1}$, and $y = w_{1,1} + w_{1,2} + w_{2,1} + w_{2,2}$. Let $x_1 = w_{1,1} + w_{1,2} + w_{2,1}$, and let $x_2 = w_{1,1}$. Then $x = x_1 + x_2$, $x_1 \leq y$, and $x_2 \leq x_1$. This finishes the case $n = 1$.

Induction step: Suppose that the conclusion of the lemma holds for a positive integer n. We show that the conclusion also holds for $n + 1$.

Since $2x \leq 2y$, it follows by the induction hypothesis that we have the following equations:

i. $x = u_1 + u_2$, where $u_1, u_2 \in G_+$.

ii. $nu_2 \leq u_1$.

iii. $y = u_1 + v_2$, where $v_2 \in G_+$.

Since $2x \leq 2y$, we must have that $2u_2 \leq 2v_2$. Hence, by another application of the induction hypothesis, we also have the following equations:

iv. $u_2 = w_1 + w_2$, where $w_1, w_2 \in G_+$.

v. $w_2 \leq w_1$.

vi. $v_2 = w_1 + z_2$, where $z_2 \in G_+$.

Now take $x_1 = u_1 + w_1$ and $x_2 =_{df} w_2$. Clearly, $x = x_1 + x_2$ and $x_1 \leq y$. Also, $(n+1)w_2 = nw_2 + w_2 \leq nu_2 + w_2 \leq u_1 + w_1$, by equations ii and v. Hence, $(n+1)x_2 \leq x_1$. This completes the induction step.

We are now in a position to prove the converse of theorem 2.4, under appropriate hypotheses.

The idea of the proof is to, for each algebra B in the given class that does not have the corona factorization property, construct a short exact sequence

$$0 \longrightarrow B \longrightarrow C \longrightarrow \mathscr{K} \longrightarrow 0$$

where C is not stable.

As mentioned earlier, it seems likely that the hypotheses can be weakened moderately, but at present we have neither a more general proof nor a counter-example.

THEOREM 5.2. *Let B be a separable, simple, stable, real rank zero C^*-algebra with cancellation of projections. Then the following conditions are equivalent:*

(1) *B has the corona factorization property.*

(2) *Every extension of B by a separable stable C^*-algebra gives a stable extension algebra.*

(3) *Every extension of B by the compact operators $\mathscr{K}$ gives a stable extension algebra.*

Before giving the proof, we mention some corollaries of the theorem. Recall that an extension of AF algebras is AF. Because the stability of AF algebras can be characterized in terms of traces, an extension of stable AF algebras is stable, and hence:

COROLLARY 5.3. *All simple AF algebras have the corona factorization property.*

Moreover, any separable simple C^*-algebra that is regular in Rørdam's sense will, if in the class covered by the above theorem, have the corona factorization property. For example, theorems 1.2 and 5.2 imply that:

COROLLARY 5.4. *Simple, separable, exact C^*-algebras, with real rank zero, stable rank one, and weakly unperforated ordered K_0 group have the corona factorization property.*

Now a lemma on strict convergence, actually a special case of a result from § 4.1 of [15].

LEMMA 5.5. *Let (p_n) be a sequence of projections that sum strictly to 1 in the multipliers of the given algebra. Then, for any bounded sequence of elements (x_n), the series*

$$\sum_1^\infty p_n x_n p_n$$

converges strictly in the multipliers.

PROOF OF THEOREM 5.2. That (1) implies (2), follows from theorem 2.4. It is clear that (2) implies (3). Hence, it suffices to show that (3) implies (1).

Suppose that B does not have the corona factorization property. We will show that (3) does not hold.

Since B is stable, we can take finite direct sums of elements, whether or not they are orthogonal, by forming Cuntz sums $\bigoplus_1^N P_i := \sum_1^N v_i P_i v_i^*$ where the v_i are generators of an unital copy of O_N in the multipliers. Infinite direct sums such as $\bigoplus_1^\infty P$ are defined to be $\sum_1^\infty v_i P v_i^*$ where the v_i are isometries generating an unital copy of O_∞ in the multipliers. In general, such sums give elements of the multipliers, with convergence following from lemma 5.5. If a sequence of projections are orthogonal, there is no distinction between sums and direct sums.

Since B does not have the corona factorization property, let P be a norm-full projection in $\mathcal{M}(B)$ such that P is not Murray-von Neumann equivalent to $1_{\mathcal{M}(B)}$. Since P is norm-full, there is a least positive integer n such that the Cuntz sum $\bigoplus_{i=1}^n P$ is Murray-von Neumann equivalent to $1_{\mathcal{M}(B)}$, and by replacing P by a suitabble multiple, we may as well suppose that $n = 2$. Similarly, we may replace B by the isomorphic algebra $(P \oplus P)B(P \oplus P)$, so that in effect $P \oplus P = 1_{\mathcal{M}(B)}$.

Let $\{p_k\}_{k=1}^\infty$ be a sequence of pairwise orthogonal projections in the real rank zero subalgebra PBP such that $P = \sum_{k=1}^\infty p_k$, where the sum converges, by lemma 5.5, in the strict topology in $\mathcal{M}(B)$.

Choosing an order-unit $p \in B$ – it can be taken to be any nonzero projection, since B is simple – we can suppose, replacing the elements p_k by finite sums of the original ones if necessary, that for each k, $p \oplus p$ is Murray-von Neumann equivalent to a subprojection of $p_k \oplus p_k$. This follows from the fact that $P \oplus P = \sum_1^\infty (p_k \oplus p_k) = 1$.

For each projection $r \in B$, let $[r]$ be the class of r in the ordered K-group $(K_0(B), K_0(B)_+)$. Hence, for each k, $2[p] \leq 2[p_k]$ in $K_0(B)$. Now, by [21, Th. 1.1 and Cor 1.6] since B has real rank zero and cancellation of projections, the ordered group $(K_0(B), K_0(B)_+)$ must have the Riesz decomposition property. Hence, by lemma 5.1, for each k, there exists a projection q_k in $K_0(B)_+$ (hence in B) such that

$$k[q_k] \leq [p_k]$$
$$[p] \leq [p_k] + [q_k]$$

in $K_0(B)$. Since B has cancellation of projections, in terms of comparison of projections, we have that for each index k,

(1) the sum $\bigoplus_{i=1}^{k} q_k$ is Murray-von Neumann equivalent to a subprojection of p_k, and

(2) p is Murray-von Neumann equivalent to a subprojection of $p_k \oplus q_k$.

Since this statement only involves equivalence classes of projections, we can replace the p_k and the q_k by Murray-von Neumann equivalent families of orthogonal projections.

Thus, we now have sequences (p_k) and (q_k) such that:

(3) the (p_k) and (q_k) are pairwise orthogonal familes of projections.

Since p is equivalent to a subprojection of $p_k + q_k$, it follows that the Cuntz sum $\bigoplus_1^\infty p$ is equivalent to a subprojection of $\sum_1^\infty (p_k + q_k)$. But, by remark 4.6, the Cuntz sum $\bigoplus_1^\infty p$ is equivalent to $1_{\mathcal{M}(B)}$.

In the multipliers of a stable algebra, a projection that majorizes a projection equivalent to 1 is itself equivalent to 1. This is because a projection equivalent to 1 is properly infinite and full, as is any projection majorizing it. Then one applies the K-theoretical properties of properly infinite projections (due to Cuntz) and the triviality of the K_0-group of the multipliers of a stable algebra. It follows that

(4) $\sum_1^\infty p_k + q_k \sim 1_{\mathcal{M}(B)}$.

Recall that for each k, $\bigoplus_{i=1}^{k} q_k$ is Murray-von Neumann equivalent to a subprojection of p_k. Hence, for each k, define q_k^1 to be q_k, and let $q_k^2, q_k^3, \ldots, q_k^k$ be equivalent pairwise orthogonal projections subordinate to p_k. We thus have the following array of projections, with each projection orthogonal to every other, and all projections in a column mutually equivalent:

$$\begin{matrix} q_1^1 & q_2^1 & q_3^1 & q_4^1 & \cdots \\ & q_2^2 & q_3^2 & q_4^2 & \cdots \\ & & q_3^3 & q_4^3 & \cdots \\ & & & q_4^4 & \cdots \end{matrix} \left.\vphantom{\begin{matrix} a \\ a \\ a \end{matrix}}\right\} \quad \text{where the } k^{\text{th}} \text{ column is} \leq p_k.$$

Let v_k^{ij} be the partial isometry linking q_k^i to q_k^j. Let $S^{ii} := \sum_{j=i}^{\infty} q_j^i$ be the multiplier projection obtained by summing the rows in the diagram, and let $V^{i,i+1}$ be the partial isometry obtained by summing the partial isometries $v_k^{i,i+1}$ linking the i^{th} and $(i+1)^{th}$ rows.

Let us pause for a moment to discuss the convergence of these two sums. Since the q_j^i, as functions of j, are majorized by the p_j, lemma 5.5 gives the convergence of the sum defining the S^{ii}. Furthermore, both the source and the range projections of $v_k^{i,i+1}$ are majorized by p_k, so that $p_k v_k^{i,i+1} p_k = v_k^{i,i+1}$, and we again have a situation where lemma 5.5 can be applied.

Now that convergence has been established, we have that, for example, $V^{1,2}$ implements an equivalence of S^{22} and $S^{11} - q_1^1$. We thus see, by transitivity of the equivalence relation, that the S^{ii} are mutually equivalent (by $V^{ij} \in \mathcal{M}(B)$) in the quotient $\mathcal{M}(B)/B$. This gives us a copy of $\mathcal{K}$ in the corona $\mathcal{M}(B)/B$, where, for example, the usual rank-one elementary operator e_{ij} corresponds to V^{ij} mod B. Let $C = C^*(V^{ij}, B)$ be the C^*-algebra generated by the V^{ij} and B.

We thus have an exact sequence

$$0 \longrightarrow B \longrightarrow C \longrightarrow \mathcal{K} \longrightarrow 0.$$

Now consider $S^{11} \in C$. Note that $1_{\mathcal{M}(B)} - S^{11}$ is equal to $\sum_{k=1}^{\infty} p_k = P$. If C were stable, then by corollary 2.2, the hereditary subalgebra $\overline{(1 - S^{11})C(1 - S^{11})}$ is stable, and hence the projection $1 - S^{11}$ is the unit of a copy of O_∞. It follows that $1 - S^{11}$ is properly infinite in $\mathcal{M}(C)$, but since $\mathcal{M}(C)$ is a (unital) subalgebra of $\mathcal{M}(B)$, of course $1 - S^{11}$ is also properly infinite as an element of $\mathcal{M}(B)$. Thus P is properly infinite, contrary to assumption.

Hence the algebra C gives an extension of B by $\mathcal{K}$, but is not itself stable.

A proof similar to that of theorem 5.2 shows:

COROLLARY 5.6. *Suppose that B is a simple, separable, stable, exact, real rank zero C^*-algebra with cancellation of projections. Then the following properties of B are equivalent:*

(1) *B has the corona factorization property.*

(2) *Suppose that whenever*

$$0 \longrightarrow B \longrightarrow C \longrightarrow \mathcal{K} \longrightarrow 0.$$

is an essential, quasidiagonal extension, with C having real rank zero, then C is necessarily stable.

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