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DIMENSION GROUPS ASSOCIATED TO β-EXPANSIONS

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(Dedicated to the memory of Gert K. Pedersen)

Abstract

Completing work by Shultz on one hand and by Katayama, Matsumoto, and Watatani on the other, we prove that *a priori* different dimension groups associated to β-expansions in fact coincide.

1. Introduction

Two different constructions associate dimension groups to β-expansions. Here, a *dimension group* ([9], [10]) is an ordered abelian group which is unperforated and has the Riesz properties, and β*-expansions* ([18], [17], [1], [19]) are elements of $\{0, \dots, \lceil\beta\rceil - 1\}^{\mathsf{N}_0}$ of the form

$$(b_n)_{n\in\mathsf{N}_0} = (\lfloor\beta T_\beta^n(x)\rfloor)_{n\in\mathsf{N}_0}$$

with $x \in [0, 1)$, where $\beta > 1$ is fixed and T_β is the β*-transformation*

$$T_\beta : [0, 1) \to [0, 1) \qquad T_\beta(x) = \beta x - \lfloor\beta x\rfloor.$$

The terminology is motivated by the observation

$$x = \sum_{n=0}^{\infty} b_n\beta^{-n-1}$$

and we shall work mainly with the closure of the set of all such β-expansions which is denoted as the β*-shift* X_β, thinking of this as a symbolic representation of orbits under T_β as indicated in Figure 1.

The first such construction, considered in [12], involves the fixed point algebra $\mathscr{F}_\beta^\infty$ for the so-called *gauge action* of the C^*-algebras associated by Matsumoto to any shift space ([14]). As noted in [5, Corollary 3.3], in this case the two different ways to build such C^*-algebras coincide, but the reader

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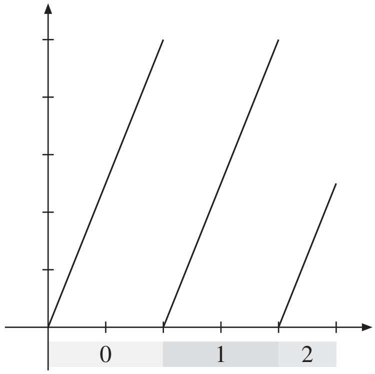

FIGURE 1. $\beta = 5/2$

may still find the alternative presentations in [3] and [6] useful. This fixed point algebra is an *approximately finite* (or "AF", cf. [2]) C^*-algebra, and its ordered K_0-group is a dimension group.

The second construction, introduced in [20] and studied further in [21], is a special case of a direct construction valid for any interval map, specialized to the map T_β defined above.

Since dimension groups with distinguished order units and unital AF algebras are in 1–1 correspondence ([9], [8]) it is just a matter of perspective whether or not one considers the dimension groups or AF algebras, as made clear in [15] and [7]. We shall choose the former as it harmonizes with our proof.

Setting off from the work of Katayama *et al* we shall work with the ordered K_0-group of the fixed point algebra $\mathscr{F}_\beta^\infty$ and denote it by $(\Delta_\beta, \Delta_\beta^+)$ where Δ_β^+ denotes the cone of positive elements in Δ_β. The importance of this object (cf. [15], [16]) stems from the fact that it is a conjugacy invariant for the underlying shift space, and that it generalizes the dimension groups used in the classification of shifts of finite type up to shift equivalence.

In fact, we shall work with a triple comprising the dimension group, the automorphism τ_β acting on it, and a distinguished order unit for the dimension group. The collection $(\Delta_\beta, \Delta_\beta^+, \tau_\beta)$ is referred to as the *dimension triple* of X_β (cf. [13] and [16]; notice however that in [16] the automorphism τ_β^{-1} is used in place of τ_β).

By [21, Proposition 10.6] the two constructions give the same dimension groups when the β-shift is sofic (cf. [13]). This is seen by comparing explicit computations of the dimension groups in [21, Proposition 10.3] and [12, Theorem 6.1]. But since the dimension group of the fixed point algebras of the C^*-algebras associated by Matsumoto to the β-shift X_β is only determined as a group in [12], the problem of determining whether the dimension groups coincide in general has been left open.

It is the purpose of this note to compute the ordered dimension group in the non-sofic case, thus proving that the two dimension groups coincide in this case as well. This is achieved by a direct computation in combination with a result of Ito and Takahashi ([11]) proving, roughly speaking, that a certain sequence of numbers $\#\mathscr{P}_n(x)$ grow as β^n uniformly in x. Apart from importing that result, we are thus able to provide a self-contained proof.

We need to point out a minor inconsistency in [12] of relevance to our work. In the proof of Case 1 of [12, Theorem 1] it is stated, correctly as proved in [12, Corollary 4.6], that the dimension group in the non-sofic case is always isomorphic to a sum of infinitely many copies of Z. However, in the statement of [12, Theorem 1] it is claimed that the dimension group is isomorphic, as a group, to

$$\mathsf{Z}[\beta, \beta^{-1}]$$

and while this is true for most β, it fails for instance for rational β such as $\beta = 3/2$ where all elements are divisible by 6.

2. Preliminaries

The real numbers will be denoted by R, the integers by Z, the positive integers by N and the non-negative integers by N_0. We will denote integers by i, k, l, m, n and b.

The β-shift space introduced above (the reader is directed to [12] or [11] for a more detailed definition) will be denoted by X_β, and σ will be the shift mapping on X_β. The alphabet $\{0, 1, \dots, \lceil\beta\rceil - 1\}$ of X_β will be denoted by $\mathfrak{a}$, and we will denote a letter of $\mathfrak{a}$ by a. We will denote the lexicographic order on both n-tuples $\mathfrak{a}^n$, infinite sequences $\mathfrak{a}^{\mathsf{N}_0}$ and finite words of arbitrary length $\mathfrak{a}^*$ by $\geq$. The set X_β has a maximal element under this order (cf. e.g. [12] or [1]) which we shall denote by ζ. Other elements of X_β will be denoted by η, ξ and ε. If $\eta \in X_\beta$ and $n \in \mathsf{N}_0$, then we will by η_n denote the $n+1$st letter of η. We denote the language of X_β by W, elements of W by u and the length of these elements by $|u|$. If $k \in \{1, 2, \dots, |u|\}$, then u_k will denote the kth letter of u. Let for $n \in \mathsf{N}_0$ the set W_n be given by

$$W_n = \{u \in W \mid |u| = n\}.$$

We then have (cf. [1, p. 136]) that the following identity holds:

$$W_n = \{u_1 u_2 \cdots u_n \in \mathfrak{a}^n \mid u_k u_{k+1} \cdots u_n \leq \zeta_1 \zeta_2 \cdots \zeta_{n+1-k} \text{ for all } 1 \leq k \leq n\}.$$

For a set A, we will by 1_A denote the characteristic function of A. We write, for $\eta \in X_\beta$,

$$\mathscr{P}_n(\eta) = \{u \in W_n \mid u\eta \in X_\beta\}$$

and call this set the *set of pasts* of length n for η. Equality of n-pasts induces the *n-past equivalence relation* which will be denoted by $\sim_n$, and $[\eta]_n$ will be the n-past equivalence class of η. We will by Ω_{X_β} denote the space

$$\left\{([x_n]_n)_{n\in\mathsf{N}_0} \in \prod_{n\in\mathsf{N}_0} X_\beta/\sim_n \,\middle|\, \forall n \in \mathsf{N}_0 : x_n \sim_n x_{n+1}\right\}$$

which is a projective limit of the set of n-past equivalence classes, cf. [4, 2.5]. Elements of Ω_{X_β} will be denoted by x, y and $([x_n]_n)_{n\in\mathsf{N}_0}$. So x_n will always denote an element of X_β and not an element of $\mathfrak{a}$. We define an order $\geq$ on $C(\Omega_{X_\beta}, \mathsf{Z})$ by letting $f \geq g$ if $f(x) \geq g(x)$ for all $x \in \Omega_{X_\beta}$. We write $f > g$ if $f \geq g$ and $f \neq g$. If $x = ([x_n]_n)_{n\in\mathsf{N}_0} \in \Omega_{X_\beta}$ and $a \in \mathscr{P}_1(x_1)$, then $([ax_{n+1}]_n)_{n\in\mathsf{N}_0}$ will belong to Ω_{X_β}. We will denote this element by ax. We use λ to denote the map on $C(\Omega_{X_\beta}, \mathsf{Z})$ given by

$$\lambda(f)(x) = \sum_{a\in\mathscr{P}_1(x_1)} f(ax)$$

for all $x = ([x_n]_n)_{n\in\mathsf{N}_0} \in \Omega_{X_\beta}$ and all $f \in C(\Omega_{X_\beta}, \mathsf{Z})$, cf. [4].

We may by [15], [16], and [4] compute $(\Delta_\beta, \Delta_\beta^+)$ as the limit of the inductive system $(C(\Omega_{X_\beta}, \mathsf{Z}), \lambda)_{n\in\mathsf{N}_0}$ of abelian groups. If $(\kappa_n)_{n\in\mathsf{N}_0}$ denotes the sequence of maps $\kappa_n : C(\Omega_{X_\beta}, \mathsf{Z}) \to \Delta_\beta$ given according to the universal property of inductive limits, such that $\kappa_{n+1} \circ \lambda = \kappa_n$ for all $n \in \mathsf{N}_0$, then we denote by Δ_β^+ the positive cone of Δ_β defined by

$$\Delta_\beta^+ = \{\kappa_n(f) \mid n \in \mathsf{N}_0,\ f \geq 0\}.$$

The automorphism τ_β on Δ_β is then defined by requiring that $\tau_\beta \circ \kappa_n = \kappa_n \circ \lambda$ for all $n \in \mathsf{N}_0$. The element $\kappa_0(1_{\Omega_{X_\beta}})$ is a distinguished order unit in $(\Delta_\beta, \Delta_\beta^+)$ corresponding to the unit of the fixed point algebra, and we denote it by $\mathbf{1}_{\Delta_\beta}$.

For $l \geq n$, let W_n^l be the set

$$W_n^l = \left\{u \in W_n \,\middle|\, \begin{array}{l} u_1u_2\cdots(u_{n-l}+1) \in W \\ u_1u_2\cdots(u_k+1) \notin W \text{ for } n-l < k \leq n \end{array}\right\}.$$

Of course the elements will fail to be in W when u_{n-l} or u_k equals $\lceil\beta\rceil - 1$. When $l = n$ the first condition is considered vacuously true.

LEMMA 2.1. *For all $n, l \in \mathbb{N}_0$ with $l \leq n$, we have*

$$W_n^l = \{u\zeta_0\zeta_1 \cdots \zeta_{l-1} \mid u \in W_{n-l}^0\}.$$

PROOF. If $u \in W_{n-l}^0$, then $u_1 u_2 \cdots (u_{n-l} + 1) \in W$ and

$$u\zeta_0\zeta_1 \cdots (\zeta_{k-1-(n-l)} + 1) \notin W \qquad \text{for} \quad n - l < k \leq n.$$

Thus $\{u\zeta_0\zeta_1 \cdots \zeta_{l-1} \mid u \in W_{n-l}^0\} \subseteq W_n^l$.

If $u \in W_n^l$, then $u_1 u_2 \cdots u_{n-l} \in W_{n-l}^0$, and $u_{n-l+1}u_{n-l+2} \cdots u_n = \zeta_0\zeta_1 \cdots \zeta_{l-1}$ because if $u_{n-l+1}u_{n-l+2} \cdots u_k = \zeta_0\zeta_1 \cdots \zeta_{k-(n-l+1)}$ for $n - l + 1 \leq k < n$ and $u_{k+1} < \zeta_{k+1-(n-l+1)}$, then $u_1 u_2 \cdots (u_{k+1} + 1) \in W$. Thus $W_n^l \subseteq \{u\zeta_0\zeta_1 \cdots \zeta_{l-1} \mid u \in W_{n-l}^0\}$.

Let for $\eta \in X_\beta$ and $n \in \mathbb{N}_0$, $D_n(\eta)$ be the set

$$\{l \in \{0, 1, \ldots, n\} \mid \eta \leq \sigma^l(\zeta)\}.$$

LEMMA 2.2. *Let $\eta \in X_\beta$ and $n \in \mathbb{N}_0$. Then $\mathcal{P}_n(\eta)$ is the disjoint union of those W_n^l for which $l \in D_n(\eta)$.*

PROOF. Since $\{W_n^l\}_{l=0}^n$ is a disjoint partition of W_n, it is enough to show that if $l \in \{0, 1, \ldots, n\}$ and $u \in W_n^l$, then $u\eta \in X_\beta$ if and only if $\eta \leq \sigma^l(\zeta)$.

It follows from Lemma 2.1 that if $l \in \{0, 1, \ldots, n\}$, $u \in W_n^l$ and $\eta \leq \sigma^l(\zeta)$, then $u\eta \in X_\beta$. If, on the other hand, $l \in \{0, 1, \ldots, n\}$, $u \in W_n^l$ and $u\eta \in X_\beta$, then

$$\zeta_0\zeta_1 \cdots \zeta_{l-1}\eta = \sigma^{n-l}(u\eta) \leq \zeta,$$

and so $\eta \leq \sigma^l(\zeta)$.

The following notation will be useful: For $x = ([x]_n)_{n\in\mathbb{N}_0} \in \Omega_{X_\beta}$, we define $\mathcal{P}_n(x) = \mathcal{P}_n(x_n)$, and for $x, y \in \Omega_{X_\beta}$, we write $x \sim_n y$ if $\mathcal{P}_n(x) = \mathcal{P}_n(y)$. If $\eta, \xi \in X_\beta$ and $n \in \mathbb{N}_0$, then we will by $[\eta, \xi]_n$ denote the subset

$$\{x \in \Omega_{X_\beta} \mid \exists \varepsilon \in X_\beta : \eta \leq \varepsilon \leq \xi \text{ and } \mathcal{P}_n(\varepsilon) = \mathcal{P}_n(x)\}$$

of Ω_{X_β}.

REMARK 2.3. It follows from the construction of Ω_{X_β} and its topology that $C(\Omega_{X_\beta}, \mathbb{Z})$ is generated by $(1_{[\eta,\eta]_n})_{\eta\in X_\beta, n\in\mathbb{N}_0}$, and it follows from Lemma 2.2 that if $\eta \in X_\beta$, $n \in \mathbb{N}_0$ and $k, l \in \{0, 1, \ldots, n\}$ are such that $\sigma^k(\zeta) < \eta \leq \sigma^l(\zeta)$ and there exists no $i \in \{0, 1, \ldots, n\}$ such that $\sigma^k(\zeta) < \sigma^i(\zeta) < \sigma^l(\zeta)$, then

$$[\eta, \eta]_n = [0, \sigma^l(\zeta)]_n \setminus [0, \sigma^k(\zeta)]_n = [0, \sigma^l(\zeta)]_l \setminus [0, \sigma^k(\zeta)]_k.$$

So $C(\Omega_{X_\beta}, \mathbb{Z})$ is generated by $(1_{[0,\sigma^n(\zeta)]_n})_{n\in\mathbb{N}_0}$.

The following result is from [12]. We include a proof for the sake of self-containedness.

LEMMA 2.4 ([12, Lemma 4.4]). *Let $n, l \in \mathsf{N}_0$ with $l \leq n$. Then*

$$\lambda\left(1_{[0,\sigma^l(\zeta)]_n}\right) = \zeta_l 1_{\Omega_{X_\beta}} + 1_{[0,\sigma^{l+1}(\zeta)]_{n+1}}.$$

PROOF. It follows from Lemma 2.2 that if $\eta, \xi \in X_\beta$ and $k \in \mathsf{N}_0$, then $\mathscr{P}_k(\eta) = \mathscr{P}_k(\xi)$ if and only if $D_k(\eta) = D_k(\xi)$, so if $a \in \mathfrak{a}$ and $x \in \Omega_{X_\beta}$, then

$$\begin{aligned}
&ax \in [0, \sigma^l(\zeta)]_n \\
&\quad \iff \exists \eta \leq \sigma^l(\zeta) : D_n(\eta) = D_n(ax_{n+1}) \\
&\quad \iff (a < \zeta_l) \vee (a = \zeta_l \wedge \exists \xi \leq \sigma^{l+1}(\zeta) : D_{n+1}(\xi) = D_{n+1}(x_{n+1})) \\
&\quad \iff (a < \zeta_l) \vee (a = \zeta_l \wedge x \in [0, \sigma^{l+1}(\zeta)]_{n+1}),
\end{aligned}$$

from which the lemma follows.

We denote for every $n \in \mathsf{N}_0$ the function $\lambda^n\left(1_{\Omega_{X_\beta}}\right)$ by f_n.

LEMMA 2.5. *For every $n \in \mathsf{N}_0$ and every $x \in \Omega_{X_\beta}$, we have*

$$f_n(x) = \#\mathscr{P}_n(x).$$

PROOF. Easily proved by induction over n.

From this point onwards, we need to specialize to the case in point of *non-sofic X_β*. As seen in [1, Proposition 4.2] and [12, Proposition 3.8(ii)] the sofic case is characterized by ζ being eventually periodic, and it is precisely through the condition $\sigma^k(\zeta) \neq \sigma^l(\zeta)$ for $k \neq l$ that the property enters our proof below.

LEMMA 2.6. *Let $n \in \mathsf{N}_0$.*

(1) *For any $x, y \in \Omega_{X_\beta}$, if $x \sim_n y$ then $f_n(x) = f_n(y)$.*
(2) *If X_β is not sofic, then there exist $x, y \in \Omega_{X_\beta}$ for which $x \sim_n y$ and $f_{n+1}(x) \neq f_{n+1}(y)$.*

PROOF. (1) follows from Lemma 2.5. For (2) choose $l \in \{0, 1, \dots, n\}$ such that $\sigma^{n+1}(\zeta) < \sigma^l(\zeta)$ and such that there exists no $k \in \{0, 1, \dots, n\}$ such that $\sigma^{n+1}(\zeta) < \sigma^k(\zeta) < \sigma^l(\zeta)$. It then follows from Lemma 2.2 that $\mathscr{P}_n(\sigma^{n+1}(\zeta)) = \mathscr{P}_n(\sigma^l(\zeta))$ and $\mathscr{P}_{n+1}(\sigma^l(\zeta)) \subsetneq \mathscr{P}_{n+1}(\sigma^{n+1}(\zeta))$. So if we let

$$x = ([x_i]_i)_{i \in \mathsf{N}_0} \in \Omega_{X_\beta}$$

and

$$y = ([y_i]_i)_{i \in \mathsf{N}_0} \in \Omega_{X_\beta},$$

where $x_i = \sigma^{n+1}(\zeta)$ and $y_i = \sigma^l(\zeta)$ for every $i \in \mathsf{N}_0$, then $x \sim_n y$ and $f_{n+1}(x) > f_{n+1}(y)$.

LEMMA 2.7. *When X_β is not sofic, then we have for every $n \in \mathsf{N}_0$ that*

$$\{f_0, f_1, \ldots, f_n\}$$

is linearly independent in $C(\Omega_{X_\beta}, \mathsf{C})$.

PROOF. The lemma is certainly true for $n = 0$. Assume that $\{f_0, f_1, \ldots, f_n\}$ is linearly independent and that

$$b_0 f_0 + b_1 f_1 + \cdots + b_{n+1} f_{n+1} = 0.$$

Choose by property (2) of Lemma 2.6 two elements $x, y \in \Omega_{X_\beta}$ such that $x \sim_n y$ and $f_{n+1}(x) \neq f_{n+1}(y)$. Then since

$$\begin{aligned} -b_{i+1} f_{n+1}(x) &= (b_0 f_0 + \cdots + b_n f_n)(x) \\ &= (b_0 f_0 + \cdots + b_n f_n)(y) = -b_{n+1} f_{n+1}(y), \end{aligned}$$

b_{n+1} must be equal to 0, and therefore $b_0 = b_1 = \cdots = b_n = 0$ by the inductive hypothesis.

DEFINITION 2.8. Let $\Phi : \bigoplus_{k \in \mathsf{N}_0} \mathsf{Z} \to C(\Omega_{X_\beta}, \mathsf{Z})$ be the group morphism defined by

$$\Phi(b_0, b_1, \ldots) = \sum_{k \in \mathsf{N}_0} b_k f_k.$$

The following result establishes a group isomorphism like in [12, Corollary 4.3 & 4.6], but with a set of generators better suited for an analysis of the positive cone.

PROPOSITION 2.9. *The group morphism Φ is an isomorphism when X_β is not sofic.*

PROOF. That Φ is injective follows from Lemma 2.7. It follows from Remark 2.3 that in order to prove that Φ is surjective, it is enough to show that $1_{[0,\sigma^n(\zeta)]_n} \in \Phi\left(\bigoplus_{k \in \mathsf{N}_0} \mathsf{Z}\right)$ for every $n \in \mathsf{N}_0$, and since

$$\lambda\left(\Phi\left(\bigoplus_{k \in \mathsf{N}_0} \mathsf{Z}\right)\right) \subseteq \Phi\left(\bigoplus_{k \in \mathsf{N}_0} \mathsf{Z}\right),$$

this follows from Lemma 2.4.

3. Computing the dimension triple

Let $M_\beta = \sum_{n\in\mathsf{N}_0}(n+1)\zeta_n\beta^{-n-1}$ (cf. [11, Corollary 3.6]) and for every $x = ([x_l]_l)_{l\in\mathsf{N}_0} \in \Omega_{X_\beta}$, let $D(x)$ be the set $\{n \in \mathsf{N}_0 \mid \sigma^n(\zeta) \geq x_n\}$. Define (cf. the function $h_\beta : [0,1] \to \mathsf{R}$ in [17]) a function F on Ω_{X_β} by

$$F(x) = M_\beta^{-1} \sum_{n\in D(x)} \beta^{-n}.$$

Lemma 3.1. *The sequence $(\beta^{-n} f_n)_{n\in\mathsf{N}_0}$ of functions converges uniformly on Ω_{X_β} to F.*

Proof. It follows from Lemma 2.1, 2.2 and 2.5 that

$$f_n(x) = \#\mathscr{P}_n(x) = \sum_{l\in D_n(x_n)} \#W_n^l = \sum_{l\in D_n(x_n)} \#W_{n-l}^0$$

for all $n \in \mathsf{N}_0$ and $x = ([x_k]_k)_{k\in\mathsf{N}_0} \in \Omega_{X_\beta}$.

Let

$$K = \sup_{i\in\mathsf{N}_0} i\left|\beta^{-i}\#W_i^0 - M_\beta^{-1}\right|.$$

According to the proof of [11, Corollary 3.6] we have $K < \infty$. So for all $n \in \mathsf{N}_0$ and $x = ([x_k]_k)_{k\in\mathsf{N}_0} \in \Omega_{X_\beta}$ we have

$$\begin{aligned}
&|\beta^{-n} f_n(x) - F(x)| \\
&\quad = \left| \sum_{l\in D_n(x_n)} \beta^{-n}\#W_{n-l}^0 - M_\beta^{-1} \sum_{l\in D(x)} \beta^{-l} \right| \\
&\quad \leq \left(\sum_{l\in D_{n-1}(x_n)} \beta^{-l} \left|\beta^{l-n}\#W_{n-l}^0 - M_\beta^{-1}\right| \right) + \beta^{-n} + M_\beta^{-1} \sum_{l\geq n} \beta^{-l} \\
&\quad \leq \left(\sum_{l\in D_{n-1}(x_n)} \beta^{-l} K/(n-l) \right) + \beta^{-n} + M_\beta^{-1}\beta^{1-n}/(\beta-1) \\
&\quad \leq K\left(\sum_{i=1}^{n} \beta^{i-n}/i \right) + \beta^{-n} + M_\beta^{-1}\beta^{1-n}/(\beta-1) \to 0
\end{aligned}$$

as $n \to \infty$. To see that the first term converges one may for instance for each k majorize the first $n - \lceil n/k \rceil$ terms by β^{i-n} and the last $\lceil n/k \rceil$ terms by $1/n$.

It follows from Proposition 2.9 that the map ψ

$$\sum_{k\in\mathsf{N}_0} b_k f_k \mapsto \sum_{k\in\mathsf{N}_0} b_k \beta^k$$

is a well-defined linear functional from $C(\Omega_{X_\beta}, \mathsf{Z})$ to R when X_β is not sofic.

In the proof below, we use the notation $T_\beta(1) = \lim_{x \nearrow 1} T_\beta(x)$.

LEMMA 3.2. *The functional ψ has the following two properties:*

(1) *If $f, g \in C(\Omega_{X_\beta}, \mathsf{Z})$ and $f > g$, then $\psi(f) > \psi(g)$,*

(2) *if $f \in C(\Omega_{X_\beta}, \mathsf{Z})$, then $(\beta^{-n}\lambda^n(f))_{n\in\mathsf{N}_0}$ converges uniformly on Ω_{X_β} to $\psi(f)F$.*

PROOF. That property (2) holds follows from Lemma 3.1.

Let $f, g \in C(\Omega_{X_\beta}, \mathsf{Z})$ and $f > g$. It then follows from Remark 2.3 that $f - g$ can be written as a linear combination of elements of the form $1_{[0,\sigma^l(\zeta)]_l} - 1_{[0,\sigma^k(\zeta)]_k}$ with $\sigma^l(\zeta) > \sigma^k(\zeta)$ and with all the coefficients positive. It follows from Lemma 2.4 that $\psi(1_{[0,\sigma^l(\zeta)]_l}) = T_\beta^l(1)$ for all $l \in \mathsf{N}_0$, and since (cf. [11, Proposition 3.2]) $\sigma^l(\zeta) > \sigma^k(\zeta)$ implies $T_\beta^l(1) > T_\beta^k(1)$, we get that $\psi(f - g) > 0$ and thus $\psi(f) > \psi(g)$.

THEOREM 3.3. *When X_β is not sofic, the dimension triple $(\Delta_\beta, \Delta_\beta^+, \tau_\beta)$ is isomorphic to $(\mathsf{Z}[t, t^{-1}], PC_\beta, \mu)$, where*

$$PC_\beta = \{p \in \mathsf{Z}[t, t^{-1}] \mid p = 0 \text{ or } p(\beta) > 0\},$$

and $\mu : \mathsf{Z}[t, t^{-1}] \to \mathsf{Z}[t, t^{-1}]$ is multiplication by t. Under this isomorphism, the distinguished order unit $\mathbf{1}_{\Delta_\beta}$ is mapped to 1.

PROOF. For every $i \in \mathsf{N}_0$, let ι_i be the group morphism from $C(\Omega_{X_\beta}, \mathsf{Z})$ to $\mathsf{Z}[t, t^{-1}]$ defined by

$$\iota_i\Big(\sum_{k\in\mathsf{N}_0} b_k f_k\Big) = \sum_{k\in\mathsf{N}_0} b_k t^{k-i}.$$

It follows from Proposition 2.9 that ι_i is well-defined and injective. Because $\iota_{i+1} \circ \lambda = \iota_i$ for every $i \in \mathsf{N}_0$, the ι_i's induce a group morphism ι from Δ_β to $\mathsf{Z}[t, t^{-1}]$. Since ι_i is injective for every $i \in \mathsf{N}_0$, ι is injective. Further, the following equality holds:

$$\bigcup_{i\in\mathsf{N}_0} \iota_i(C(\Omega_{X_\beta}, \mathsf{Z})) = \mathsf{Z}[t, t^{-1}].$$

Consequently, ι is also surjective and thus an isomorphism, and since $\iota_i \circ \lambda = \mu \circ \iota_i$ for every $i \in \mathsf{N}_0$, we have that $\iota \circ \tau_\beta = \mu \circ \iota$.

Let $\sum_{k=-l}^{m} b_k t^k \in \mathsf{Z}[t, t^{-1}]$ with $\sum_{k=-l}^{m} b_k \beta^k > 0$, and let $f = \sum_{k=-l}^{m} b_k f_{k+l}$. We then have that

$$\psi(f) = \beta^l \sum_{k=-l}^{m} b_k \beta^k > 0,$$

and since it follows from Lemma 3.2 that $(\beta^{-n}\lambda^n(f))_{n\in\mathsf{N}_0}$ converges uniformly on Ω_{X_β} to $\psi(f)F$, and $F(x) \geq M_\beta^{-1}$ (since $0 \in D(x)$ for every $x \in \Omega_{X_\beta}$), there exists an $n \in \mathsf{N}_0$ such that $\sum_{k=-l}^{m} b_k f_{k+l+n}$ is a strictly positive function, and hence

$$\sum_{k=-l}^{m} b_k t^k = \iota_{l+n}\left(\sum_{k=-l}^{m} b_k f_{k+l+n}\right) \in \iota(\Delta_\beta^+).$$

Let us denote the linear functional $p \mapsto p(\beta)$ from $\mathsf{Z}[t, t^{-1}]$ to R by Ψ. If $f \in C(\Omega_{X_\beta}, \mathsf{Z})$ and $f > 0$, then $\psi(f) > 0$ according to Lemma 3.2, and since $\beta^n \Psi \circ \iota_n = \psi$ for every $n \in \mathsf{N}_0$, we have that $\Psi(\iota_n(f)) > 0$ for all $n \in \mathsf{N}_0$, which shows that $\iota(\Delta_\beta^+) \subseteq PC_\beta$.

Finally, we note that $\iota(\mathbf{1}_{\Delta_\beta}) = \iota_0(f_0) = 1$.

COROLLARY 3.4. *The dimension groups associated in [12] and in [21], respectively, to any β-shift, coincide.*

PROOF. The sofic case was solved in [21, Proposition 10.6], and the non-sofic case is got by comparing the previous theorem to [21, Proposition 10.5].

COROLLARY 3.5. *The AF algebras $\mathscr{F}_\beta^\infty$ and F_{T_β} defined in [12] and in [7], respectively, are isomorphic for any β.*

PROOF. The sofic case was solved in [7, Corollary 15.3], and the non-sofic case is got by comparing the dimension group in the previous theorem, along with the distinguished order unit, to that of [21, Proposition 10.6].

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