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IDEAL STRUCTURE AND C^*-ALGEBRAS OF LOW RANK

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(Dedicated to the memory of Gert K. Pedersen)

Abstract

We explore various constructions with ideals in a C^*-algebra A in relation to the notions of real rank, stable rank and extremal richness. In particular we investigate the maximum ideals of low rank. And we investigate the relationship between existence of infinite or properly infinite projections in an extremally rich C^*-algebra and non-existence of ideals or quotients of stable rank one.

1. Introduction

The concept of dimension for a topological space X originates in the basic fact that manifolds are locally homeomorphic to euclidean spaces, which have an obvious linear dimension. In the more abstract version given by Čech's covering dimension of a normal space X, the dimension gives conditions under which certain functions extend and certain cohomology groups vanish.

Regarding a C^*-algebra A as the non-commutative analogue of $C(X)$ (or $C_0(X)$) for a compact (or just locally compact) Hausdorff space X, it is natural to try to extend the notion of topological dimension of X to the analogous setting. The more so as the covering dimension of X is easily characterized in terms of elements in $C(X)$. In [34] Rieffel defined the *(topological) stable rank*, $\mathrm{tsr}(A)$, of an arbitrary C^*-algebra A, using concepts from dimension theory. Shortly after, the stable rank was identified with the Bass stable rank of A, [21], which is a purely algebraic concept. In particular, by an earlier result of Vaserstein, [41], we have $\mathrm{tsr}(C_0(X)) = \left[\frac{1}{2}\dim(X \cup \{\infty\})\right] + 1$; the factor $\frac{1}{2}$ arising from the use of complex scalars in $C_0(X)$.

The *real rank* of a C^*-algebra was introduced in [9] as an alternative to Rieffel's stable rank. Formally the only difference is that self-adjoint elements replace the general elements in Rieffel's definition, but this has unexpected consequences, especially for small values of the rank. In general one has $\mathrm{RR}(A) \leq$

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$2\,\mathrm{tsr}(A) - 1$, and – pleasing for the eye – $\mathrm{RR}(C_0(X)) = \dim(X \cup \{\infty\})$. However, in the lowest possible cases, $\mathrm{tsr}(A) = 1$ and $\mathrm{RR}(A) = 0$, the two notions are independent: one may be satisfied without the other.

One of the real surprises is the symmetry with which stable rank one and real rank zero sometimes interact with the two K-groups for a unital C^*-algebra A: If I is a closed ideal in A and $\mathrm{tsr}(A) = 1$, the natural map $K_0(I) \to K_0(A)$ is injective, whereas the map $K_1(I) \to K_1(A)$ is injective if $\mathrm{RR}(A) = 0$. Also, the natural map from Murray-von Neumann equivalence classes of projections in A to $K_0(A)$ is injective if A has stable rank one, whereas its image generates the whole group if A is of real rank zero.

Recall from [10] that a unital C^*-algebra A is *extremally rich* if the open set A_q^{-1} of *quasi-invertible* elements is dense in A. Here A_q^{-1} is defined as $A^{-1}\mathscr{E}(A)A^{-1}$, where $\mathscr{E}(A)$ denotes the set of extreme points in the closed unit ball A_1 of A. Equivalently, cf. [11], A is extremally rich if $\mathrm{conv}(\mathscr{E}(A)) = A_1$, so that – as a Banach space – A has the λ-property, cf. [31]. If $A = C(X)$, extremal richness is equivalent to $\dim(X) \le 1$. In general, extremal richness is a generalization of Rieffel's notion of stable rank one suitable for not necessarily finite C^*-algebras. Thus every purely infinite simple C^*-algebra is extremally rich, as is every von Neumann algebra.

Evidently extreme partial isometries are not as natural a class to work with as unitaries. For this reason the concept of quasi-invertibility may appear somewhat artificial. But it keeps coming up in connection with various quite natural problems; the most recent being the question of characterizing elements in C^*-algebras with persistently closed range, cf. [12, §7]. The following minor result can also be taken as an indication of the ubiquity of quasi-invertibility. Formally it may be considered as an extension of Rørdam's results in [37, §3] from prime to general C^*-algebras.

The symbol "=" denotes norm closure and $\widetilde{A}$ denotes the unitization of A ($\widetilde{A} = A$ if A is unital). Let A be a C^*-algebra and define the set of *symmetric zero-divisors*

$$\mathrm{ZD}^*(A) = \{x \in A \mid \exists y \in A,\ \|y\| = 1 : xy = yx = 0\}.$$

Similarly, define the set of *symmetric topological zero-divisors* as

$$\mathrm{ZD}^*_\infty(A) = \{x \in A \mid \exists (y_n) \subset A,\ \|y_n\| = 1 : \lim xy_n = \lim y_n x = 0\}.$$

PROPOSITION 1.1. *The following conditions on an element x in A are equivalent:*

(i) $x \in \mathrm{ZD}^*_\infty(A)$.

(ii) $x \in (\mathrm{ZD}^*(A))^=$.

(iii) $x \notin \widetilde{A}_q^{-1}$.

PROOF. (i)$\Rightarrow$(iii) If $x \in \widetilde{A}_q^{-1}$, there is by [10, Theorem 1.1] a pair of morphisms λ and ρ of $\widetilde{A}$ with $\ker\lambda \cap \ker\rho = 0$, such that $\lambda(x)$ is left invertible in $\lambda(\widetilde{A})$, but $\rho(x)$ is right invertible in $\rho(\widetilde{A})$. So $ax - \mathbf{1} \in \ker\lambda$ and $xb - \mathbf{1} \in \ker\rho$ for suitable elements a, b in $\widetilde{A}$. If now $xy_n \to 0$ and $y_n x \to 0$, then $\lambda(y_n) \to 0$ and $\rho(y_n) \to 0$. However, the morphism $\lambda \oplus \rho$ is isometric, so $y_n \to 0$. Consequently $x \notin \mathrm{ZD}^*_\infty(A)$.

(iii)$\Rightarrow$(ii) If $x \notin \widetilde{A}_q^{-1}$ we distinguish two cases:

(a) Zero is isolated in $\mathrm{sp}(|x|)$. In that case we have a polar decomposition $x = v|x|$ with v a partial isometry in A. Let $p_+ = \mathbf{1} - v^*v$ and $p_- = \mathbf{1} - vv^*$. If $p_+Ap_- = 0$ then A must be unital and $x \in A_q^{-1}$, contrary to the hypothesis. There is therefore a non-zero element y in p_+Ap_-. Evidently

$$xy = (v|x|v^*v)(p_+y) = 0 \qquad \text{and} \qquad yx = (yp_-)(vv^*v|x|) = 0;$$

so $x \in \mathrm{ZD}^*(A)$.

(b) Zero is a limit point in $\mathrm{sp}(|x|)$. In that case let (f_n) and (g_n) be sequences of continuous functions on R_+ such that $0 \le f_n \le 1$, $f_n(0) = 0$, $f_n(t) = 1$ for $2^{-n-1} \le t \le 2^{-n}$, $f_n(t) = 0$ for $t \ge 2^{-n+1}$, $g_n(t) = 0$ for $0 \le t \le 2^{-n+1}$, $g_n(t) = t$ for $t \ge 2^{-n+2}$, and $0 \le g_n(t) \le t$. If $x = v|x|$ is the polar decomposition of x (now with v in A''), then $y_n = f_n(|x|)v^* \in A$, $\|y_n\| = 1$ if $\mathrm{sp}(|x|) \cap [2^{-n-1}, 2^{-n}] \ne \emptyset$, $x_n = vg_n(|x|) \in A$, and $\|x_n\| \le \|x\|$. Passing if necessary to a subsequence we may assume that $\|y_n\| = 1$ for all n. Now

$$x_n y_n = vg_n(|x|)f_n(|x|)v^* = 0 \qquad \text{and} \qquad y_n x_n = f_n(|x|)v^*vg_n(|x|) = 0.$$

Hence $x_n \in \mathrm{ZD}^*(A)$ and $x \in (\mathrm{ZD}^*(A))^=$.

(ii)$\Rightarrow$(i) If $x = \lim x_n$ and $x_n y_n = y_n x_n = 0$ for a normalized sequence (y_n), then $\lim xy_n = \lim y_n x = 0$, as desired.

REMARK 1.2. Note that when A is non-unital we have $A \cap \widetilde{A}_q^{-1} = \emptyset$, because $\widetilde{A}_q^{-1}/A = \mathbf{C}\mathbf{1} \setminus \{0\}$. So every element in A is a symmetric topological zero-divisor.

The plan of the paper is as follows: In Section 2 we treat the maximum ideal of real rank zero, $I_{\mathrm{RR}\,0}(A)$, and the "maximum" extremally rich ideal, $I_{\mathrm{er}}(A)$, parallel to Rørdam's treatment [36] of $I_{\mathrm{tsr}\,1}(A)$, the maximum ideal of stable rank one. In fact, in general A has no largest extremally rich ideal. But there are two ways to characterize $I_{\mathrm{tsr}\,1}(A)$ other than that it is the largest ideal of stable rank one (see 2.2(ii) and 2.14(ii)/2.16), and $I_{\mathrm{er}}(A)$ has properties exactly parallel to these. For extremally rich C^*-algebras we also discuss the sense in which $\mathrm{tsr}(A) > 1$ or $I_{\mathrm{tsr}\,1}(A) = 0$ implies infinite behavior. In Section 3 we discuss defect ideals, which measure the stable rank one quotients of extremally rich C^*-algebras. We give three different results with hypotheses of increasing strength showing that lack of stable rank one quotients implies existence of properly infinite projections. Also we define *purely properly infinite* for non-simple C^*-algebras. In Section 4 we discuss isometric richness, a concept intermediate between stable rank one and extremal richness, and its relationship to ideal structure.

The authors previously announced a paper entitled, "Extremally rich ideals in C^*-algebras." The present paper and [13] constitute an expanded version of that paper.

2. Maximum ideals of minimal rank

All three concepts, stable rank one, extremal richness and real rank zero, are formulated as attempts to describe low-dimensional behaviour in a non-commutative setting. In this section we shall explore constructions that lead to hereditary C^*-subalgebras and even ideals of low rank in general C^*-algebras. Our first result is a typical sample.

THEOREM 2.1. *Let B be a hereditary C^*-subalgebra of a C^*-algebra A.*

(i) *If $\widetilde{B} \subset (\widetilde{A}_q^{-1})^=$ then B is extremally rich.*

(ii) *If $\widetilde{B} \subset (\widetilde{A}^{-1})^=$ then B has stable rank one.*

(iii) *If $\widetilde{B}_{sa} \subset (\widetilde{A}_{sa}^{-1})^=$ then B has real rank zero.*

PROOF. In essence the argument is contained in the second half of the proof of [10, Theorem 3.5]. However, for the convenience of the reader we give the details.

(i) We may assume that A is unital. Given now an element $\tilde{b}$ in $\widetilde{B}$ we may assume for the purpose of approximation that it has the form $\tilde{b} = \mathbf{1} + b$ for some b in B. By assumption we can for any $\epsilon > 0$ find a in A_q^{-1} such that $\|\mathbf{1} + b - a\| < \epsilon$. For ϵ small enough we can then define the elements $d = \mathbf{1} - (a-b)^{-1}$ and

$$\begin{aligned} c &= (a-b)^{-1}a(\mathbf{1}-db)^{-1} = (\mathbf{1}-d)(a-b+b)(\mathbf{1}-db)^{-1} \\ &= (\mathbf{1}+(\mathbf{1}-d)b)(\mathbf{1}-db)^{-1} = \mathbf{1}+b(\mathbf{1}-db)^{-1}. \end{aligned}$$

By construction $c \in A_q^{-1}$; and since $b(\mathbf{1}-db)^{-1} = \sum_{n=0}^{\infty} b(db)^n$ and B is hereditary we see that $b(\mathbf{1}-db)^{-1} \in B$, whence $c \in \widetilde{B}$. Consequently $c \in A_q^{-1} \cap \widetilde{B} = \widetilde{B}_q^{-1}$. Finally, since $\|d\| \le (1-\epsilon)^{-1}\epsilon$ we can estimate

$$\begin{aligned} \|\mathbf{1}+b-c\| = \left\| \sum_{n=1}^{\infty} b(db)^n \right\| &\le \|b\|^2(1-\|b\|\|d\|)^{-1}\|d\| \\ &\le \|b\|^2(1-\epsilon-\|b\|\epsilon)^{-1}\epsilon, \end{aligned}$$

which shows that c is an approximant to $1 + b$. Thus $\widetilde{B}_q^{-1}$ is dense in $\widetilde{B}$, as required.

(ii) Replacing quasi-invertibles with invertibles in the argument above we obtain a proof of (ii).

(iii) Replacing quasi-invertibles with self-adjoint invertibles in the argument for (i) we will get a proof of (iii), but a little caution is required. Both a and b are now self-adjoint so $d = d^*$ as constructed. But then the computation

$$b(\mathbf{1} - db)^{-1} = b\sum_{n=0}^{\infty}(db)^n = \sum_{n=0}^{\infty}(bd)^n b = (\mathbf{1} - bd)^{-1}b = (b(\mathbf{1} - db)^{-1})^*$$

shows that also $c = c^*$, as desired.

DEFINITIONS 2.2. (i) Unless expressly mentioned, the word *ideal* will in this paper designate a closed (and therefore *-invariant) ideal in a C^*-algebra.

(ii) Rørdam shows in [36, 4.1–4.3] that in every C^*-algebra A there is a largest ideal $I_{\mathrm{tsr}\,1}(A)$ of stable rank one, given by

$$I_{\mathrm{tsr}\,1}(A) = \{x \in A \mid \forall y \in \widetilde{A} : \alpha(x + y) = \alpha(y)\},$$

where $\alpha(y) = \mathrm{dist}(y, \widetilde{A}^{-1})$. Equivalently,

$$I_{\mathrm{tsr}\,1}(A) = \{x \in A \mid x + \widetilde{A}^{-1} \subset (\widetilde{A}^{-1})^{=}\}.$$

A similar construction is possible with respect to ideals of real rank zero:

THEOREM 2.3. *For a given C^*-algebra A let $\alpha_r(z) = \mathrm{dist}(z, \widetilde{A}_{sa}^{-1})$ and define*

$$\begin{aligned} R &= \{x \in A_{sa} \mid \forall y \in \widetilde{A}_{sa} : \alpha_r(x + y) = \alpha_r(y)\} \\ &= \{x \in A_{sa} \mid x + \widetilde{A}_{sa}^{-1} \subset (\widetilde{A}_{sa}^{-1})^{=}\}. \end{aligned}$$

Then $I_{\mathrm{RR}\,0}(A) = R + iR$ is an ideal of real rank zero in A, and the largest such.

PROOF. If an element x in A_{sa} satisfies the first condition, then we have $\alpha_r(x + y) = \alpha_r(y) = 0$, for each y in $\widetilde{A}_{sa}^{-1}$, whence $x + y \in (\widetilde{A}_{sa}^{-1})^{=}$. Conversely, if x satisfies the second condition and $y \in \widetilde{A}_{sa}$, then for each $\epsilon > 0$ there is a z in $\widetilde{A}_{sa}^{-1}$ such that

$$\alpha_r(x + y) + \epsilon > \|x + y + z\| = \|y + (x + z)\| \geq \alpha_r(y),$$

since $x + z \in (\widetilde{A}_{sa}^{-1})^{=}$. Thus $\alpha_r(x + y) \geq \alpha_r(y)$. Since $R = -R$ we can replace x with $-x$ and then y with $x + y$ to obtain the reverse inequality, hence equality.

The set R defined by the two equivalent conditions is clearly a a closed real subspace of A_{sa} contained in $(\widetilde{A}_{sa}^{-1})^{=}$. If $x \in R$ and $y \in \widetilde{A}_{sa}$, then for each real t and z in $\widetilde{A}_{sa}^{-1}$ we have

$$(\mathbf{1}+ity)x(\mathbf{1}-ity)+z=(\mathbf{1}+ity)(x+(\mathbf{1}+ity)^{-1}z(\mathbf{1}-ity)^{-1})(\mathbf{1}-ity)\\ \in(\mathbf{1}+ity)(\widetilde{A}_{sa}^{-1})^{=}(\mathbf{1}-ity)=(\widetilde{A}_{sa}^{-1})^{=}.$$

Thus $(\mathbf{1}+ity)x(\mathbf{1}-ity)\in R$. Since R is a closed subspace we get by first and second order expansions in t that $i(yx-xy)\in R$ and that $yxy\in R$. Applied with $y=\mathbf{1}\pm z$ for some z in A_{sa}, the second fact shows that $zx+xz\in R$. In conjunction with the stability under commutators this implies that $I_{\mathrm{RR}\,0}(A)=R+iR$ is a closed ideal of A. Since $(I_{\mathrm{RR}\,0}(A))^{\sim}_{sa}=\widetilde{R}_{sa}\subset(\widetilde{A}_{sa}^{-1})^{=}$ by definition, it follows from Theorem 2.1 that $\mathrm{RR}(I_{\mathrm{RR}\,0}(A))=0$.

Assume now that I is an ideal of A with $\mathrm{RR}(I)=0$. For each x in I_{sa} and y in $\widetilde{A}_{sa}^{-1}$ with polar decomposition $y=u|y|$ (so that $u=u^*$) we let B denote the C^*-subalgebra of $\widetilde{A}$ generated by $\widetilde{I}$ and the projection $p=2u-\mathbf{1}$. Then $pBp=p\widetilde{I}p$, which is isomorphic to $(pIp)^{\sim}$ and therefore of real rank zero. Similarly $\mathrm{RR}((\mathbf{1}-p)\widetilde{I}(\mathbf{1}-p))=0$, and it follows from [9, Theorem 2.5] that $\mathrm{RR}(B)=0$. Therefore, with $x_0=|y|^{-1/2}x|y|^{-1/2}$ in I, we have

$$x+y=|y|^{1/2}(x_0+u)|y|^{1/2}\in|y|^{1/2}B_{sa}|y|^{1/2}\\ \subset|y|^{1/2}(B_{sa}^{-1})^{=}|y|^{1/2}\subset|y|^{1/2}(\widetilde{A}_{sa}^{-1})^{=}|y|^{1/2}=(\widetilde{A}_{sa}^{-1})^{=}.$$

This means that $x\in R$, whence $I\subset I_{\mathrm{RR}\,0}(A)$.

Our construction from [12, Proposition 5.3] carries over (changing 1 to 0) to produce a largest ideal of real rank zero whose K_0-group vanishes in $K_0(A)$. The following easy lemma is a translation $(1\to 0)$ of [12, Lemma 5.2].

Lemma 2.4. *For each ideal I and every C^*-subalgebra B of a unital C^*-algebra A consider the induced maps*

$$\iota_0\colon K_0(I)\oplus K_0(B)\longrightarrow K_0(I+B)\qquad\textit{and}\qquad \iota_1\colon K_1(I)\longrightarrow K_1(I+B).$$

If $\mathrm{RR}(B)=0$ *then ι_0 is surjective and ι_1 is injective.*

Proof. Without loss of generality we may assume that $I+B$ is unital. If now p is a projection in $I+B$ and $\pi\colon I+B\longrightarrow(I+B)/I$ denotes the quotient morphism then $\pi(p)$ is a projection in $(I+B)/I=B/(I\cap B)$. Since $\mathrm{RR}(B)=0$ projections lift from every quotient of B, cf. [9], so there is a projection q in B such that $\pi(q)=\pi(p)$. Thus the K-theory exact sequence implies that $[p]-[q]$ is in $\iota_0(K_0(I))$. This argument passes to matrix algebras over $B+I$ and proves that $K_0(I+B)=\iota_0(K_0(I))+\iota_0(K_0(B))$.

Replacing in the argument above $\pi(p)$ with an arbitrary projection in $B/(I\cap B)$ shows that the induced morphism $\pi_0\colon K_0(I+B) \longrightarrow K_0((I+B)/I)$ is surjective. By the six-term-exact sequence this means that $\partial_0\colon K_0((I+B)/I) \longrightarrow K_1(I)$ is the zero map, whence $\iota_1\colon K_1(I) \longrightarrow K_1(I+B)$ must be injective.

PROPOSITION 2.5. *For every C^*-algebra A there is a largest ideal $I_{0,0}(A)$ of real rank zero such that the induced map $\iota_0\colon K_0(I_{0,0}(A)) \longrightarrow K_0(A)$ is zero.*

PROOF. Let $\mathscr{I}_{0,0}(A)$ denote the class of ideals I such that $\mathrm{RR}(I) = 0$ and $\iota_0\colon K_0(I) \longrightarrow K_0(A)$ is zero. If I and J belong to $\mathscr{I}_{0,0}(A)$ we consider the extension

$$0 \longrightarrow J \longrightarrow I + J \longrightarrow (I + J)/J\ (= I/(I \cap J)) \longrightarrow 0.$$

Since projections lift from $I/(I \cap J)$ to I it follows that $\mathrm{RR}(I + J) = 0$. Moreover, by Lemma 2.4,

$$\iota_0(K_0(I + J)) = \iota_0(K_0(I)) + \iota_0(K_0(J)) = \{0\}.$$

Thus $I + J \in \mathscr{I}_{0,0}(A)$. This means that $\mathscr{I}_{0,0}(A)$ is inductively ordered under inclusion, and since both real rank and K-groups are stable under inductive limits we can define

$$I_{0,0}(A) = \varinjlim I, \qquad I \in \mathscr{I}_{0,0}(A).$$

Note that the first half of the argument provides another proof of the existence of $I_{\mathrm{RR}\,0}(A)$ as the largest ideal of real rank zero. It does not, however, give its other characteristics.

DEFINITIONS 2.6. (i) If $I_{\mathrm{tsr}\,1}(A) = 0$ we say that A has *no ideals of stable rank one*. In the presence of extremal richness this forces A to exhibit a highly infinite behaviour, as we shall see.

(ii) Recall that a projection p in a C^*-algebra A is *finite* if it is not Murray-von Neumann equivalent to a proper subprojection. Following tradition we say that a unital C^*-algebra A is *finite* if it contains no infinite projections; i.e. if every isometry is unitary. For non-prime C^*-algebras this definition is not optimal, but we will not argue with tradition. Fortunately the discrepancies disappear when we define A to be *absolutely finite* if no (primitive) quotient of A contains any infinite projections. If this even holds for all matrix algebras over A we say that A is *absolutely stably finite*.

PROPOSITION 2.7. *For an extremally rich unital C^*-algebra A the following conditions are equivalent:*

(i) *A is absolutely finite.*

(ii) *A is absolutely stably finite.*

(iii) $\operatorname{tsr}(A) = 1$.

(iv) $\mathscr{E}(A) = \mathscr{U}(A)$.

PROOF. (i) ⇒ (iv) If $v \in \mathscr{E}(A) \setminus \mathscr{U}(A)$ there must be some irreducible representation $(\pi, \mathscr{H})$ of A in which $\pi(v)$ is not unitary. Since $\pi(A)$ is primitive, $\pi(v)$ is either an isometry or a co-isometry, contradicting (i).

(iv) ⇒ (iii) By assumption $A_q^{-1} = A^{-1}$, and since A is extremally rich this set is dense, whence $\operatorname{tsr}(A) = 1$.

(iii) ⇒ (ii) Being of stable rank one is a stable property preserved under quotient maps, so no (primitive) quotient of $\mathsf{M}_n(A)$ can contain a non-unitary isometry.

Evidently (ii) ⇒ (i).

THEOREM 2.8. *An extremally rich C^*-algebra A has no ideals of stable rank one if and only if:*

(i) *Every non-zero hereditary C^*-subalgebra of A contains a non-zero projection, and*

(ii) *Every non-zero projection in A supports a non-unitary extreme partial isometry.*

PROOF. Only the forward implication needs proof, and for this we may evidently assume that A is unital. Now let B be a non-zero hereditary C^*-subalgebra of A. If $I(B)$ denotes the closed ideal of A generated by B, then B and $I(B)$ are Rieffel-Morita equivalent by [8]. By assumption $\operatorname{tsr}(I(B)) > 1$, and consequently also $\operatorname{tsr}(B) > 1$, cf. [10, Corollary 5.8]. Since $\widetilde{B}$ is extremally rich by [10, Theorem 3.5], viz. Theorem 2.1, we deduce from Proposition 2.7 that $\widetilde{B}$ contains a non-unitary extreme partial isometry v. We may assume that $v = \mathbf{1} + b$ for some b in B, which implies that both defect projections of v belong to B. One (or both) of them is non-zero, which establishes condition (i) in the theorem.

For condition (ii), note that if p is a non-zero projection in A then pAp is a unital, hereditary C^*-subalgebra of A. As before this implies that $\operatorname{tsr}(pAp) > 1$, and since pAp is extremally rich we can find w in $\mathscr{E}(pAp) \setminus \mathscr{U}(pAp)$, whence

$$(*) \qquad (p - ww^*)A(p - w^*w) = 0,$$

and either $ww^* \neq p$ or $w^*w \neq p$.

COROLLARY 2.9. *A simple C^*-algebra is extremally rich if and only if it is either purely infinite or has stable rank one.*

PROPOSITION 2.10. *If A is any simple, unital C^*-algebra and B is a UHF-algebra then $A \otimes B$ is extremally rich.*

PROOF. If A is stably finite Rørdam showed that $A \otimes B$ has stable rank one, cf. [37, 6.7], and therefore is extremally rich. Otherwise, by [37, 6.8], $A \otimes B$ is purely infinite and thus extremally rich by [31, Theorem 10.1].

REMARK 2.11. If A is not prime we cannot deduce that every non-zero projection p in A is infinite under the circumstances in Theorem 2.8. However, if in equation $(*)$ in the proof of 2.8 we have both $ww^* \neq p$ and $w^*w \neq p$, it follows that pAp contains a C^*-subalgebra isomorphic to the extended Toeplitz algebra $\mathscr{T}_e$, cf. [10, Proposition 6.10]. Consequently, if p is finite pAp contains two centrally orthogonal sequences of mutually equivalent, orthogonal projections (p_n) and (q_n) (corresponding to the ideal $\mathsf{K} \oplus \mathsf{K}$ in $\mathscr{T}_e$). Even worse, each projection p_n (or q_n) has the same properties as p. In particular, the ideal structure of A must be rich: Every non-zero closed ideal of pAp contains an orthogonal pair of non-zero ideals if p is a finite projection. We proceed to show that this behavior actually can occur.

2.12. Infinite Tensor Products

Let (A_n) be a sequence of unital C^*-algebras and for each n let $A^{(n)} = \bigotimes_{k=1}^n A_k$ denote the spatial tensor product. There is a natural embedding $A^{(n)} \longrightarrow A^{(n+1)}$ given by $a_n \longrightarrow a_n \otimes \mathbf{1}$, and as usual we define

$$A = \bigotimes_{n=1}^{\infty} A_n = \varinjlim A^{(n)}.$$

Then A is a unital C^*-algebra which is separable and nuclear provided that all the A_n's are; and its ideal structure can – in principle – be determined from that of the A_n's. In particular we note that if $\pi_n \colon A_n \longrightarrow Q_n$ is a sequence of unital morphisms there is a unique morphism

$$\pi \colon \bigotimes_{n=1}^{\infty} A_n \longrightarrow \bigotimes_{n=1}^{\infty} Q_n$$

given by $\pi | A^{(n)} = \otimes_{k=1}^n \pi_k$ for all n. We shall write $\pi = \otimes_{n=1}^{\infty} \pi_n$, and we note that $\ker \pi$ is the ideal in A generated by elements of the form

$$A(\pi_n) = \{a = \otimes_{k=1}^{\infty} a_k \mid a_k = \mathbf{1} \text{ for } k \neq n \text{ and } a_n \in \ker \pi_n\}.$$

Assume now that in each A_n we have chosen an ideal I_n. In the applications I_n

will always be essential in A_n and often simple. We define the C^*-subalgebras

$$B_n = \left(\bigotimes_{k=1}^{n-1} I_k\right) \otimes A_n \otimes \mathbf{1} \subset A^{(n)} \otimes \mathbf{1} \subset A.$$

Observe now that $B_n B_m \subset B_m$ if $n \le m$. We can therefore define

$$B^{(n)} = B_1 + B_2 + \cdots + B_n = C^*(B_1 \cup B_2 \cup \cdots \cup B_n) \subset A^{(n)} \subset A.$$

Note that each $B^{(n)}$ contains the ideal $I^{(n)} = \bigotimes_{k=1}^{n} I_n$. Finally we put

$$B = \varinjlim B^{(n)} = C^*\big(\cup_{n=1}^{\infty} B_n\big) \subset A.$$

The idea behind the construction is that we can find all the irreducible representations of each $B^{(n)}$, hence ultimately also the primitive ideals of B, as either coming from the ideal $I^{(n)}$ or arising from one and only one of the summands B_k in $B^{(n)}$ and corresponding to a representation of the quotient $B_k/I^{(k)} = I^{(k-1)} \otimes (A_k/I_k)$.

We shall employ the tensor product construction above with all the A_n's being equal to one of three algebras: The (ordinary) Toeplitz algebra $\mathscr{T}$, the extended Toeplitz algebra $\mathscr{T}_e$ (cf. [31, 9.3–9.5]) or the trivial Toeplitz algebra $\mathscr{T}_t$. If s demotes the unilateral shift on ℓ^2 and K the algebra of compact operators on ℓ^2 then $\mathscr{T} = C^*(s)$. Moreover, $\mathscr{T}_e = C^*(s \oplus s^*)$ (on $\ell^2 \oplus \ell^2$), and $\mathscr{T}_t = \mathscr{T}_e + \mathsf{K}(\ell^2 \oplus \ell^2)$. It follows that each of these algebras is an extension, viz.

$$0 \longrightarrow \mathsf{K} \longrightarrow \mathscr{T} \longrightarrow C(\mathsf{S}) \longrightarrow 0;$$
$$0 \longrightarrow \mathsf{K} \oplus \mathsf{K} \longrightarrow \mathscr{T}_e \longrightarrow C(\mathsf{S}) \longrightarrow 0;$$
$$0 \longrightarrow \mathsf{K}(\ell^2 \oplus \ell^2) \longrightarrow \mathscr{T}_t \longrightarrow C(\mathsf{S}) \longrightarrow 0.$$

The first two are non-trivial, but in the third we notice that since $s \oplus s^*$ is a compact perturbation of the bilateral shift u on $\ell^2 \oplus \ell^2 = \ell^2(\mathsf{Z})$, the algebra is a split extension, $\mathscr{T}_t = C^*(u) + \mathsf{K}(\ell^2(\mathsf{Z}))$. In particular, $\mathscr{T}_t$ has stable rank one, whereas $\mathscr{T}$ and $\mathscr{T}_e$ are only extremally rich. Even so, $\mathscr{T}$ is isometrically rich (see Section 4), whereas $\mathscr{T}_e$ is our pet example of an algebra that is extremally rich, but not isometrically rich.

The algebras obtained will be denoted B_I, B_{II}, B_{III}, respectively. All three are extremally rich with no ideals of real rank zero. Also B_I and B_{II} have no ideals of stable rank one, whereas $\operatorname{tsr}(B_{III}) = 1$. The algebras B_I and B_{III} are primitive. Hence B_I is isometrically rich and 2.8(ii) implies that every non- zero projection in B_I is infinite. On the other hand, $B_{II} \subset B_{III}$, whence B_{II} is stably finite in the classical sense, though by 2.8 it should be regarded as highly infinite. Despite this infinite behavior, it is true that every non-zero hereditary C^*-subalgebra of B_I or B_{II} has a non-trivial stable rank one quotient (cf. Section 3, below).

The primitive ideal spaces of B_I and B_{III}, aside from the dense point, consist of an infinite sequence (T_n) of circles. The closure of each point of T_n contains $\mathsf{T}_1, \ldots, \mathsf{T}_{n-1}$. Because there are two natural maps from $\mathscr{T}_e$ onto $\mathscr{T}$, there is a family of maps from B_{II} onto B_I indexed by the Cantor set. This gives a Cantor set "at the bottom" of the primitive ideal space $B_{II}^{\vee}$ instead of a dense point. And instead of a sequence of circles there is a binary tree of circles. Each T_n is replaced by the union of 2^{n-1} disjoint circles. Each point in the Cantor set corresponds to an infinite path in the tree, and its closure contains just the circles on this path. And the closure of a point on one of the circles contains the ancestor circles.

We provide a few indications of proof, but many details are left to the reader. Because $I^{(n)}$ is essential in $B^{(n)}$ (even in $A^{(n)}$), it is easy to see that the inclusion of $B^{(n-1)}$ in $B^{(n)}$ is extreme point preserving (e.p.p.). Hence to prove B extremally rich, it suffices to prove by induction that each $B^{(n)}$ is. To do this we use [10, Theorem 6.1] for the extension in which the ideal is B_n and the quotient is $B^{(n-1)}/B^{(n-1)} \cap B_n = B^{(n-1)}/I^{(n-1)}$. Clearly extremal partial isometries lift, and each minor defect projection is in $I^{(n)}$ (which is either K or the direct sum of 2^n copies of K). It is easy to see that for any projections P in $M(I^{(n)})$ and Q in $I^{(n)}$ the bimodule $PI^{(n)}Q$ is extremally rich. Thus the hypotheses of [10, Theorem 6.1] are verified.

Because $I^{(n)}$ is essential in $B^{(n)}$, each non-trivial ideal of $B^{(n)}$ contains $I^{(n)}$ or one of its simple summands. Thus every non-zero ideal of B contains an ideal isomorphic to $\mathsf{K} \otimes B$. Since B has neither stable rank one (except in case III) nor real rank zero, the assertions on non-existence of ideals of low rank are justified.

Suppose π is an irreducible representation of B. In case II for each n $\pi_{|I^{(n)}}$ must vanish on all but one of the simple summands. Thus π is the pullback of an irreducible representation of B_I by one of the maps from B_{II} onto B_I mentioned above. Now if $\pi_{|I^{(n)}} \neq 0$ for all n, we see that π is faithful in cases I and III; and in case II the kernel of π is the kernel of the map from B_{II} onto B_I. Otherwise, choose the smallest value of n such that $\pi_{|I^{(n)}} = 0$. Then π vanishes on $B_{n+1}, B_{n+2}, \ldots$ but not on B_n. Thus π is determined by an irreducible representation of $B_n/I^{(n)}$ (recall that $(B_n + B_{n+1} + \ldots)^=$ is an ideal of B). This gives us the circle T_n, or the n'th level of the tree in case II.

LEMMA 2.13. *Let I be a closed ideal in a unital C^*-algebra A. For each u in $\mathscr{E}(A)$ and x in I such that $u + x \in (A_q^{-1})^=$, there is a sequence (a_n) in A_q^{-1} converging to $u + x$, such that $a_n - u \in I$ for all n. In particular, if $a_n = u_n|a_n|$ is the polar decomposition of a_n in A, then $u_n \in \mathscr{E}(A)$ and $u - u_n \in I$.*

Analogous statements hold if $u \in \mathscr{U}(A)$ and $u + x \in (A^{-1})^=$, and when $u \in \mathscr{U}(A)_{sa}$ (so u is a symmetry) and $u + x \in (A_{sa}^{-1})^=$ (so $x \in I_{sa}$).

Proof. By assumption there is a sequence (b_n) in A_q^{-1} converging to $u+x$. If $\pi\colon A \longrightarrow A/I$ denotes the quotient morphism we see that $\pi(b_n) \to \pi(u)$. We can therefore eventually write $\pi(b_n) = v_n e_n$, with v_n in $\mathscr{E}(A/I)$ and e_n in $(A/I)_+^{-1}$, such that $v_n \to \pi(u)$ and $e_n \to \mathbf{1}$. By [12, Theorem 2.1] there are sequences $\pi(w_n)$ and $\pi(w'_n)$ of unitaries in $\mathscr{U}_0(A/I)$ converging to $\mathbf{1}$, such that $\pi(w_n)v_n\pi(w'_n) = \pi(u)$; and by a standard lifting argument we may assume that (w_n) and (w'_n) are sequences in $\mathscr{U}_0(A)$ converging to $\mathbf{1}$. We may also choose a sequence (d_n) in A_+^{-1} converging to $\mathbf{1}$ such that $\pi(d_n) = e_n$. Replacing b_n with $a_n = w_n b_n d_n^{-1} w'_n$ we still have a_n in A_q^{-1} and $a_n \to u + x$, but now $\pi(a_n) = \pi(w_n)v_n e_n e_n^{-1}\pi(w'_n) = \pi(u)$. Therefore, if $a_n = u_n|a_n|$ is the polar decomposition in A, then $|a_n| - u^*u \in I$, whence $a_n - u_n \in I$, so $u - u_n \in I$, as claimed.

The analogous statements for the stable rank one or real rank zero situations are proved in exactly the same manner, using the well-known facts that two unitaries or two symmetries that are close are also homotopic.

Proposition 2.14. *Let I be an ideal in a unital C^*-algebra A, and let B be a unital C^*-subalgebra of A.*

(i) *If I and B are extremally rich and B is extreme-point-preservingly (e.p.p.) embedded in A, then $I + B$ is e.p.p. embedded in A. If moreover $I + B \subset (A_q^{-1})^=$ then $I + B$ is extremally rich.*

(ii) *If I and B have stable rank one, then $I + B$ has stable rank one.*

(iii) *If I and B have real rank zero, then $I + B$ has real rank zero.*

Proof. (i) Take v in $\mathscr{E}(I+B)$ with defect projections p_+ and p_-. If $(\pi, \mathscr{H})$ is an irreducible representation of A there are only two possibilities: For one, $I \subset \ker\pi$, in which case

$$\pi(v) \in \mathscr{E}(\pi(B)) = \pi(\mathscr{E}(B)) \subset \pi(\mathscr{E}(A)) \subset \mathscr{E}(\pi(A)),$$

since B is extremally rich and e.p.p. embedded in A. Thus $\pi(p_+Ap_-) = 0$. Otherwise $I \not\subset \ker\pi$, in which case $\pi(I)$ is strongly dense in $\mathsf{B}(\mathscr{H})$, and again

$$\pi(p_+Ap_-) \subset \pi(p_+)\pi(I)^{-s}\pi(p_-) \subset (\pi(p_+(I+B)p_-))^{-s} = 0.$$

Consequently $\pi(p_+Ap_-) = 0$ for all π, whence $p_+Ap_- = 0$, so that $v \in \mathscr{E}(A)$ as claimed.

Assume now that $I + B \subset (A_q^{-1})^=$, and consider the extension

$$(*) \qquad 0 \longrightarrow I \longrightarrow I + B \xrightarrow{\rho} (I+B)/I \longrightarrow 0.$$

Both I and $(I + B)/I$ (being isomorphic to a quotient of B) are extremally rich, and for every w in $\mathscr{E}((I+B)/I)$ there is a u in $\mathscr{E}(B)$ such that $\rho(u) = w$; so extreme partial isometries lift. To show that $I + B$ is extremally rich we need therefore, by condition (iv) in [10, Theorem 6.1], only check that $u + x \in ((I + B)_q^{-1})^=$ for every u in $\mathscr{E}(I + B)$ and x in I.

By the first part of the argument $u \in \mathscr{E}(A)$ and by assumption there is a sequence (a_n) in A_q^{-1} converging to $u + x$. This means that $a_n = u_n|a_n|$ with u_n in $\mathscr{E}(A)$ and zero an isolated point in $\mathrm{sp}(|a_n|)$. Using Lemma 2.13 we may assume that $u_n - u \in I$ and $|a_n| - u^*u \in I$. It follows that $|a_n| \in I + B$ and also $u_n \in \mathscr{E}(I + B)$, whence $a_n \in (I + B)_q^{-1}$ as desired.

(ii) This is proved in the same manner as above, but now we do not have to worry about extreme points being correctly embedded.

(iii) This is proved as above.

DEFINITIONS 2.15. Applying Theorem 2.1 to ideals instead of hereditary C^*-subalgebras we see, as in [36, Proposition 4.2], that Rørdam's ideal $I_{\mathrm{tsr}\,1}(A)$ may be characterized as the largest ideal I of A such that $\widetilde{I} \subset (\widetilde{A}^{-1})^=$. Similarly, $I_{\mathrm{RR}\,0}(A)$ is the largest ideal I of A for which $\widetilde{I}_{sa}$ is contained in $(\widetilde{A}_{sa}^{-1})^=$. An analogous characterization is not possible for ideals in the closure of the quasi-invertible elements. Looking for large ideals that are extremally rich inside a C^*-algebra A, we shall instead mimic Rørdam's construction (cf. 2.2)) and define

$$I_{\mathrm{er}}(A) = \{x \in A \mid x + \widetilde{A}_q^{-1} \subset (\widetilde{A}_q^{-1})^=\}.$$

THEOREM 2.16. *The set $I_{\mathrm{er}}(A)$ is an extremally rich ideal of A, and the largest ideal such that $I_{\mathrm{er}}(A) + B$ is extremally rich for every extremally rich C^*-subalgebra B of A such that $\widetilde{B}$ is is e.p.p embedded in $\widetilde{A}$. In particular $I_{\mathrm{tsr}\,1}(A) \subset I_{\mathrm{er}}(A)$. Moreover, with $\alpha_q(x) = \mathrm{dist}(x, \widetilde{A}_q^{-1})$ we have*

$$I_{\mathrm{er}}(A) = \{x \in A \mid \forall y \in \widetilde{A} : \alpha_q(x - y) = \alpha_q(y)\}.$$

PROOF. For the last assertion, take x in $I_{\mathrm{er}}(A)$ and y in $\widetilde{A}$. Then for each $\epsilon > 0$ there is a z in $\widetilde{A}_q^{-1}$ such that

$$\alpha_q(x - y) = \mathrm{dist}(x - y, \widetilde{A}_q^{-1}) \geq \|x - y + z\| - \epsilon \geq \alpha_q(y) - \epsilon,$$

since $x + z \in (\widetilde{A}_q^{-1})^=$. Thus $\alpha_q(x - y) \geq \alpha_q(y)$. Replacing y with $x - y$ we get $\alpha_q(y) \geq \alpha_q(x - y)$, and thus the desired equality. Conversely, if an element x in A satisfies this equality for every y, then by taking y in $\widetilde{A}_q^{-1}$ we get

$$\alpha_q(x + y) = \alpha_q(-y) = 0,$$

whence $x + y \in (\widetilde{A}_q^{-1})^=$; so that $x \in I_{\mathrm{er}}$.

From the two equivalent definitions of $I_{\mathrm{er}}(A)$ it is now clear that it is a closed, *-invariant subspace contained in $(\widetilde{A}_q^{-1})^=$. Since $y\widetilde{A}_q^{-1} = \widetilde{A}_q^{-1}$ if $y \in \widetilde{A}^{-1}$ we see that if furthermore $x \in I_{\mathrm{er}}(A)$, then

$$yx + \widetilde{A}_q^{-1} = y(x + \widetilde{A}_q^{-1}) \subset y(\widetilde{A}_q^{-1})^= = (\widetilde{A}_q^{-1})^=,$$

whence $yx \in I_{\mathrm{er}}(A)$. It follows that $\widetilde{A}^{-1}I_{\mathrm{er}}(A) = I_{\mathrm{er}}(A)$ (and $I_{\mathrm{er}}(A)\widetilde{A}^{-1} = I_{\mathrm{er}}(A)$ by *-invariance), from which we conclude that $I_{\mathrm{er}}(A)$ is a closed ideal of A.

Since $\widetilde{I}_{\mathrm{er}}(A) = I_{\mathrm{er}}(A) + \mathbf{C}\mathbf{1} \subset (\widetilde{A}_q^{-1})^=$ it follows from Theorem 2.1 that $I_{\mathrm{er}}(A)$ is extremally rich. Moreover, from the definition of $I_{\mathrm{er}}(A)$ we see that if B is any unital, extremally rich and e.p.p. embedded C^*-subalgebra of $\widetilde{A}$, then

$$I_{\mathrm{er}}(A) + B = I_{\mathrm{er}}(A) + (B_q^{-1})^= \subset I_{\mathrm{er}}(A) + (\widetilde{A}_q^{-1})^= \subset (\widetilde{A}_q^{-1})^=,$$

whence $I_{\mathrm{er}}(A) + B$ is extremally rich by Proposition 2.14.

Conversely, if I is an ideal of A that satisfies the conditions above, take v in $\mathscr{E}(\widetilde{A})$, and let B denote the unital C^*-subalgebra of $\widetilde{A}$ generated by v. Then B is isomorphic to the extended Toeplitz algebra $\mathscr{T}_e$ or one of its quotients, and therefore extremally rich and e.p.p. embedded in $\widetilde{A}$, cf. [10, Proposition 6.10]. By Proposition 2.14 the C^*-algebra $I + B$ is then also e.p.p. embedded in $\widetilde{A}$, and by assumption $I + B$ is extremally rich. Thus for every x in $\widetilde{A}^{-1}$ we have

$$I + vx = (I + v)x \subset ((I + B)_q^{-1})^= x \subset (\widetilde{A}_q^{-1})^= x = (\widetilde{A}_q^{-1})^=.$$

Since this holds for every v in $\mathscr{E}(\widetilde{A})$ and every x in $\widetilde{A}^{-1}$ it follows that

$$I + \widetilde{A}_q^{-1} \subset (\widetilde{A}_q^{-1})^=,$$

whence $I \subset I_{\mathrm{er}}(A)$.

The second sentence now follows from [10, 6.3].

EXAMPLE 2.17. The characterization of $I_{\mathrm{er}}(A)$ as the largest "well-behaved" extremally rich ideal in A cannot be improved, since in general there is no largest extremally rich ideal in a C^*-algebra A. A specific counterexample, already mentioned in [10, 6.12] and [12, 5.8], is available. Here $I_{\mathrm{er}}(A) = I_{\mathrm{tsr}\,1}(A)$; actually I_{er} is equal to the largest ideal $I_{1,0}(A)$ of stable rank one in A such that $\iota_1\colon K_1(I_{1,0}(A)) \to K_1(A)$ is the zero map, cf. [12, Proposition 5.3]. There are two ideals in A:

$$I_1 = \begin{pmatrix} A_1 & A_0 \\ A_0 & A_0 \end{pmatrix} \qquad \text{and} \qquad I_2 = \begin{pmatrix} A_0 & A_0 \\ A_0 & A_2 \end{pmatrix},$$

both extensions of the form

$$0 \longrightarrow I_{\mathrm{er}}(A) \longrightarrow I_i \longrightarrow C(\mathsf{T}) \longrightarrow 0,$$

and both extremally rich. However, $A = I_1 + I_2$ is not extremally rich.

The ideal $I_{\mathrm{er}}(A)$ can be used to reformulate one of our main results from [10], Theorem 6.1(iv).

COROLLARY 2.18. *Let I be an ideal in a C^*-algebra A. Then A is extremally rich if and only if the following conditions hold:*

(i) $I \subset I_{\mathrm{er}}(A)$, *i.e.* $I + \widetilde{A}_q^{-1} \subset (\widetilde{A}_q^{-1})^=$,

(ii) A/I *is extremally rich, i.e.* $\widetilde{A}/I = ((\widetilde{A}/I)_q^{-1})^=$,

(iii) *Quasi-invertibles lift, i.e.* $(\widetilde{A}/I)_q^{-1} = \widetilde{A}_q^{-1}/I$.

PROOF. The three conditions are evidently necessary. To prove sufficiency, take x in $\widetilde{A}$. Since $I + \widetilde{A}_q^{-1} \subset (\widetilde{A}_q^{-1})^=$ we get

$$\begin{aligned}\alpha_q(x) &= \inf \left\| x + \widetilde{A}_q^{-1} \right\| = \inf \left\| x + \widetilde{A}_q^{-1} + I \right\| = \inf \left\| (x + I) + \widetilde{A}_q^{-1}/I \right\| \\ &= \inf \left\| (x + I) + (\widetilde{A}/I)_q^{-1} \right\| = 0,\end{aligned}$$

as desired.

3. Defect ideals

DEFINITION 3.1. For each C^*-algebra A we define the *defect ideal* of A to be the ideal $\mathscr{D}(A)$ generated by all defect projections arising from elements in $\mathscr{E}(\widetilde{A})$. Evidently the possibilities $\mathscr{D}(A) = A$ and $\mathscr{D}(A) = 0$ are not excluded. Even though the definition of $\mathscr{D}(A)$ involves $\widetilde{A}$ the defect projections all belong to A. Thus $\mathscr{D}(\widetilde{A}) = \mathscr{D}(A) \subset A$. The term "defect ideal" was used in [12, Definition 5.9] to designate the possibly larger ideal of A obtained from $\mathscr{D}(A \otimes \mathsf{K})$, but we shall here prefer the non-stable version. For extremally rich C^*-algebras there is no difference, cf. 3.3.

We claim that $\mathscr{D}(A)$ has the following minimality characterizations:

PROPOSITION 3.2. *For every C^*-algebra A the defect ideal $\mathscr{D}(A)$ is the smallest ideal I such that*

$$\mathscr{E}(\widetilde{A})/I \subset \mathscr{U}(\widetilde{A}/I)\ .$$

If moreover A is extremally rich then $\mathscr{D}(A)$ is the smallest ideal such that $\mathrm{tsr}(A/\mathscr{D}(A)) = 1$.

PROOF. The first condition is evident from the definition of $\mathscr{D}(A)$. If now A is extremally rich

$$\mathscr{E}(\widetilde{A})/I = \mathscr{E}(\widetilde{A}/I)$$

for every ideal I of A by [10, Theorem 6.1]. The first condition therefore translates as

$$\mathscr{E}(\widetilde{A}/I) = \mathscr{U}(\widetilde{A}/I)\ ,$$

which by Proposition 2.7 is equivalent to $\mathrm{tsr}(\widetilde{A}/I) = 1$.

DISCUSSION 3.3. Note that when A is extremally rich we have

$$\mathscr{D}(A/I) = (\mathscr{D}(A) + I)/I = \mathscr{D}(A)/(\mathscr{D}(A) \cap I)$$

for every ideal I of A, because $\mathscr{E}(\widetilde{A}/I) = (\mathscr{E}(\widetilde{A}) + I)/I$ by [10, Theorem 6.1].

Observe also that in the presence of extremal richness the defect ideal is invariant under Rieffel-Morita equivalence. Thus whenever A and B are extremally rich C^*-algebras and $\sim_M$ denotes Rieffel-Morita equivalence $A \sim_M B$ implies that $\mathscr{D}(A) \sim_M \mathscr{D}(B)$ and $A/\mathscr{D}(A) \sim_M B/\mathscr{D}(B)$. In particular,

$$\mathscr{D}(A) \otimes \mathsf{K} = \mathscr{D}(A \otimes \mathsf{K}) \qquad \text{and} \qquad \mathscr{D}(B) = \mathscr{D}(A) \cap B$$

for every full, hereditary C^*-subalgebra B of A.

If I and J are ideals of a C^*-algebra A such that $\mathrm{tsr}(A/I) = \mathrm{tsr}(A/J) = 1$, then also $\mathrm{tsr}(A/(I \cap J)) = 1$. This is easy to verify since we may realize the quotient algebra as a surjective pullback,

$$A/(I \cap J) = A/I \oplus_{A/(I+J)} A/J\ ,$$

cf. [39, Proposition 3.16]. Consequently the set $\mathscr{I}$ of ideals I such that $\mathrm{tsr}(A/I) = 1$ is directed under reverse inclusion. In general $\mathscr{I}$ will not contain a minimal element (the closed unit disk, for example, does not have a maximal closed subset of dimension one), but when A is extremally rich a minimal ideal in $\mathscr{I}$ exists by Proposition 3.2, viz. the ideal $\mathscr{D}(A)$.

It may happen, of course, that $\mathscr{D}(A)$ has a non-zero quotient of stable rank one, so in the general case we obtain a descending chain of ideals $\{I_\alpha \mid 0 \leq \alpha \leq \beta\}$, indexed by a segment of the ordinals, such that

$$\begin{aligned}
&I_0 = A. \\
&I_{\alpha+1} = \mathscr{D}(I_\alpha) && \text{for each} \quad \alpha < \beta. \\
&I_\alpha = \bigcap_{\gamma<\alpha} I_\gamma && \text{if } \alpha \text{ is a limit ordinal.} \\
&\mathscr{D}(I_\beta) = I_\beta && \text{or} \quad I_\beta = 0.
\end{aligned}$$

DEFINITIONS 3.4. (i) We say that a C^*-algebra A has *no quotients of stable rank one* if $\operatorname{tsr}(A/I) > 1$ for every proper ideal I of A. In case A is extremally rich this is equivalent to the demand that $\mathscr{D}(A) = A$. In view of Proposition 3.2 this condition implies (for an extremally rich algebra) that every primitive quotient of A contains infinite projections. [In the non-unital case a little work is needed to prove this last statement. The ideas in the proof of Theorem 2.8 are relevant as is the theory of $I_{\operatorname{tsr} 1}(A)$.]

(ii) As in [12, 4.1] we shall allow an abbreviated notation for Murray-von Neumann equivalence of projections in matrix algebras over a C^*-algebra A: If p and q are projections in A we write $np \precsim mq$, if for some $k \geq \max(n, m)$ we have

$$\left(\oplus_{i=1}^{n} p\right) \oplus \left(\oplus_{i=1}^{k-n} 0\right) \precsim \left(\oplus_{i=1}^{m} q\right) \oplus \left(\oplus_{i=1}^{k-m} 0\right) \qquad \text{in} \quad \mathsf{M}_k(A).$$

LEMMA 3.5. *Let A be an extremally rich, unital C^*-algebra such that $\mathscr{D}(A) = A$. Then there is a positive integer m such that*

$$(n+m)\mathbf{1} \precsim m\mathbf{1}$$

for all n in N.

PROOF. Since the primitive ideal space of A is compact, we can find a finite set $\{u_j\}$ in $\mathscr{E}(A)$ such that A is generated as an ideal by the set of projections $\{p_j\}$, where $p_j = \mathbf{1} - u_j^* u_j$. Since A is linearly generated by its unitaries we can therefore find elements w_k in $\mathscr{U}(A)$, $1 \leq k \leq m$, such that

$$\mathbf{1} \leq \sum_{k=1}^{m} w_k p_k w_k^*,$$

with p_k in $\{p_j\}$, possibly with repetitions. But then $\mathbf{1} \precsim \bigoplus_{k=1}^{m} p_k$. For each n set $q_k = \mathbf{1} - (u_k^*)^{m+n} u_k^{m+n}$. Then

$$q_k = \sum_{i=0}^{m+n-1} u_k^{*i} p_k u_k^i,$$

so that $q_k \sim (m+n)p_k$ for every k. Consequently

$$(m+n)\mathbf{1} \precsim (m+n) \bigoplus_{k=1}^{m} p_k \sim \bigoplus_{k=1}^{m} q_k \leq m\mathbf{1}.$$

REMARK 3.6. Taking $n = m$ in the previous lemma shows that in $\mathsf{M}_m(A)$ (where now $2\mathbf{1}_m \precsim \mathbf{1}_m$) there is a pair of isometries with orthogonal ranges, which by definition means that $\mathsf{M}_m(A)$ is properly infinite. This prompts the question whether we can always take $m = 1$ in Lemma 3.5. Equivalently formulated: If an extremally rich, unital C^*-algebra A satisfies $\mathscr{D}(A) = A$ (i.e. has no quotients of stable rank one), does it follow that A is properly infinite (so that $2\mathbf{1} \precsim \mathbf{1}$)? We shall see in [14] that the answer is yes when A has weak cancellation; in particular when A is isometrically rich (extreme partial isometries are either isometries or co-isometries), cf. Theorem 4.7. In the general case we can at the moment only answer the question when stronger conditions of infinite behaviour are put on A.

Definition 3.7. Returning to the discussion of extremal richness as a substitute for stable rank one we wish to fix some notation relating to infinite projections. Following von Neumann algebra terminology we say that a projection in a C^*-algebra A is *properly infinite* if $2p \precsim p$, i.e. if pAp contains two isometries with orthogonal ranges.

Lemma 3.8. *In an extremally rich C^*-algebra A the following conditions are equivalent:*

(i) *Every non-zero projection in A is properly infinite;*

(ii) *$\mathscr{D}(I) = I$ for every ideal I of A that contains a full projection;*

(iii) *$\mathscr{D}(pAp) = pAp$ for every projection p in A.*

Proof. (i)$\Rightarrow$(ii) If p is a full projection in I and properly infinite, then $p \precsim q$ for some defect projection q, whence $\mathscr{D}(I) = I$.

(ii)$\Rightarrow$(iii) If I denotes the ideal generated by p then as in 3.3

$$\mathscr{D}(pAp) = pAp \cap \mathscr{D}(I) = pAp \cap I = pAp.$$

(iii)$\Rightarrow$(i) Let p be a non-zero projection in A. Since $\mathscr{D}(pAp) = pAp$ there is then a finite set $\{v_1, v_2, \dots, v_n\}$ in $\mathscr{E}(pAp)$ such that pAp is ideally generated by the projections $p_i = p - v_i^* v_i$, $1 \le i \le n$. Let I_i denote the ideal of pAp generated by p_i. We shall now recursively construct an increasing sequence of projections $q_0, q_1, \dots, q_n$ in pAp, with $q_0 = 0$, such that q_i generates the ideal $J_i = I_1 + \cdots + I_i$ for $i > 0$, and such that $2q_i \precsim q_i$ for every i.

Assume that q_i has been constructed, and put $B = (p-q_i)A(p-q_i)$. Since

$$B/B \cap J_i = (B + J_i)/J_i = pAp/J_i,$$

the extreme point $v_{i+1} + J_i$ in pAp/J_i can be lifted to v in $\mathscr{E}(B)$. Taking $e_i = p - q_i - v^*v$ we see that $e_i + J_i = p_{i+1} + J_i$, so that $e_i + q_i$ generates J_{i+1} as an ideal. Since $e_i \le p - q_i$ we have

$$\mathscr{D}(e_i B e_i) = \mathscr{D}(e_i A e_i) = e_i A e_i = e_i B e_i.$$

By Lemma 3.5 there is therefore an m such that $2me_i \precsim me_i$. Let

$$e = p - q_i - v^{*m}v^m = \sum_{k=0}^{m-1} v^{*k}e_iv^k,$$

so that $e \sim me_i$, and let $q_{i+1} = q_i + e$. Since $2q_i \precsim q_i$ by assumption and $2e \precsim e$ by construction we have the desired conclusion $2q_{i+1} \precsim q_{i+1}$; and evidently the ideal generated by q_{i+1} is J_{i+1} since $e_i + q_i \leqq q_{i+1}$.

In the end we find a projection q_n that generates pAp (= J_n) as an ideal, so for some k we have $p \precsim kq_n$. Therefore

$$2p \precsim 2kq_n \precsim q_n \leq p.$$

THEOREM 3.9. *In an extremally rich C^*-algebra A the following conditions are equivalent:*

(i) *Every non-zero hereditary C^*-subalgebra of A is generated as an ideal by its properly infinite projections;*

(i′) *Every non-zero hereditary C^*-subalgebra of A is generated hereditarily by its properly infinite projections.*

(ii) $\mathscr{D}(I) = I$ *for every ideal I of A;*

(iii) $\mathrm{tsr}(I/J) > 1$ *for every pair of distinct ideals of A with $J \subset I$;*

(iv) *No non-zero ideal of A has quotients of stable rank one.*

(v) *No non-zero quotient of A has ideals of stable rank one.*

Under these conditions every non-zero projection in A is properly infinite.

PROOF. (i) ⇒ (v) If the first condition is satisfied for A, then it also holds for any non-zero quotient of A. No non-zero quotient of A can therefore contain any non-zero ideal of stable rank one, because each ideal (indeed, each hereditary C^*-subalgebra) contains a properly infinite projection.

(v) ⇔ (iii) ⇔ (ii) ⇔ (iv) The first equivalence is obvious, and the next two follow by applying Proposition 3.2 to arbitrary ideals of A.

(ii) ⇒ (i) By Lemma 3.8 every (non-zero) projection in A is properly infinite. If now B is a non-zero hereditary C^*-subalgebra of A, let I denote the ideal of A generated by B. By assumption we have $\mathscr{D}(J) = J$ for every ideal J of I. Using Rieffel-Morita equivalence the same is therefore true for B. In particular, B is generated as an ideal by its projections.

(i) ⇔ (i′) If a hereditary C^*-subalgebra is invariant under inner automorphisms, it must be an ideal by [30, 5.2.1].

DISCUSSION 3.10. (i) Let us agree to call an arbitrary C^*-algebra A *purely properly infinite* if it satisfies condition (i) of Theorem 3.9. Even if A is not extremally rich it is still true that every non-zero projection p of such an algebra is properly infinite. To see this, apply (i) to $B = pAp$. If q is a properly infinite projection in B then $v^*v = p$ and $q \precsim p - vv^*$ for some v in B. Take now $q_1, \ldots, q_n$ and $v_1, \ldots, v_n$ as above so that B is ideally generated by the q_i's. Then with $v = v_1 \cdots v_n$, the defect projection $q = p - vv^*$ generates B as an ideal, and $2p \precsim p - v^m v^{*m}$ for m sufficiently large. It follows from [34, Proposition 5.6] that $\text{tsr}(pAp) = \infty$ for every non-zero projection p, and thus $\text{tsr}(I) > 1$ for every non-zero ideal of A. Also it is easy to see that hereditary subalgebras, ideals and quotients of a purely properly infinite C^*-algebra A are again purely properly infinite. In particular, any simple quotient of such an algebra will be purely infinite. Finally we note that any C^*-algebra which is Rieffel-Morita equivalent to A is again purely properly infinite. (This follows from the criterion for Rieffel-Morita equivalence in terms of linking algebras, [8], and the fact, Cuntz [16], that the hereditary C^*-subalgebras generated by x^*x and xx^* are isomorphic.)

(ii) There are several equivalent ways to define purely infinite for simple C^*-algebras that do not lead to equivalent concepts in the general case. The present definition is one of them. Another, less fortunate, as we shall see, is to apply the exact words of Cuntz's definition in [16] to any non-simple C^*-algebra A. Under this definition A is "purely infinite" if every non-zero hereditary C^*-subalgebra contains an infinite projection. The algebra B_I constructed in 2.12 is extremally rich and "purely infinite" in this sense, by Theorem 2.8 and the fact that B_I is primitive. However, B_I has a quotient isomorphic to $C(\mathsf{S}^1)$. Thus, although it is infinite, it does not satisfy any intuitive notion of being *purely* infinite. Note also that $\bigcap \mathscr{D}^n(B_I) = \{0\}$, so that B_I contains no properly infinite projections.

Taking this phenomenon further we observe that any C^*-algebra A would be "purely infinite" provided only that it contained an essential ideal I which was "purely infinite". Here A/I could be finite in any conceivable sense.

(iii) Kirchberg and Rørdam, [23], [24], have recently given definitions of purely infinite – in many cases equivalent to absorption of $\mathcal{O}_\infty$ under tensoring – which extend the concept from the simple case and do not have the disadvantages of the "purely infinite" concept mentioned above. It can be shown that the main Kirchberg-Rørdam concept is equivalent to our purely proper infinite for C^*-algebras having "enough" projections. What is required is that every hereditary C^*-subalgebra of A is generated as an ideal by its projections. This is almost a standard abundance-of-projections concept. The *ideal property* used in [28] and [40] requires only that every ideal of A should be (ideally) generated by projections, so the condition we want is that every hereditary C^*-subalgebra of A should have the ideal property. Rørdam [38, Theorem 3.2] constructs an AH-algebra of stable rank one which is purely infinite in the Kirchberg-Rørdam sense, but (necessarily) contains no non-trivial projections. We are grateful to G. A. Elliott and M. Rørdam for discussions related to these remarks.

EXAMPLE 3.11. There exists an extremally rich, unital C^*-algebra A which is not purely properly infinite but satisfies the conditions of Lemma 3.8, and every non-zero hereditary C^*-subalgebra of A contains a non-zero projection. (Note that these conditions lead to a fourth concept of purely infinite extending the simple concept: Every non-zero hereditary C^*-subalgebra contains a non-zero projection and all non-zero projections are properly infinite.) The algebra A has a composition series $\{\{0\}, I, J, A\}$ such that I is purely infinite simple and essential, $J/I = C_0(]0, 1]) \otimes \mathsf{K}$ and A/J is again purely infinite and simple. The Jacobson topology of the primitive ideal space $A^\vee$ of A is such that the primitive ideal J is contained in the closure of each point of $]0, 1]$ (identified with $(J/I)^\vee$). It follows that $B \cap I \neq 0$ whenever B is a non-zero hereditary C^*-subalgebra of A, whence B contains non-zero projections. Since J/I has no non-zero projections, any projection p in A not belonging to I is not in J. From the topology of $A^\vee$ it then follows that p generates A ideally. Also, $\mathscr{D}(A) = A$ since A has no quotients of stable rank one. Thus A satisfies condition (ii) in Lemma 3.8.

To construct A take any non-unital, purely infinite simple C^*-algebra I_0 and let J_0 be a split essential extension of I_0 by $C_0(]0, 1])$. Thus the Busby invariant is given by an injective homomorphism $\tau\colon C_0(]0, 1]) \longrightarrow M(I_0)/I_0$, i.e. by choosing a positive element in $M(I_0)/I_0$ with spectrum equal to $[0, 1]$. Tensoring this extension with K we obtain a short exact sequence:

$$0 \longrightarrow I \longrightarrow J \longrightarrow C_0(]0, 1]) \otimes \mathsf{K} \longrightarrow 0.$$

Now take any unital, purely infinite simple C^*-algebra I_1 and define A to be a trivial homogeneous extension of J by I_1 in the sense of Pimsmer-Popa-Voiculescu, [33]. The Busby invariant now comes from a homomorphism

$$\sigma\colon I_1 \longrightarrow \mathbf{1} \otimes \mathsf{B}(\mathscr{H}) \subset M(J_0 \otimes \mathsf{K}),$$

where σ is unital and faithful modulo $\mathbf{1} \otimes \mathsf{K}$.

The facts that I_0 is extremally rich and $C_0(]0, 1])$ has stable rank one and is projective easily imply that J_0 is extremally rich. Consequently also J is extremally rich. To complete the proof that A is extremally rich we need only show that pAq is an extremally rich bimodule when q is a minor defect projection and p a defect projection of an isometry in $\mathbf{1} \otimes \mathsf{B}(\mathscr{H})$, cf. [10, Theorem 6.1]. Thus $p = \mathbf{1} \otimes p_0$, where p_0 has infinite rank. We may assume that $q = q_0 \otimes e_{11}$ for some defect projection q_0 in I_0, since in any case q is a non-zero projection in I whose K_0−class is zero in $K_0(I)$, and all such projections are equivalent by [17]. Thus $pAq = pIq \cong I_0q \otimes \ell^2$, which is extremally rich.

4. Isometrically rich C^*-algebras

4.1. Overview

A unital C^*-algebra A is called *isometrically rich* if the set $A_\ell^{-1} \cup A_r^{-1}$ of left or right invertible elements is dense in A. As for extremal richness we circumvent the non-unital case by declaring A to be isometrically rich if $\widetilde{A}$ is isometrically rich.

A whole theory parallel to that of extreme richness could be developed for isometrically rich C^*-algebras. We leave it to the interested reader to check that the main results in sections 3–6 of [10] remain true when $A_\ell^{-1} \cup A_r^{-1}$ and the set $\mathscr{E}_i(A) \cup \mathscr{E}_i(A)^*$ of isometries and co-isometries are substituted for A_q^{-1} and $\mathscr{E}(A)$. Actually some statements (and many proofs) become simpler: Isometries and co-isometries are preserved under unital embeddings, so that [10, Proposition 5.2] now takes the simple form that any unital, inductive limit of isometrically rich C^*-algebras is isometrically rich. But inspection of [10, Example 6.12] reveals that the extension theory does not simplify. We warn the reader that although Proposition 4.3 involves isometric richness for the quotient, it only guarantees extremal richness for the extension. The extended Toeplitz algebra (cf. [31, Propositions 9.3–9.5] and [10, Proposition 6.10]) is a specific (counter) example because it is *not* isometrically rich (having no proper isometries or co-isometries).

Of course, when the C^*-algebra A is prime, then $A_q^{-1} = A_\ell^{-1} \cup A_r^{-1}$ and $\mathscr{E}(A) = \mathscr{E}_i(A) \cup \mathscr{E}_i(A)^*$, so that the notions of extremal and isometric richness coalesce. This easier case was considered in [31, §8], and the concept of isometric richness is also implicit in Rørdam's paper [36, 3.3].

PROPOSITION 4.2. *A unital C^*-algebra A is isometrically rich if and only if A is extremally rich and*

$$\mathscr{E}(A) = \mathscr{E}_i(A) \cup \mathscr{E}_i(A)^* .$$

PROOF. It suffices to assume $A = (A_l^{-1})^= \cup (A_r^{-1})^=$ and then to show that $\mathscr{E}(A) = \mathscr{E}_i(A) \cup \mathscr{E}_i(A)^*$. For this, take v in $\mathscr{E}(A)$ and assume that $v \in (A_l^{-1})^=$. By [31, Corollary 7.2] this means that for any continuous function f on R_+ vanishing in a neighbourhood of zero there is an isometry w in $\mathscr{E}_i(A)$ such that $vf(|v|) = wf(|v|)$. Since v is a partial isometry $f(|v|) = |v|$, provided only that $f(0) = 0$ and $f(1) = 1$, so we actually have $v = wv^*v$. Multiplying from left and right with w^* and v^*, respectively, this gives $w^*vv^* = v^*$, so that also $v = vv^*w$. However, $v \in \mathscr{E}(A)$ and thus

$$0 = (\mathbf{1} - vv^*)w(\mathbf{1} - v^*v) = (w - v)(\mathbf{1} - v^*v) = w - v,$$

whence $v = w \in \mathscr{E}_i(A)$ as desired.

PROPOSITION 4.3. *Suppose that*

$$0 \longrightarrow \mathsf{K} \longrightarrow A \xrightarrow{\rho} B \longrightarrow 0$$

is a short exact sequence, where K *as usual denotes the algebra of compact operators. If B is isometrically rich then A is extremally rich.*

PROOF. Since $\mathsf{B}(\mathscr{H})$ is the multiplier algebra of K with corona algebra $Q = \mathsf{B}(\mathscr{H})/\mathsf{K}$ we obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathsf{K} & \longrightarrow & A & \xrightarrow{\rho} & B & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle\eta} & & \downarrow{\scriptstyle\tau} & & \\ 0 & \longrightarrow & \mathsf{K} & \longrightarrow & \mathsf{B}(\mathscr{H}) & \xrightarrow{\pi} & Q & \longrightarrow & 0 \end{array}$$

in which the right square is a pullback diagram, cf. [32, Remark 3.2].

Assuming, as we may, that A and B and all the morphisms are unital we take u in $\mathscr{E}(B)$. Since B is isometrically rich, u is either an isometry or a co-isometry. Assuming the former we see that $\tau(u)$ is an isometry in Q. Now by standard techniques we can choose w in $B(H)$ such that $\pi(w) = \tau(u)$ and w is an isometry or co-isometry. The co-isometry case occurs only if $\tau(u)$ is a unitary with positive index. Since $A = B \oplus_Q \mathsf{B}(\mathscr{H})$ it follows that the element $v = (u, w)$ is in $\mathscr{E}(A)$. (In the case where w is not an isometry, the right defect projection of v is in K and the left defect projection is in the kernel of η.)

This shows that $\rho(A_q^{-1}) = B_q^{-1}$, and since B is extremally rich, whereas K has stable rank one we conclude from [10, Corollary 6.3] that A is extremally rich.

The pullback method used above is capable of considerable generalization, cf. [13, Section 4]. In particular K may be replaced with any dual C^*-algebra.

EXAMPLE 4.4. If B is a unital C^*-algebra on a (separable) Hilbert space $\mathscr{H}$ such that $\mathscr{E}(B)$ contains an extreme partial isometry, neither of whose defect projections is compact, then the C^*-algebra $A = \mathsf{K} + B$ is not extremally rich. If it were, then

$$\mathscr{E}(A)/\mathsf{K} = \mathscr{E}(A/\mathsf{K})$$

by [10, Theorem 6.1]. But elements in $\mathscr{E}(A)$ are either isometries or co-isometries because A is prime; whereas $\mathscr{E}(A/\mathsf{K})$ contains an element which is neither isometric nor co-isometric. This elementary (counter-)example shows the necessity of demanding B isometrically rich (not just extremally rich) in Proposition 4.3.

The following proposition offers one way of reducing questions about isometric richness to extremal richness, cf. 4.1.

PROPOSITION 4.5. *A C^*-algebra A is isometrically rich if and only if A is a quotient of an extremally rich, primitive C^*-algebra B.*

PROOF. If $A = B/I$ as stated, then A is isometrically rich because B is, and obviously isometric richness passes to quotients.

Conversely, if A is isometrically rich let $\rho\colon A \to \mathsf{B}(\mathscr{H})$ be a faithful representation such that

$$\rho(A) \cap \mathsf{K} = \{0\},$$

where K again denotes the algebra of compact operators on $\mathscr{H}$. One such can be obtained by choosing a faithful representation $(\sigma, \mathscr{H}_0)$, and then letting ρ be the infinite amplification of σ on $\mathscr{H} = \mathscr{H}_0 \otimes \ell^2$. If $B = \rho(A) + \mathsf{K}$ then Proposition 4.3 applies to show that B is extremally rich, and obviously B is primitive.

LEMMA 4.6. *Let A be an isometrically rich C^*-algebra. Then for each finite set $\{p_i\}$ of defect projections in A there is some defect projection p, such that*

$$\bigoplus p_i \sim p.$$

PROOF. We have isometries u_i such that $p_i = \mathbf{1} - u_i u_i^*$ for $i = 1, \dots, n$. Define the isometry $u = u_1 u_2 \dots u_n$. Then with $v_i = u_1 u_2 \dots u_i$ (and $v_0 = \mathbf{1}$) we have

$$\mathbf{1} - uu^* = \sum_{i=1}^{n} v_{i-1}(\mathbf{1} - u_i u_i^*) v_{i-1}^* \sim \bigoplus_{i=1}^{n} p_i,$$

so we can take $p = \mathbf{1} - uu^*$.

THEOREM 4.7. *Every isometrically rich, unital C^*-algebra A with no quotients of stable rank one is properly infinite. In particular,* $\operatorname{tsr}(A) = \infty$.

PROOF. As in the proof of Lemma 3.5 we can find defect projections $p_1, \dots, p_m$ such that $\mathbf{1} \precsim \bigoplus p_k$. By Lemma 4.6 there is a defect projection p such that $p \sim \bigoplus 2p_k$. Consequently,

$$2\mathbf{1} \precsim \bigoplus 2p_k \sim p \leq \mathbf{1}\ .$$

COROLLARY 4.8. *Every primitive quotient of an extremally rich, unital C^*-algebra A with no quotients of stable rank one is properly infinite. In particular,* $\mathrm{tsr}(A) = \infty$.

4.9. Hindsight

Concluding our discussion of isometrically rich C^*-algebras we wish to point out a detracting element. Being extremally rich for a unital C^*-algebra A with closed unit ball A_1 can be expressed by the condition

$$A_1 = \mathrm{conv}(\mathscr{E}(A)),$$

or by demanding that A has the (uniform) $\lambda-$property, cf. [11, Theorem 3.7]. Any isometric linear map between unital C^*-algebras will therefore preserve extremal richness. By contrast, isometric richness (density of $A_\ell^{-1} \cup A_r^{-1}$) is not an isometric invariant, not even invariant under Jordan *-isomorphisms.

To be specific let $j\colon \mathscr{T} \to \mathscr{T}$ denote the Jordan *-automorphism of the Toeplitz algebra $\mathscr{T}$ obtained by transposition. In particular, $j(s) = s^*$, where s denotes the unilateral shift on ℓ^2 (so that $\mathscr{T} = C^*\langle s\rangle$). If $\pi\colon \mathscr{T} \longrightarrow C(\mathsf{T})$ denotes the quotient morphism of $\mathscr{T}$ with kernel K and θ is the *-automorphism of $C(\mathsf{T})$ given by $\theta(f)(t) = f(t^{-1})$ we see that j is a lift of θ, i.e. $\pi(j(x)) = \theta(\pi(x))$.

We realize the extended Toeplitz algebra as a pullback, cf. [10, 6.10],

$$\mathscr{T}_e = \{(x, y) \in \mathscr{T} \oplus \mathscr{T} \mid \pi(x) = \theta(\pi(y))\}$$

and obtain a Jordan *-isomorphism k of $\mathscr{T}_e$ onto the *double Toeplitz algebra*

$$\mathscr{T}_d = \{(x, y) \in \mathscr{T} \oplus \mathscr{T} \mid \pi(x) = \pi(y)\}$$

by the formula $k(x, y) = (x, j(y))$. Since $\mathscr{T}_d$ is an extension of $\mathsf{K} \oplus \mathsf{K}$ by $C(\mathsf{T})$ it follows from simple index considerations that $\mathscr{T}_d$ is isometrically rich. By contrast, $\mathscr{T}_e$ is our canonical example of an extremally rich, but *not* isometrically rich C^*-algebra.

4.10. AW^*-Algebras

An *AW^*-algebra* is a (necessarily unital) C^*-algebra A such that for each subset $S \subset A$ the (left and right) ideals of left and right annihilators of S, denoted by $^\perp S$ and $S^\perp$, respectively, are principal. Thus for some (unique) projections p and q in A we have

$$^\perp S = Ap \qquad \text{and} \qquad S^\perp = qA.$$

AW^*-algebras and their purely algebraic counterparts *Baer *-rings* were introduced by Kaplansky, see [22, pp. 71–86], in order to axiomatize the intrinsic (i.e. non-spatial) theory of von Neumann algebras. The program was quite successful and led to substantial simplification and better insight towards the original material. (Despite the line in a contemporary Chicago student revue: "We're at sea; Capt'n Lansky has lost his bearings!") The monograph [5] remains the standard reference for AW^*-algebras.

Every element in an AW^*-algebra A has a polar decomposition, [5, §21, Proposition 2]. Moreover, since the projections in A enjoy "generalized comparability", every partial isometry extends to an extremal partial isometry, [5, §14, Exercise 19A]. Using the formula

$$z = \tfrac{1}{2}(z + i(\mathbf{1} - z^2)^{\frac{1}{2}}) + \tfrac{1}{2}(z - i(\mathbf{1} - z^2)^{\frac{1}{2}})$$

for $0 \le z \le \mathbf{1}$, it follows that every element x in the closed unit ball of A can be written as $x = \frac{1}{2}(v + w)$ with v, w in $\mathscr{E}(A)$, cf. [5, §21, Exercise 9A]. The elements v and w can even be taken homotopic to each other. Thus AW^*-algebras are extremally rich in the same strong way as von Neumann algebras are. In particular, they have the uniform λ-property with $\lambda(x) \ge \frac{1}{2}$ for every x, cf. [11, Theorem 3.7].

4.11. Rickart Algebras

A *Rickart C^*-algebra* is a (unital) C^*-algebra A such that the AW^*-condition holds only for singleton sets. Thus, for each element x in A there are (unique) projections p and q in A such that

$$^{\perp}\{x\} = Ap \qquad \text{and} \qquad \{x\}^{\perp} = qA.$$

These algebras are sequential analogues of Kaplansky's AW^*-algebras, in the sense that only countably many projections can be added at a time, see [5] or [22]; but whereas AW^*-algebras are extremally rich, this is not true for all Rickart C^*-algebras, cf. Proposition 4.12. However, many of them certainly are, and in any case their von Neumann algebraic tendencies are so strong that they will satisfy most of the properties expected for extremally rich C^*-algebras of real rank zero.

In [1, §1] Ara introduced the ideal $\mathscr{I}(A)$ of a Rickart C^*-algebra A as the closed ideal generated by what we might call the "infinitesimal" projections in A, i.e. projections p such that $p \oplus \mathbf{1} \precsim 0 \oplus \mathbf{1}$ in $\mathsf{M}_2(A)$. He proceeded to show that $\mathscr{I}(A)$ is a Rickart ideal, and the smallest closed ideal such that $A/\mathscr{I}(A)$ is finite, [1, Theorem 1.5]. The ideal was then used in [1] and the subsequent papers [2], [3], [4] to establish a number of interesting results for Rickart C^*-algebras by dividing the problem into the finite case (where techniques from von Neumann regular rings apply) and the infinite case (where most of the obstructions for equivalence vanish). Of particular interest in this context is [2, Theorem 3.5] which states that every quotient C^*-algebra of a Rickart C^*-algebra has K_1-surjectivity, cf. [14]. We are indebted to Ara and Goodearl for this information as well as for the idea for the example in 4.12. Examples like this appear in [19, 14.35] and in Section IV of [20].

Evidently our defect ideal must equal Ara's ideal $\mathscr{I}(A)$ for every extremally rich Rickart C^*-algebra A, cf. Proposition 3.2. However, easy computations with operator-valued 2×2-matrices show that every infinitesimal projection is sub-equivalent to a defect projection arising from an isometry. Thus, for a general (unital) C^*-algebra A, the ideal $\mathscr{I}(A)$ is generated by defect projections from the isometries in A, whence $\mathscr{I}(A) \subset \mathscr{D}(A)$. This inclusion may be strict. For example, if we take the extended Toeplitz algebra $\mathscr{T}_e$, which has no non-unitary isometries by [31, Proposition 9.4], then

$$\mathscr{I}(\mathscr{T}_e) = 0 \qquad \text{whereas} \qquad \mathscr{D}(\mathscr{T}_e) = \mathsf{K} \oplus \mathsf{K}.$$

On the other hand, if the C^*-algebra A is isometrically rich, then $\mathscr{I}(A) = \mathscr{D}(A)$, and all the defect projections are infinitesimal. To some extent this may explain why the theory of isometrically rich C^*-algebras is easier to handle than the general extremally rich case.

PROPOSITION 4.12. *There exists a primitive Rickart C^*-algebra which is not extremally rich.*

PROOF. Let $\mathscr{H}$ and $\mathscr{H}_\omega$ denote a separable and a nonseparable Hilbert space, respectively. With $B = \mathsf{B}(\mathscr{H}) \oplus \mathsf{B}(\mathscr{H})$ we consider a unital embedding $\rho\colon B \to \mathsf{B}(\mathscr{H}_\omega)$ such that $\rho(B) \cap \mathsf{B}_1(\mathscr{H}_\omega) = 0$, where $\mathsf{B}_1(\mathscr{H}_\omega)$ denotes the Rickart ideal of operators on $\mathscr{H}_\omega$ with separable ranges. Specifically, we take $\rho(B) = \mathbf{1} \otimes B$ referring to a decomposition of $\mathscr{H}_\omega$ as $\mathscr{H}_\omega \otimes (\mathscr{H} \oplus \mathscr{H})$.

Let $A = \rho(B) + \mathsf{B}_1(\mathscr{H}_\omega)$. Then it is easy to verify that A is a Rickart C^*-algebra, evidently primitive. If it were extremally rich then

$$\mathscr{E}(A)/\mathsf{B}_1(\mathscr{H}_\omega) = \mathscr{E}(B)$$

by [10, Theorem 6.1]. However, each extreme partial isometry of the prime C^*-algebra A must be either an isometry or a co-isometry, contradicting the fact that $\mathscr{E}(B)$ contains the element $s \oplus s^*$, where s denotes the unilateral shift on $\mathscr{H}$.

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