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MULTIPLICATIVE PROPERTIES OF POSITIVE MAPS

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(Dedicated to the memory of Gert K. Pedersen)

Abstract

Let ϕ be a positive unital normal map of a von Neumann algebra M into itself. It is shown that with some faithfulness assumptions on ϕ there exists a largest Jordan subalgebra C_ϕ of M such that the restriction of ϕ to C_ϕ is a Jordan automorphism and each weak limit point of $(\phi^n(a))$ for $a \in M$ belongs to C_ϕ.

1. Introduction

In the study of positive linear maps of C^*-algebras the multiplicative properties of such maps have been studied by several authors, see e.g. [9], [2], [3], [4], [6]. If $\phi: A \to B$ is a positive unital map between C^*-algebras A and B an application of Kadison's Schwarz inequality, [8] to the operators $a + a^*$ and $i(a - a^*)$ yields the inequality [10]

$$\phi(a \circ a^*) \geq \phi(a) \circ \phi(a)^*, \qquad a \in A, \tag{1}$$

where $a \circ b = \frac{1}{2}(ab + ba)$ is the Jordan product. Thus one obtains an operator valued sesquilinear form

$$\langle a, b \rangle = \phi(a \circ b^*) - \phi(a) \circ \phi(b)^*, \qquad a, b \in A. \tag{2}$$

If we apply the Cauchy-Schwarz inequality to $\omega(\langle a, b \rangle)$ for all states ω of B it was noticed in [6] that if $\phi(a \circ a^*) = \phi(a) \circ \phi(a)^*$ then $\langle a, b \rangle = 0$ for all $b \in A$. We call the set

$$A_\phi = \{a \in A : \phi(a \circ a^*) = \phi(a) \circ \phi(a)^*\}$$

the *definite set* of ϕ. It is a Jordan subalgebra of A, and if $a \in A_\phi$ then $\phi(a \circ b) = \phi(a) \circ \phi(b)$ for all $b \in A$.

In the present paper we shall develop the theory further. We first study positive unital normal, i.e. ultra weakly continuous, maps $\phi: M \to M$, where

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M is a von Neumann algebra. We mainly study properties of the definite set M_ϕ and some of its Jordan subalgebras of M plus convergence properties of the orbits $(\phi^n(a))$ for $a \in M$. We shall show that when there exists a faithful family $\mathscr{F}$ of ϕ-invariant normal states there is a largest Jordan subalgebra C_ϕ of M called the multiplicative core of M, on which ϕ acts as a Jordan automorphism. Furthermore if $a \in M$ then every weak limit point of the orbit $(\phi^n(a))$ lies in C_ϕ, and if $\rho(a \circ b) = 0$ for all $b \in C_\phi$, then $\phi^n(a) \to 0$ weakly.

Much of the above work was inspired by a theorem of Arveson, [1]. In the last section we study the C^*-algebra case and the relation of our discussion with Arveson's work. Then $\phi\colon A \to A$ is a positive unital map, and we assume the orbits $(\phi^n(a))$ with $a \in A$ are norm relatively compact and that there exists a faithful family $\mathscr{F}$ of ϕ-invariant states. It is then shown that the multiplicative core C_ϕ of ϕ equals the set of main interest in [1], namely the norm closure of the linear span of all eigenoperators $a \in A$ with $\phi(a) = \lambda a$, $|\lambda| = 1$, and that $\lim_{n\to\infty} \|\phi^n(a)\| = 0$ if and only if $\rho(a \circ b) = 0$ for all $b \in C_\phi$ and $\rho \in \mathscr{F}$.

2. Maps on von Neumann algebras

Throughout this section M denotes a von Neumann algebra, $\phi\colon M \to M$ is a positive normal unital map. M_ϕ denotes the definite set of ϕ and $\langle , \rangle$ the operator valued sesquilinear form $\langle a, b\rangle = \phi(a \circ b^*) - \phi(a) \circ \phi(b)^*$, $a, b \in A$.

LEMMA 2.1. *Let assumptions be as above, and suppose (a_α) is a bounded net in M which converges weakly to $a \in M$. If $\langle a_\alpha, a_\alpha\rangle \to 0$ weakly, then $a \in M_\phi$, and $\phi(a \circ b) = \phi(a) \circ \phi(b)$ for all $b \in M$.*

PROOF. Let ω be a normal state on M. By the Cauchy-Schwarz inequality, if $b, c \in M$ we have

$$|\omega(\langle b, c\rangle)|^2 \le \omega(\langle b, b\rangle)\omega(\langle c, c\rangle).$$

By assumption, if a_α and a are as in the statement of the lemma, and $b \in M$ then

$$\begin{aligned} |\omega(\langle a, b\rangle)|^2 &= \lim_\alpha |\, \omega(\langle a_\alpha, b\rangle)|^2 \\ &\le \lim_\alpha \omega(\langle a_\alpha, a_\alpha\rangle)\omega(\langle b, b\rangle) = 0. \end{aligned}$$

Since this holds for all normal states ω, $\langle a, b\rangle = 0$, completing the proof.

In analogy with the definition of G-finite for automorphism groups we introduce

DEFINITION 2.2. With ϕ as above we say M is *ϕ-finite* if there exists a faithful family $\mathscr{F}$ of ϕ-invariant normal states on the von Neumann algebra generated by the image $\phi(M)$.

LEMMA 2.3. *Assume M is ϕ-finite. Then for $a \in M$ we have*

(i) *Every weak limit point of the orbit $(\phi^n(a))$ of a belongs to M_ϕ.*

(ii) *If $\rho(\phi^n(a) \circ b) = 0$ for all $b \in M_\phi$, $\rho \in \mathscr{F}$, then $\phi^n(a) \to 0$ weakly.*

PROOF. If $\rho \in \mathscr{F}$ denote by $\|.\|_\rho$ the seminorm $\|x\|_\rho = \rho(x \circ x^*)^{\frac{1}{2}}$. Then by the inequality (1)

$$\begin{aligned}\|\phi^{n+1}(a)\|_\rho^2 &= \rho(\phi^{n+1}(a) \circ \phi^{n+1}(a)^*)\\ &\leq \rho(\phi(\phi^n(a) \circ \phi^n(a)^*))\\ &= \|\phi^n(a)\|_\rho^2.\end{aligned}$$

Thus the sequence $\|\phi^n(a)\|_\rho^2$ is decreasing, hence $\|\phi^n(a)\|_\rho^2 - \|\phi^{n+1}(a)\|_\rho^2 \to 0$. We have

$$\begin{aligned}\rho(\langle\phi^n(a), \phi^n(a)\rangle) &= \rho(\phi(\phi^n(a) \circ \phi^n(a)^*) - \phi(\phi^n(a)) \circ \phi(\phi^n(a)^*))\\ &= \rho(\phi^n(a) \circ \phi^n(a)^* - \phi^{n+1}(a) \circ \phi^{n+1}(a)^*)\\ &= \|\phi^n(a)\|_\rho^2 - \|\phi^{n+1}(a)\|_\rho^2 \to 0.\end{aligned}$$

Since this hold for all $\rho \in \mathscr{F}$ and $\mathscr{F}$ is faithful, $\langle\phi^n(a), \phi^n(a)\rangle \to 0$ weakly. By Lemma 2.1, if a_0 is a weak limit point of $(\phi^n(a))$ then $a_0 \in M_\phi$, proving (i).

To show (ii) suppose $\rho(\phi^n(a) \circ b) = 0$ for all $b \in M_\phi$, $\rho \in \mathscr{F}$. Let a_0 be a weak limit point of $(\phi^n(a))$. Then $\rho(a_0 \circ b) = 0$ for all $b \in M_\phi$, in particular by part (i) $\rho(a_0 \circ a_0) = 0$. Since $\mathscr{F}$ is faithful on the von Neumann algebra generated by $\phi(M)$, $a_0 = 0$. Thus 0 is the only weak limit point of $(\phi^n(a))$, so $\phi^n(a) \to 0$ weakly. The proof is complete.

It is not true in general that $\phi(M_\phi) \subseteq M_\phi$. We therefore introduce the following auxiliary concept. If $\phi: A \to A$ is positive unital with A a C^*-algebra, then $A_\Phi = \{a \in A_\phi : \phi^k(a) \in A_\phi, k \in \mathbf{N}\}$.

LEMMA 2.4. *Let M be ϕ-finite and M_Φ defined as above. Then M_Φ is a weakly closed Jordan subalgebra of M_ϕ such that $\phi(M_\Phi) \subseteq M_\Phi$, and if $a \in M$ then every weak limit point of $(\phi^n(a))$ belongs to M_Φ. Furthermore, if $\rho(\phi^n(a) \circ b) = 0$ for all $b \in M_\Phi$, $\rho \in \mathscr{F}$, then $\phi^n(a) \to 0$ weakly.*

PROOF. Since M is weakly closed and ϕ is weakly continuous on bounded sets M_Φ is weakly closed. Since ϕ and its powers ϕ^k are Jordan homomorphisms on M_Φ it is straightforward to show M_Φ is a Jordan subalgebra of M. Furthermore it is clear from its definition that $\phi(M_\Phi) \subseteq M_\Phi$.

If $a \in M$ and a_0 is a weak limit point of $(\phi^n(a))$, then $a_0 \in M_\phi$ by Lemma 2.3. Then $\phi(a_0)$ is a weak limit point of $(\phi^{n+1}(a))$, hence belongs to M_ϕ, again by Lemma 2.3. Iterating we have $\phi^k(a_0) \in M_\phi$ for all $k \in \mathsf{N}$. Thus $a_0 \in M_\Phi$. The last statement follows exactly as in Lemma 2.3. The proof is complete.

It is not true that $\phi(M_\Phi) = M_\Phi$. To remedy this problem we introduce yet another Jordan subalgebra.

DEFINITION 2.5. Let $\phi: A \to A$ be positive unital with A a C^*-algebra. The *multiplicative core* of ϕ is the set

$$C_\phi = \bigcap_{n=0}^{\infty} \phi^n(A_\Phi).$$

LEMMA 2.6. *C_ϕ satisfies the following:*

(i) *C_ϕ is a Jordan subalgebra of A.*

(ii) *$\phi(C_\phi) = C_\phi$.*

Suppose the restriction of ϕ to C_ϕ is faithful. Then we have

(iii) *The restriction of ϕ to C_ϕ is a Jordan automorphism.*

(iv) *C_ϕ is the largest Jordan subalgebra of A on which the restriction of ϕ is a Jordan automorphism.*

PROOF. As in Lemma 2.4 C_ϕ is clearly a Jordan subalgebra of A such that $\phi(C_\phi) \subseteq C_\phi$ and is weakly closed in the von Neumann algebra case. Furthermore, since $\phi(A_\Phi) \subseteq A_\Phi$, we have $\phi^n(A_\Phi) \subseteq \phi^{n-1}(A_\Phi)$, so that the sequence $(\phi^n(A_\Phi))$ is decreasing. Thus

$$C_\phi = \bigcap_{n=0}^{\infty} \phi^{n+1}(A_\Phi) = \phi(C_\phi),$$

so (i) and (ii) are proved.

We next show (iii). By (ii) the restriction of ϕ to C_ϕ is a Jordan homomorphism of C_ϕ onto itself. In particular since ϕ is faithful on C_ϕ, it is a Jordan automorphism of C_ϕ, proving (iii).

To show (iv) let B be a Jordan subalgebra of A such that $\phi|_B$ is a Jordan automorphism of B. Then clearly $B \subseteq A_\Phi$, and $\phi^n(B) = B$, so that

$$B = \bigcap_{n=0}^{\infty} \phi^n(B) \subseteq \bigcap_{n=0}^{\infty} \phi^n(A_\Phi) = C_\phi.$$

The proof is complete.

We can now prove our main result.

THEOREM 2.7. *Let M be ϕ-finite, and $\mathscr{F}$ a set of normal ϕ-invariant states which is faithful on the von Neumann algebra generated by $\phi(M)$. Let $a \in M$. Then we have*

(i) *Every weak limit point of $(\phi^n(a))$ lies in C_ϕ.*

(ii) *If $\rho(a \circ b) = 0$ for all $b \in C_\phi$, $\rho \in \mathscr{F}$, then $\phi^n(a) \to 0$ weakly.*

PROOF. (i) Let a_0 be a weak limit point of $(\phi^n(a))$. By Lemma 2.4 $a_0 \in M_\Phi$. Choose a subnet $(\phi^{n_\alpha}(a))$ which converges weakly to a_0. Let $k \in \mathsf{N}$, and let $(\phi^{m_\beta}(a))$ be a subnet of $(\phi^{n_\alpha - k}(a))$ which converges weakly to $a_1 \in M_\Phi$ (again using Lemma 2.4, since $(\phi^{m_\beta}(a))$ will be a subnet of $(\phi^n(a))$). Each m_β is of the form $n_{\alpha_j} - k$. The net $(\phi^{n_{\alpha_j}}(a))$ converges to a_0, since it is a subnet of the converging net $(\phi^{n_\alpha}(a))$. Thus we have

$$\begin{aligned}\phi^k(a_1) &= \lim \phi^k(\phi^{m_\beta}(a))\\ &= \lim \phi^{k+(n_{\alpha_j}-k)}(a)\\ &= \lim \phi^{n_{\alpha_j}}(a)\\ &= a_0.\end{aligned}$$

Thus $a_0 \in \phi^k(M_\Phi)$ for all $k \in \mathsf{N}$, hence $a_0 \in C_\phi$.

To show (ii) suppose $\rho(a \circ b) = 0$ for all $\rho \in \mathscr{F}$, $b \in \mathsf{C}_\phi$. Since $\phi^k(C_\phi) = C_\phi$ there exists $c \in C_\phi$ such that $b = \phi^k(c)$. Thus

$$\begin{aligned}\rho(\phi^k(a) \circ b) &= \rho(\phi^k(a) \circ \phi^k(c))\\ &= \rho(\phi^k(a \circ c))\\ &= \rho(a \circ c) = 0.\end{aligned}$$

By part (i) every weak limit point a_0 of $(\phi^n(a))$ lies in C_ϕ, so it follows by the above that $\rho(a_0 \circ b) = 0$ for all $b \in C_\phi$. In particular $\rho(a_0 \circ a_0) = 0$, so by faithfulness of $\mathscr{F}$, $a_0 = 0$, hence $\phi^n(a) \to 0$ weakly. The proof is complete.

One might believe that the converse of part (ii) in the above theorem is true. This is false. Indeed, let M_0 be a von Neumann algebra with a faithful normal tracial state τ_0. Let $M_i = M_0$, $\tau_i = \tau_0$, $i \in \mathsf{Z}$, and let $M = \bigotimes_{-\infty}^{\infty}(M_i, \tau_i)$. Let ϕ be the shift to the right. Then $C_\phi = M$. However, if $a = \ldots 1 \otimes a_0 \otimes 1 \ldots \in M$ with $a_0 \in M_0$, then $\lim_{n\to\infty} \phi^n(a) = \tau_0(a_0)1$, so if $\tau_0(a_0) = 0$, then the weak limit is 0. But $\tau(a \circ b) \neq 0$ for some $b \in M = C_\phi$.

If we assume convergence in the strong-* topology then the converse holds, as we have

PROPOSITION 2.8. *Let M be ϕ-finite. Let $a \in M$ and suppose the sequence $(\phi^n(a))$ converges in the strong-* topology. Then $\rho(a \circ b) = 0$ for all $b \in C_\phi$, $\rho \in \mathscr{F}$ if and only if $\phi^n(a) \to 0$ *-strongly.*

PROOF. If $\rho(a \circ b) = 0$ for all $b \in C_\phi$, $\rho \in \mathscr{F}$ then $\phi^n(a) \to 0$ weakly by the theorem. Since the sequence converges *-strongly the limit must be 0.

Conversely, if $\phi^n(a) \to 0$ *-strongly, then for all $b \in C_\phi$, $\rho \in \mathscr{F}$

$$\rho(a \circ b) = \rho(\phi^n(a \circ b)) = \rho(\phi^n(a) \circ \phi^n(b)) \to 0,$$

since multiplication is *-strongly continuous on bounded sets. The proof is complete.

We have not in general found a nice description of the complement of C_ϕ in M, i.e. a subspace D such that M is a direct sum of C_ϕ and D. In the finite case with a faithful normal ϕ-invariant trace this can be done.

PROPOSITION 2.9. *Suppose M has a faithful normal ϕ-invariant tracial state. Then there exists a faithful normal positive projection $P: M \to C_\phi$ which commutes with ϕ. Let $D = \{a - P(a) : a \in M\}$. Then $M = C_\phi + D$ is a direct sum, and if $a \in D$ then $\phi^n(a) \to 0$ weakly.*

PROOF. Since M is finite the same construction as that of trace invariant conditional expectations onto von Neumann subalgebras yields the existence of a faithful trace invariant positive normal projection $P: M \to C_\phi$, see [7]. Let τ be the trace alluded to in the proposition. Since τ is faithful and ϕ-invariant, ϕ has an adjoint map $\phi^*: M \to M$ defined by $\tau(a\phi^*(b)) = \tau(\phi(a)b)$ for $a, b \in M$. Clearly ϕ^* is τ-invariant, positive, unital, and normal, and its extension $\bar{\phi}^*$ to an operator on $L^2(M, \tau)$ is the usual adjoint of the extension $\bar{\phi}$ of ϕ. Since the restriction of $\bar{\phi}$ to the closure C_ϕ^- of C_ϕ in $L^2(M, \tau)$ is an isometry of C_ϕ^- onto itself, so is $\bar{\phi}^*$. It follows that $\phi P = P\phi P = (P\phi^* P)^* = (\phi^* P)^* = P\phi$.

It is clear that $M = C_\phi + D$ is a direct sum. Suppose $a \in D$, i.e. $P(a) = 0$. Then $\tau(a \circ b) = 0$ for all $b \in C_\phi$. If we let $\mathscr{F} = \{\tau|_{C_\phi} \circ P\}$ then, since P commutes with ϕ, $\mathscr{F}$ is a faithful family of normal ϕ-invariant states. By Theorem 2.7 $\phi^n(a) \to 0$ weakly, proving the proposition.

3. Maps of C^*-algebras

Arveson [1] proved the following result.

THEOREM 3.1 (Arveson). *Let A be a C^*-algebra, $\phi: A \to A$ a completely positive contraction such that the orbit $(\phi^n(a))$ is norm relative compact for all $a \in A$. Then there exists a completely positive projection $P: A \to A$ onto the norm closed linear span E_ϕ of the eigenoperators $a \in A$ with $\phi(a) = \lambda a$, with $|\lambda| = 1$, and $\alpha = \phi|_{E_\phi}$ is a complete isometry of E_ϕ onto itself. We have*

$$\lim_{n\to\infty} \|\phi^n(a) - (\alpha \circ P)^n(a)\| = 0,$$

and A is the direct sum of E_ϕ and the set $\{a \in A : \lim_n \|\phi^n(a)\| = 0\}$.

We shall now show how our previous results yield a result which is in a sense complementary to Arveson's theorem.

THEOREM 3.2. *Let A be a unital C^*-algebra and $\phi\colon A \to A$ a positive unital map such that the orbit $(\phi^n(a))$ is norm relative compact for all $a \in A$. Let C_ϕ be the multiplicative core for ϕ in A, and let E_ϕ denote the set of eigenoperators $a \in A$ such that $\phi(a) = \lambda a$, with $|\lambda| = 1$. Assume there exists a set $\mathcal{F}$ of ϕ-invariant states which is faithful on the C^*-algebra generated by $\phi(A)$. Then we have*

(i) *$E_\phi = C_\phi$ is a Jordan subalgebra of A.*

(ii) *The restriction $\phi|_{E_\phi}$ is a Jordan automorphism of E_ϕ.*

(iii) *Let $a \in A$. Then $\rho(a \circ b) = 0$ for all $\rho \in \mathcal{F}, b \in C_\phi$ if and only if $\lim_{n\to\infty} \|\phi^n(a)\| = 0$.*

PROOF. We first show (ii). If $\phi(a) = \lambda a$ then $\phi(a^*) = \bar{\lambda} a^*$, so E_ϕ is self-adjoint. Furthermore by inequality (1)

$$\phi(a \circ a^*) \geq \phi(a) \circ \phi(a^*) = \lambda a \circ \bar{\lambda} a^* = a \circ a^*.$$

Composing by $\rho \in \mathcal{F}$ and using that $\mathcal{F}$ is faithful on $C^*(\phi(A))$ it follows that $\phi(a \circ a^*) = \phi(a) \circ \phi(a^*)$, so $a \in A_\phi$, the definite set of ϕ. Since $a \in E_\phi$ is an eigenoperator, so is a^2, hence E_ϕ is a Jordan subalgebra of A_ϕ. Note that if $\phi(a) = \lambda a$ then $\phi(\phi(a)) = \phi(\lambda a) = \lambda\phi(a)$, so $\phi(a) \in E_\phi$. Thus $\phi\colon E_\phi \to E_\phi$. If $a = \sum \mu_i a_i \in E_\phi$ where $\phi(a_i) = \lambda_i a_i$, then $a = \sum \mu_i \bar{\lambda}_i \phi(a_i) \in \phi(E_\phi)$, so by density of such a's, $\phi(E_\phi) = E_\phi$. Thus by faithfulness of $\mathcal{F}$ the restriction $\phi|_{E_\phi}$ is a Jordan automorphism, proving (ii).

It follows from Lemma 2.6 that $E_\phi \subseteq C_\phi$. To show the converse inclusion we use that the orbit $(\phi^n(a))_{n\in\mathbb{N}}$ is norm relative compact for all $a \in A$. By Lemma 2.6 the restriction of ϕ to C_ϕ is a Jordan automorphism, hence in particular an isometry. We assert that if $a \in C_\phi$ then the orbit $(\phi^n(a))_{n\in\mathbb{Z}}$ is relative norm compact. For this it is enough to show that the set $(\phi^{-n}(a))_{n\in\mathbb{N}}$ is relative norm compact, or equivalently that each sequence $(\phi^{-n_k}(a))$ has a convergent subsequence. By assumption $(\phi^{n_k}(a))$ has a convergent subsequence $(\phi^{m_l}(a))$. Since this sequence is Cauchy, and

$$\|\phi^{-n}(a) - \phi^{-m}(a)\| = \|\phi^{n+m}(\phi^{-n}(a) - \phi^{-m}(a))\| = \|\phi^n(a) - \phi^m(a)\|,$$

it follows that $(\phi^{-m_l}(a))$ is Cauchy, and therefore converges. Thus the set $(\phi^{-n}(a))_{n\in\mathbb{N}}$ is relative norm compact, as is $(\phi^n(a))_{n\in\mathbb{Z}}$. By a well known result on almost periodic groups, see e.g. Lemma 2.8 in [1], $\phi|_{C_\phi}$ has pure point spectrum. Thus $C_\phi \subseteq E_\phi$, proving (i).

It remains to show (iii). As in the proof of Lemma 2.3 we find that every norm limit point a_0 of $(\phi^n(a))$ belongs to A_ϕ, and by the proof of Lemma 2.4 $a_0 \in A_\Phi = \{x \in A_\phi : \phi^k(x) \in A_\phi, k \in \mathsf{N}\}$. A straightforward modification of the proof of Theorem 2.7(i), replacing weak by norm, shows that $a_0 \in C_\phi$. Let $a \in A$ satisfy $\rho(a \circ b) = 0$ for all $b \in C_\phi$, $\rho \in \mathscr{F}$. Then by the proof of Theorem 2.7(ii), every norm limit point of $(\phi^n(a))$ is 0. Thus there is a subsequence $(\phi^{n_k}(a))$ of $(\phi^n(a))$ such that for all $\varepsilon > 0$ there is k_0 such that $\|\phi^{n_k}\| \leq \varepsilon$ when $k \geq k_0$. But then $n > n_k$ for $k \geq k_0$ implies

$$\|\phi^n(a)\| = \|\phi^{n-n_k}(\phi^{n_k}(a)\| \leq \|\phi^{n_k}\| < \varepsilon.$$

Thus $\|\phi^n(a)\| \to 0$.

Conversely, if $\|\phi^n(a)\| \to 0$ then for $b \in C_\phi$, $\rho \in \mathscr{F}$

$$\rho(a \circ b) = \rho(\phi^n(a \circ b)) = \rho(\phi^n(a) \circ \phi^n(b)) \to 0,$$

for $n \to \infty$. Thus $\rho(a \circ b) = 0$, completing the proof of the theorem.

It was shown in [5] that if A is a C^*-algebra, and $P\colon A \to A$ is a faithful positive unital projection then the image $P(A)$ is a Jordan subalgebra of A. The following corollary proves more.

COROLLARY 3.3. *Let A be a C^*-algebra and $P\colon A \to A$ a faithful positive unital projection. Then $E_P = C_P = P(A)$. Hence $P(A)$ is in particular a Jordan subalgebra of A.*

PROOF. Since $P^2 = P$ the orbit of each $a \in A$ is finite, so compact. Since P is faithful the set of states $\mathscr{F} = \{\omega|_{P(A)} \circ P\}$ with ω a state on A, is a faithful family of P-invariant states. Thus by Theorem 3.2 we have $E_P = C_P$. Since P is a projection the only nonzero eigenvalue of P is 1, and the corresponding eigenoperators are the elements in $P(A)$. Thus $E_P = P(A)$, proving the corollary.

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