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MEANS OF UNITARY OPERATORS, REVISITED

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(Dedicated to the memory of Gert K. Pedersen by the first-named author, his pupil, and the second-named author, his mentor, with admiration, affection, and respect)

Abstract

It is proved that an operator with bound not exceeding $(n-2)n^{-1}$ in a C^*-algebra is the mean of n unitay operators in that algebra.

1. Introduction

In [3], it is proved that if $\|A\| < 1 - \frac{2}{n}$, then $A = \frac{1}{n}(U_1 + \cdots + U_n)$, where A lies in a C^*-algebra $\mathfrak{A}$ and $U_1, \dots, U_n$ are in the unitary group $\mathscr{U}(\mathfrak{A})$ of $\mathfrak{A}$. The Russo-Dye theorem [6], "each A in $(\mathfrak{A})_1$, the closed unit ball ($\{A : \|A\| \le 1,\ A \in \mathfrak{A}\}$) in $\mathfrak{A}$, is the norm limit of convex combinations of unitary operators in $\mathfrak{A}$," is an immediate consequence of this much sharper result. The launch platform for the investigation in [3] was the observation by L. T. Gardner [1] that

$$(*) \qquad \mathscr{U}(\mathfrak{A}) + (\mathfrak{A})_1^o \subseteq \mathscr{U}(\mathfrak{A}) + \mathscr{U}(\mathfrak{A}),$$

where $(\mathfrak{A}_1)^o = \{A : \|A\| < 1,\ A \in \mathfrak{A}\}$ (the *open* unit ball of $\mathfrak{A}$). To see this, note that, with T in $(\mathfrak{A})_1^o$ and V in $\mathscr{U}(\mathfrak{A})$, $\frac{1}{2}(V + T) = \frac{1}{2}V(I + V^*T)$ and $\|V^*T\| = \|T\| < 1$. Thus $I + V^*T$, and hence $\frac{1}{2}(V + T)$ are invertible. So, $\frac{1}{2}(V+T) = UH$, with U in $\mathscr{U}(\mathfrak{A})$ and $H \ge 0$ in $\mathfrak{A}$. Now, $\|H\| = \|UH\| \le 1$, whence $H = \frac{1}{2}(U_1+U_2)$, with U_1 and U_2 in $\mathscr{U}(\mathfrak{A})$. Thus $V+T = UU_1+UU_2$, with UU_j in $\mathscr{U}(\mathfrak{A})$, and $(*)$ follows. Gardner proceeds from this observation to his short proof of the Russo-Dye theorem.

At a lecture, about Gardner's proof, to the Operator Algebra Seminar in Copenhagen on 7 October 1983, the second-named author noted that a different departure from Gardner's observation allowed one to conclude that each T in $(\mathfrak{A})_1^o$ is a finite, convex combination of elements in $\mathscr{U}(\mathfrak{A})$, from which the Russo-Dye theorem is immediate. A few days of discussion after that lecture,

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led to the following argument from (∗) to the result in [3] noted at the beginning of this article. With V in $\mathscr{U}(\mathfrak{A})$ and T in $(\mathfrak{A})_1^o$,

$$\begin{aligned} V + (n-1)T &= V + T + (n-2)T \\ &= U_1 + V_1 + (n-2)T \\ &= U_1 + U_2 + V_2 + (n-3)T \qquad (**) \\ &= \cdots = U_1 + \cdots + U_{n-2} + V_{n-2} + T \\ &= U_1 + \cdots + U_{n-2} + U_{n-1} + U_n, \end{aligned}$$

with each U_j and V_j in $\mathscr{U}(\mathfrak{A})$. If $n \geq 3$ and $S \in (1 - \frac{2}{n})(\mathfrak{A})_1^0$, then $\|(n-1)^{-1}(nS - I)\| \leq (n-1)^{-1}(n\|S\| + 1) < 1$. Replacing T by $(n-1)^{-1}(nS - I)$ and V by I in (∗∗), we have

$$nS = U_1 + \cdots + U_n \qquad (U_n \in \mathscr{U}(\mathfrak{A})).$$

As noted in [3], n is as good an estimate as possible of the least number of elements of $\mathscr{U}(\mathfrak{A})$ needed in a convex sum equal to T in $\mathfrak{A}$ when $\|T\| < 1 - \frac{2}{n}$, for with V a non-unitary isometry on a Hilbert space $\mathscr{H}$, and $1 - \frac{2}{n-1} < a_n < 1 - \frac{2}{n}$, $a_n V$ has norm a_n and is a mean of n unitary operators on $\mathscr{H}$ but no fewer. There are a number of other topics discussed, results proved, and questions raised in [3]. Those questions are answered in a hail of further results by M. Rørdam in his brilliant [7]. One question, raised by C. Olsen and G. K. Pedersen in [4], remained unanswered: Is T in $\mathfrak{A}$ a mean of n elements of $\mathscr{U}(\mathfrak{A})$ when $\|T\| = 1 - \frac{2}{n}$? For $\mathfrak{A}$ a von Neumann algebra, this question is answered in the affirmative in [4]; indeed, the "unitary rank" of each T in $(\mathfrak{A})_1$ is determined as well in terms of Olsen's index for T [5] and the distance of T from the group of invertible elements in $\mathfrak{A}$. For the general C^*-algebra $\mathfrak{A}$, this question was daunting to many of us. There were partial results; for example, the first-named author answered the question affirmatively when $\mathfrak{A}$ is commutative. (See Proposition 3.6 of [4].) The second-named author proved (unpublished notes) that if $\{z : |z| \leq 1 - \frac{2}{n}\}$ is not the spectrum of T (that is, if a single point of this disk is missing from the spectrum of T), Then T in $\mathfrak{A}$ is the mean of n elements of $\mathscr{U}(\mathfrak{A})$ when $\|T\| = 1 - \frac{2}{n}$. The argument was intricate. It could be made much simpler using later results and techniques of Rørdam [7]. The full conjecture, however, remained elusive until the first-named author proved it [2] (at the end of 1987). That proof was quite involved. Pedersen, on receiving a copy of that proof, was able to simplify it considerably. The "simplified" proof was still so complex that Pedersen remarked to the second-named author, that despite having "simplified it," he still did not understand it. When the Pedersen version reached the second-named author, it was simplified and restructured further. It became "understandable," well-motivated, but still not "simple." This last version of the first-named author's proof is the one we present in the next section. It remains attached to the same structure as the original argument of the first-named author.

2. The proof

We begin with some notation, in addition to the notation established in the preceding section. Throughout, $\mathfrak{A}$ is a unital C^*-algebra, $(\mathfrak{A})_1^+ = \{H : H \geq 0, H \in (\mathfrak{A})_1\}$, and $\mathscr{P} = \{UH : H \in (\mathfrak{A})_1^+\}$. We denote by 'sp$(T)$' the spectrum of T (in $\mathfrak{A}$ relative to $\mathfrak{A}$). We prove the main theorem of this article in what follows.

THEOREM. *If $A \in \mathfrak{A}$ and $\|A\| \leq 1 - \frac{2}{n}$ $(n = 3, 4, \ldots)$, then $A = \frac{1}{n}(U_1 + \cdots + U_n)$ with $U_1, \ldots, U_n$ in $\mathscr{U}(\mathfrak{A})$.*

With the aid of the lemma that follows:

LEMMA 1. *If $T \in (\mathfrak{A})_1$ and H is in $(\mathfrak{A})_1^+$, then*

$$T + 2H = U + V + V^*$$

for some U and V in $\mathscr{U}(\mathfrak{A})$, where $\mathrm{sp}(U^*V) \subseteq \left\{e^{i\theta} : -\frac{\pi}{2} \leq \theta \leq \pi\right\}$, *we can prove:*

LEMMA 2. *If $T \in (\mathfrak{A})_1$ and $S \in \mathscr{P}$, then*

$$T + 2S = U + 2R,$$

where $U \in \mathscr{U}(\mathfrak{A})$ and $R \in \mathscr{P}$.

With the aid of Lemma 2, we can prove our theorem. We prove the theorem from Lemma 2 first.

PROOF OF THEOREM. Let B be $\frac{n}{n-2}A$. Then $B \in (\mathfrak{A})_1$. From Lemma 2, with S in $\mathscr{P}$,

$$\begin{aligned} nA + 2S = (n-2)B + 2S &= (n-3)B + B + 2S = (n-3)B + U_1 + 2S_1 \\ &= U_1 + (n-4)B + B + 2S_1 = U_1 + U_2 + (n-4)B + 2S_2 \\ &= \cdots = U_1 + \cdots + U_{n-2} + 2S_{n-2}, \end{aligned}$$

where each $U_j \in \mathscr{U}(\mathfrak{A})$ and each $S_j \in \mathscr{P}$.

When $T \in \mathscr{P}$, $T = UH$, with U in $\mathscr{U}(\mathfrak{A})$ and H in $(\mathfrak{A})_1^+$, whence $2T = UV + UV^*$, where $V = H + i(I - H^2)^{\frac{1}{2}} \in \mathscr{U}(\mathfrak{A})$. Thus $2S_{n-2} = U_{n-1} + U_n$, with U_{n-1} and U_n in $\mathscr{U}(\mathfrak{A})$, and

$$nA + 2S = U_1 + U_2 + \cdots + U_n.$$

As $0 \in \mathscr{P}$ and S is an arbitrary element of $\mathscr{P}$, we may use 0 for S. Then $A = \frac{1}{n}(U_1 + \cdots + U_n)$.

PROOF OF LEMMA 2. Since $S \in \mathscr{P}$, $S = VH$ for some V in $\mathscr{U}(\mathfrak{A})$ and H in $(\mathfrak{A})_1^+$. From Lemma 1,

$$T + 2S = V(V^*T + 2H) = V(W + V_0 + V_0^*)$$

for some W and V_0 in $U(\mathfrak{A})$, where $\mathrm{sp}(W^*V_0) \subseteq \mathsf{C}_0$ and $\mathsf{C}_0 = \left\{e^{i\theta} : -\frac{\pi}{2} \leq \theta \leq \pi\right\}$. The function f on C_0, defined by $f(e^{i\theta}) = e^{\frac{1}{2}i\theta}$, is continuous. Thus $f(W^*V_0)$ is an element U_0 in $\mathfrak{A}$, $U_0^2 = W^*V_0$, and $\mathrm{sp}(U_0)$ lies in the right half-plane. Thus $U_0 + U_0^* = 2K$, where $K \in (\mathfrak{A})_1^+$ and

$$\begin{aligned} T + 2S &= V(W + V_0 + V_0^*) = VW(I + W^*V_0 + W^*V_0^*) \\ &= VW(I + U_0^2 + W^*V_0^*) = VWU_0(U_0^* + U_0 + U_0^*W^*V_0^*) \\ &= VWU_0(2K + U_0^*W^*V_0^*) = VV_0^* + 2VWU_0K \\ &= U + 2R, \end{aligned}$$

where $U = VV_0^* \in \mathscr{U}(\mathfrak{A})$ and $VWU_0K = R \in \mathscr{P}$.

PROOF OF LEMMA 1. If we have found U and V, then $T - U = V + V^* - 2H$, which is self-adjoint. Thus $\frac{1}{2i}(U - U^*)$ must be B, where $T = A + iB$ with A and B self-adjoint. Define U to be $B' + iB$, where the notation D' will be used to denote $(I - D^2)^{\frac{1}{2}}$, when $-I \leq D \leq I$. Then $T + 2H - U = A - B' + 2H = V + V^*$. Define V to be $C + iC'$, where $C = \frac{1}{2}(A - B' + 2H)$. For this, we must show that $-I \leq C \leq I$. Since $A = \frac{1}{2}(T + T^*)$ and $B = \frac{1}{2i}(T - T^*)$, we have that

$$A^2 + B^2 = \frac{1}{2}(TT^* + T^*T) \leq I,$$

since $\|T\| \leq 1$ (so that $TT^* \leq I$ and $T^*T \leq I$). Thus $A^2 \leq B'^2$ and $|A| \leq B'$. In particular, $A \leq B'$, whence $A - B' + 2H \leq 2H$, and $C \leq H \leq I$. At the same time, $C \geq \frac{1}{2}(A - B') \geq -I$, since $\|A - B'\| \leq 2$ (for $\|A\| \leq \|T\| \leq 1$ and $\|B'\| \leq 1$).

To establish the spectrum condition on U^*V, we assume that $-\cos\theta - i\sin\theta\ (= \lambda)$ is in $\mathrm{sp}(U^*V)$, where $0 < \theta < \frac{1}{2}\pi$. Then $U^*V - \lambda I$ and, hence, $V - \lambda U$ are not invertible in $\mathfrak{A}$. Some maximal left or right ideal in $\mathfrak{A}$ contains $V - \lambda U$, so that $0 = \rho(V - \lambda U)$ for some (pure) state ρ of $\mathfrak{A}$. Now,

$$V - \lambda U = C + \cos\theta B' - \sin\theta B + i(C' + \cos\theta B + \sin\theta B').$$

Since ρ is a state,

$$\rho(C + \cos\theta B' - \sin\theta B) = 0 = \rho(C' + \cos\theta B + \sin\theta B').$$

Thus

$$\rho(\cos\theta C - \cos\theta\sin\theta B + \cos^2\theta B') = 0$$
$$\rho(\sin\theta C' + \sin\theta\cos\theta B + \sin^2\theta B') = 0,$$

and

$$0 = \rho(\cos\theta C + B' + \sin\theta C') = \rho(\cos\theta(C + B') + (1 - \cos\theta)B' + \sin\theta C').$$

Note that $C + B' = \frac{1}{2}(A + B' + 2H) \geq 0$, since $-A \leq |A| \leq B'$, from our earlier observations. By assumption $0 < \theta < \frac{1}{2}\pi$, so that $\cos\theta$, $1 - \cos\theta$, and $\sin\theta$ are positive numbers. As $C + B'$, B', and C' are positive operators and ρ is a state, we have that

$$\rho(C + B') = \rho(B') = \rho(C') = 0,$$

and $0 = \rho(C'^2) = 1 - \rho(C^2)$. Hence

$$0 = \rho(CB') = \rho(B'C) = \rho(B'^2) = \rho((C + B')^2).$$

But then

$$0 = \rho((C + B')^2) = \rho(C^2 + CB' + B'C + B'^2) = \rho(C^2) = 1,$$

a contradiction. Thus λ, of the form described, is not in $\mathrm{sp}(U^*V)$.

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