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BROWN MEASURES OF UNBOUNDED OPERATORS AFFILIATED WITH A FINITE VON NEUMANN ALGEBRA

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(Dedicated to the memory of Gert K. Pedersen)

Abstract

In this paper we generalize Brown's spectral distribution measure to a large class of unbounded operators affiliated with a finite von Neumann algebra. Moreover, we compute the Brown measure of all unbounded R-diagonal operators in this class. As a particular case, we determine the Brown measure $z = xy^{-1}$, where (x, y) is a circular system in the sense of Voiculescu, and we prove that for all $n \in \mathsf{N}$, $z^n \in L^p(\mathscr{M}, \tau)$ if and only if $0 < p < \frac{2}{n+1}$.

1. Introduction

Let $\mathscr{M}$ be a finite von Neumann algebra with a faithful, normal, tracial state τ, and let

$$\Delta(T) = \exp\biggl(\int_0^\infty \log t \, d\mu_{|T|}(t)\biggr)$$

denote the corresponding Fuglede-Kadison determinant. L. G. Brown proved in [3] that for every $T \in \mathscr{M}$, there exists a unique, compactly supported measure $\mu_T \in \mathrm{Prob}(\mathsf{C})$ with the property that

$$\log \Delta(T - \lambda \mathbf{1}) = \int_{\mathsf{C}} \log|z - \lambda| \, d\mu_T(z), \qquad \lambda \in \mathsf{C}.$$

This measure is called Brown's spectral distribution measure (or just the Brown measure) of T. It was computed in a number of special cases in [9], [2], [5], and [1]. In particular, it was proven in [9, Theorem 4.5] that if $T \in \mathscr{M}$ is R-diagonal in the sense of Nica and Speicher [16], then μ_T can be determined from the S-transform of the distribution $\mu_{|T|^2}$. For simplicity, assume that $T \in \mathscr{M}$ is an R-diagonal element which is not proportional to a unitary and for which

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$\ker(T) = 0$. Then μ_T is the unique probability measure on C which is invariant under the rotations $z \mapsto \gamma z$, $\gamma \in \mathsf{T}$, and which satisfies

$$\mu_T\big(B\big(0, \mathscr{S}_{\mu_{|T|^2}}(t-1)^{-\frac{1}{2}}\big)\big) = t, \qquad 0 < t < 1.$$

In this paper we extend the Brown measure to all operators in the set $\mathscr{M}^\Delta$ of closed, densely defined operators T affiliated with $\mathscr{M}$ satisfying

$$\int_0^\infty \log^+ t \, d\mu_{|T|}(t) < \infty,$$

where $\log^+ t = \max\{\log t, 0\}$. Moreover, we extend [9, Theorem 4.5] to all R-diagonal operators in $\mathscr{M}^\Delta$. Finally, we will study a particular example of an unbounded R-diagonal element, namely the operator $z = xy^{-1}$, where (x, y) is a circular system in the sense of Voiculescu.

The material in this paper is organized as follows: In section 2 we introduce the class $\mathscr{M}^\Delta$ and generalize the Brown measure to all $T \in \mathscr{M}^\Delta$ by proving, that for such T, there is a unique $\mu_T \in \operatorname{Prob}(\mathsf{C})$ satisfying

$$\int_{\mathsf{C}} \log^+ |z| \, d\mu_T(z) < \infty$$

and

$$\log \Delta(T - \lambda \mathbf{1}) = \int_{\mathsf{C}} \log |z - \lambda| \, d\mu_T(z), \qquad \lambda \in \mathsf{C}.$$

Moreover, we extend Weil's inequality

$$\int_{\mathsf{C}} |z|^p \, d\mu_T(z) \leq \|T\|_p^p$$

to all $T \in L^p(\mathscr{M}, \tau)$. The main results in section 2 are stated in the appendix of Brown's paper [3] without proofs or with very sketchy proofs. Since the results of the remaining sections of this paper and of our forthcoming paper [10] rely heavily on these statements, we have decided to include complete proofs. We will follow a different route than the one outlined in [3]. For instance, we do not use the functions $\Lambda_t(T)$ and $s_T(t)$ from [3, section 1].

In section 3 we introduce unbounded R-diagonal operators and we prove the following generalization of [9, section 3]: The powers $(S^n)_{n=1}^\infty$ of an R-diagonal operator are R-diagonal, and the sum $S + T$ and the product ST of $*$-free R-diagonal operators are again R-diagonal. Moreover,

$$\begin{aligned}
\mu_{|S^n|^2} &= \mu_{|S|^2}^{\boxtimes n}, \\
\tilde{\mu}_{|S+T|} &= \tilde{\mu}_{|S|} \boxplus \tilde{\mu}_{|T|}, \\
\mu_{|ST|^2} &= \mu_{|S|^2} \boxtimes \mu_{|T|^2},
\end{aligned}$$

where $\tilde{\mu} = \frac{1}{2}(\mu + \check{\mu})$ denotes the symmetrization of a measure $\mu \in \mathrm{Prob}(\mathsf{R})$, and $\boxplus$ ($\boxtimes$, resp.) denotes the additive (multiplicative, resp.) free convolution of measures (cf. [4]). These results are applied in section 4 to determine the Brown measure of R-diagonal operators in $\mathcal{M}^{\Delta}$.

In section 5 we consider the operator $z = xy^{-1}$, where (x, y) is a circular system in the sense of Voiculescu, and we prove that the Brown measure of z is given by

$$d\mu_z(s) = \frac{1}{\pi(1+|s|^2)^2}\, d\,\mathrm{Re}\, s\, d\,\mathrm{Im}\, s.$$

Moreover, we show that for all $n \in \mathsf{N}$, $z^n, z^{-n} \in L^p(\mathcal{M}, \tau)$ iff $0 < p < \frac{2}{n+1}$, and when this holds,

$$\|z^n\|_p^p = \|z^{-n}\|_p^p = \frac{(n+1)\sin\left(\frac{\pi p}{2}\right)}{\sin\left(\frac{(n+1)\pi p}{2}\right)},$$

and

$$\|(z^n - \lambda \mathbf{1})^{-1}\|_p \leq \|z^{-n}\|_p, \qquad \lambda \in \mathsf{C}.$$

The last two formulas play a key role in our forthcoming paper [10] on invariant subspaces for operators in a general II_1-factor.

2. The Brown measure of certain unbounded operators

In [3, Appendix] Brown described in outline how to define a Brown measure for certain *unbounded* operators affiliated with $\mathcal{M}$, where $\mathcal{M}$ is a von Neumann algebra equipped with a faithful, normal, semifinite trace.

In this section we give a more detailed exposition on the subject in the case where $\mathcal{M}$ is a finite von Neumann algebra with faithful, tracial state τ. To be more explicit, we show how one can extend the definition of the Brown measure to a class $\mathcal{M}^{\Delta}$ of closed, densely defined operators affiliated with $\mathcal{M}$. We also prove that many of the properties of the Brown measure for bounded operators carry over to the unbounded case.

We let $\tilde{\mathcal{M}}$ denote the set of closed, densely defined operators affiliated with $\mathcal{M}$. Recall that every operator $T \in \tilde{\mathcal{M}}$ has a polar decomposition

$$T = U|T| = U \int_0^{\infty} t\, dE_{|T|}(t), \tag{2.1}$$

where $U \in \mathcal{M}$ is a unitary, and the spectral measure $E_{|T|}$ takes values in $\mathcal{M}$. In particular, for $T \in \tilde{\mathcal{M}}$ we may define $\mu_{|T|} \in \mathrm{Prob}(\mathsf{R})$ by

$$\mu_{|T|}(B) = \tau(E_{|T|}(B)), \qquad (B \in \mathsf{B}). \tag{2.2}$$

DEFINITION 2.1. We denote by $\mathcal{M}^{\Delta}$ the set of operators $T \in \tilde{\mathcal{M}}$ fulfilling the condition

$$\tau(\log^{+}|T|) = \int_0^{\infty} \log^{+}(t)\, d\mu_{|T|}(t) < \infty. \tag{2.3}$$

For $T \in \mathcal{M}^{\Delta}$, the integral

$$\int_0^{\infty} \log t\, d\mu_{|T|}(t) \in \mathsf{R} \cup \{-\infty\}$$

is well-defined, and we define the *Fuglede-Kadison determinant* of T, $\Delta(T) \in [0, \infty)$, by

$$\Delta(T) = \exp\Big(\int_0^{\infty} \log t\, d\mu_{|T|}(t)\Big). \tag{2.4}$$

Note that for $T \in \mathcal{M}$, $\Delta(T)$ is the usual Fuglede-Kadison determinant of T.

REMARK 2.2. If $T \in L^p(\mathcal{M}, \tau)$ for some $p \in (0, \infty)$, then

$$\int_0^{\infty} t^p\, d\mu_{|T|}(t) < \infty,$$

implying that

$$\int_1^{\infty} \log t\, d\mu_{|T|}(t) = \frac{1}{p}\int_1^{\infty} \log(t^p)\, d\mu_{|T|}(t) \le \frac{1}{p}\int_1^{\infty} t^p\, d\mu_{|T|}(t) < \infty,$$

and hence $T \in \mathcal{M}^{\Delta}$.

LEMMA 2.3. *If $T \in \mathcal{M}^{\Delta}$ and $\Delta(T) > 0$, then T is invertible in $\tilde{\mathcal{M}}$, $T^{-1} \in \mathcal{M}^{\Delta}$, and $\Delta(T^{-1}) = \frac{1}{\Delta(T)}$.*

PROOF. If $T \in \mathcal{M}^{\Delta}$ and $\Delta(T) > 0$, then

$$\int_0^{1} |\log t|\, d\mu_{|T|}(t) < \infty.$$

Hence, $\tau(E_{|T|}(\{0\})) = \mu_{|T|}(\{0\}) = 0$, so that $\ker(T) = \{0\}$. Since $\mathcal{M}$ is finite, also $\ker(T^*) = \{0\}$, which implies that T has a closed, densely defined inverse $T^{-1} \in \tilde{\mathcal{M}}$. Take a unitary $U \in \mathcal{M}$ such that $T = U|T|$. Then

$$|T^{-1}| = U|T|^{-1}U^*.$$

Hence, $\mu_{|T^{-1}|} = \mu_{|T|^{-1}}$. Since $\mu_{|T|^{-1}}$ is the push-forward measure of $\mu_{|T|}$ via the map $t \mapsto \frac{1}{t}$, we now have that

$$\int_1^\infty \log t \, d\mu_{|T^{-1}|}(t) = \int_1^\infty \log t \, d\mu_{|T|^{-1}}(t) = \int_0^1 \log\left(\frac{1}{t}\right) d\mu_{|T|}(t)$$
$$= -\int_0^1 \log t \, d\mu_{|T|}(t) < \infty.$$

Hence, $T^{-1} \in \mathcal{M}^\Delta$ and

$$\log \Delta(T^{-1}) = \int_0^\infty \log\left(\frac{1}{t}\right) d\mu_{|T|}(t) = -\log \Delta(T),$$

i.e. $\Delta(T^{-1}) = \frac{1}{\Delta(T)}$.

LEMMA 2.4. *Let $T \in \tilde{\mathcal{M}}$. Then the following are equivalent:*

(a) *$T \in \mathcal{M}^\Delta$, i.e. $\int_0^\infty \log^+(t)\, d\mu_{|T|}(t) < \infty$.*

(b) *$T = AB^{-1}$ for some $A, B \in \mathcal{M}$ with $\Delta(B) > 0$.*

(c) *$T = C^{-1}D$ for some $C, D \in \mathcal{M}$ with $\Delta(C) > 0$.*

Moreover, if $T \in \mathcal{M}^\Delta$ and $T = AB^{-1} = C^{-1}D$ for some $A, B, C, D \in \mathcal{M}$ with $\Delta(B), \Delta(C) > 0$, then

$$\Delta(T) = \frac{\Delta(A)}{\Delta(B)} = \frac{\Delta(D)}{\Delta(C)}. \tag{2.5}$$

PROOF. If $T \in \mathcal{M}^\Delta$, then $T = U|T|$ for some unitary $U \in \mathcal{M}$, and $T = AB^{-1}$, where

$$A = U|T|(|T|^2 + \mathbf{1})^{-\frac{1}{2}} \in \mathcal{M} \tag{2.6}$$

and

$$B = (|T|^2 + \mathbf{1})^{-\frac{1}{2}} \in \mathcal{M}. \tag{2.7}$$

Since $\frac{1}{2}\log(t^2+1) \le \log(2t)$ when $t \ge 1$, we get that

(2.8)
$$\log \Delta(B) = -\frac{1}{2}\int_0^\infty \log(t^2+1)\, d\mu_{|T|}(t)$$
$$\ge -\frac{1}{2}\int_{[0,1[} \log 2 \, d\mu_{|T|}(t) - \int_{[1,\infty[} \log(2t)\, d\mu_{|T|}(t) > -\infty,$$

that is, $\Delta(B) > 0$.

Also, $T = U|T|U^*U$, and with

$$S = U|T|U^*, \tag{2.9}$$

$$C = (S^2 + \mathbf{1})^{-\frac{1}{2}} \in \mathcal{M}, \tag{2.10}$$

and

$$D = S(S^2 + \mathbf{1})^{-\frac{1}{2}} \in \mathcal{M}, \tag{2.11}$$

we have that $T = C^{-1}DU$. Moreover,

$$\begin{aligned} \log \Delta(C) &= -\frac{1}{2}\int_0^\infty \log(t^2+1)\, d\mu_S(t) \\ &= -\frac{1}{2}\int_0^\infty \log(t^2+1)\, d\mu_{|T|}(t) > -\infty, \end{aligned}$$

i.e. $\Delta(C) > 0$.

Now we have shown that (a) implies (b) and (c). On the other hand, if $T = AB^{-1}$ for some $A, B \in \mathcal{M}$ with $\Delta(B) > 0$, then we may assume that $B \geq 0$. Then

$$\tau(\log^+ |T|) \leq \tau(\log(\mathbf{1} + |T|^2)) = \tau(\log(\mathbf{1} + B^{-1}A^*AB^{-1})).$$

Since $B^{-1}A^*AB^{-1} \leq \|A\|^2 B^{-2}$, and since $t \mapsto \log(1+t)$ is operator monotone on $[0, \infty)$, we get that

$$\begin{aligned} \tau(\log^+ |T|) &\leq \tau(\log(\mathbf{1} + \|A\|^2 B^{-2})) \\ &\leq \tau(\log((1 + \|A\|^2)(\mathbf{1} + B^{-2}))) \\ &= \log(1 + \|A\|^2) + \tau(\log(\mathbf{1} + B^{-2})). \end{aligned}$$

Since B is bounded and $\Delta(B) > 0$,

$$\begin{aligned} \tau(\log(\mathbf{1} + B^{-2})) &= \tau(\log(B^2 + \mathbf{1})) - 2\tau(\log B) \\ &\leq \log(\|B\|^2 + 1) - 2\Delta(B) \\ &< \infty. \end{aligned}$$

This shows that $T \in \mathcal{M}^\Delta$, i.e. (b) implies (a). It follows that if $T = C^{-1}D$ for some $C, D \in \mathcal{M}$ with $\Delta(C) > 0$, then $T^* \in \mathcal{M}^\Delta$. Take a unitary $U \in \mathcal{M}$ such that $T = U|T|$. Then $|T^*| = U|T|U^*$, implying that $\mu_{|T^*|} = \mu_{|T|}$. Hence T belongs to $\mathcal{M}^\Delta$ as well, and (c) implies (a).

Now, let $T \in \mathcal{M}^{\Delta}$. Then $T = AB^{-1} = C^{-1}D$ for some $A, B, C, D \in \mathcal{M}$ with $\Delta(B), \Delta(C) > 0$. Moreover, for all such choices of A, B, C and D,

$$CA = C(AB^{-1})B = C(C^{-1}D)B = DB.$$

Since Δ is multiplicative on $\mathcal{M}$ (cf. [7]), it follows that

$$\Delta(C)\Delta(A) = \Delta(CA) = \Delta(DB) = \Delta(D)\Delta(B).$$

Hence,

$$\frac{\Delta(A)}{\Delta(B)} = \frac{\Delta(D)}{\Delta(C)}. \tag{2.12}$$

In particular, with A and B as in (2.6) and (2.7), respectively, we have that $\Delta(B) > 0$, $T = AB^{-1}$, and

$$\log \Delta(A) = \int_0^{\infty} \log\left(\frac{t}{\sqrt{t^2+1}}\right) d\mu_{|T|}(t),$$

and

$$\log \Delta(B) = \int_0^{\infty} \log\left(\frac{1}{\sqrt{t^2+1}}\right) d\mu_{|T|}(t),$$

so that

$$\log \Delta(T) = \log \Delta(A) - \log \Delta(B).$$

Then by (2.12), for all choices of $C, D \in \mathcal{M}$ with $\Delta(C) > 0$ and $T = C^{-1}D$,

$$\frac{\Delta(D)}{\Delta(C)} = \frac{\Delta(A)}{\Delta(B)} = \Delta(T).$$

Then finally, by (2.12), for all choices of $A, B \in \mathcal{M}$ with $\Delta(B) > 0$ and $T = AB^{-1}$, we also have that

$$\frac{\Delta(A)}{\Delta(B)} = \Delta(T).$$

PROPOSITION 2.5. *If $S, T \in \mathcal{M}^{\Delta}$, then $ST \in \mathcal{M}^{\Delta}$, and*

$$\Delta(ST) = \Delta(S)\Delta(T). \tag{2.13}$$

PROOF. Let $S, T \in \mathcal{M}^{\Delta}$. Take $A, B, C, D \in \mathcal{M}$ with $\Delta(B), \Delta(C) > 0$, such that $T = AB^{-1}$ and $S = C^{-1}D$. Then

$$ST = C^{-1}DAB^{-1},$$

where $DAB^{-1} \in \mathcal{M}^\Delta$. Hence there exist $E, F \in \mathcal{M}$ with $\Delta(E) > 0$ such that $DAB^{-1} = E^{-1}F$. It follows that

$$ST = C^{-1}E^{-1}F = (EC)^{-1}F, \tag{2.14}$$

where $EC, F \in \mathcal{M}$, and $\Delta(EC) = \Delta(E)\Delta(C) > 0$. That is, ST belongs to $\mathcal{M}^\Delta$.

To prove (2.13), we let A, B, C, D, E, F be as above. Applying (2.5) to $ST = (EC)^{-1}F$, $S = C^{-1}D$ and $T = AB^{-1}$, we get that

$$\begin{aligned}\Delta(ST) &= \frac{\Delta(F)}{\Delta(EC)} = \frac{\Delta(F)}{\Delta(E)\Delta(C)} = \frac{\Delta(DA)}{\Delta(B)}\frac{1}{\Delta(C)} \\ &= \frac{\Delta(A)}{\Delta(B)}\frac{\Delta(D)}{\Delta(C)} = \Delta(S)\Delta(T).\end{aligned}$$

PROPOSITION 2.6. *$\mathcal{M}^\Delta$ is a subspace of $\tilde{\mathcal{M}}$. In particular, for $T \in \mathcal{M}^\Delta$ and $\lambda \in \mathsf{C}$, $T - \lambda\mathbf{1} \in \mathcal{M}^\Delta$.*

PROOF. Clearly, if $T \in \mathcal{M}^\Delta$ and $\alpha \in \mathsf{C}$, then $\alpha T \in \mathcal{M}^\Delta$. If $S, T \in \mathcal{M}$, choose $A, B, C, D \in \mathcal{M}$ with $\Delta(B) > 0$, $\Delta(C) > 0$ and such that

$$S = C^{-1}D, \qquad T = AB^{-1}.$$

Then

$$S + T = C^{-1}(DB + CA)B^{-1},$$

where $DB + CA \in \mathcal{M}$ and $B^{-1}, C^{-1} \in \mathcal{M}^\Delta$ (cf. Lemma 2.3). Then, by Proposition 2.5, $S + T \in \mathcal{M}^\Delta$.

In the following we consider a fixed operator $T \in \mathcal{M}^\Delta$. Then we define $f : \mathsf{C} \to [-\infty, \infty)$ by

$$f(\lambda) = L(T - \lambda\mathbf{1}) := \log\Delta(T - \lambda\mathbf{1}), \qquad (\lambda \in \mathsf{C}). \tag{2.15}$$

The next thing we want to prove is:

THEOREM 2.7. *f given by* (2.15) *is subharmonic in C, and*

$$d\mu_T = \frac{1}{2\pi}\nabla^2 f\, d\lambda \tag{2.16}$$

(taken in the distribution sense) defines a probability measure on $(\mathsf{C}, \mathsf{B}_2)$. μ_T is the unique probability measure on $(\mathsf{C}, \mathsf{B}_2)$ satisfying

(i)
$$\int_{\mathsf{C}} \log^+ |z|\, d\mu_T(z) < \infty,$$

(ii)

$$\forall \lambda \in \mathsf{C}: \quad L(T - \lambda \mathbf{1}) = \int_{\mathsf{C}} \log|\lambda - z| \, d\mu_T(z). \tag{2.17}$$

Moreover,

(iii)

$$\int_{\mathsf{C}} \log^+ |z| \, d\mu_T(z) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \, d\theta. \tag{2.18}$$

The following lemma was proven by F. Larsen in his unpublished thesis (cf. [14, section 2]). For the convenience of the reader we include a (somewhat different) proof.

LEMMA 2.8. *Let $a, b \in \mathcal{M}$ and let $\varepsilon > 0$. Define $g_\varepsilon, g : \mathsf{C} \to \mathsf{R}$ by*

$$g_\varepsilon(\lambda) = \tfrac{1}{2}\tau(\log((a - \lambda b)^*(a - \lambda b) + \varepsilon \mathbf{1})),$$

and

$$g(\lambda) = \log \Delta(a - \lambda b).$$

Then g_ε is subharmonic, and if $g(\lambda) > -\infty$ for some $\lambda \in \mathsf{C}$, then g is subharmonic as well.

PROOF. Let $\lambda_1 = \mathrm{Re}(\lambda)$, $\lambda_2 = \mathrm{Im}(\lambda)$, $\lambda \in \mathsf{C}$. At first we show that $(\lambda_1, \lambda_2) \mapsto g_\varepsilon(\lambda_1 + i\lambda_2)$ is a C^2-function in R^2. Fix $\varepsilon > 0$, and define $h, k : \mathsf{C} \to \mathcal{M}$ by

$$h(\lambda) = (a - \lambda b)^*(a - \lambda b) + \varepsilon \mathbf{1},$$
$$k(\lambda) = (a - \lambda b)(a - \lambda b)^* + \varepsilon \mathbf{1}.$$

Then h and k are second order polynomials in (λ_1, λ_2) with coefficients in $\mathcal{M}$, and $h(\lambda) \geq \varepsilon \mathbf{1}$, $k(\lambda) \geq \varepsilon \mathbf{1}$ for all $\lambda \in \mathsf{C}$. Hence, by [11, Lemma 4.6],

$$g_\varepsilon(\lambda) = \tfrac{1}{2}\tau(\log h(\lambda)), \qquad \lambda \in \mathsf{C},$$

has continuous partial derivatives given by

$$\frac{\partial g_\varepsilon}{\partial \lambda_j} = \frac{1}{2}\tau\left(h^{-1}\frac{\partial h}{\partial \lambda_j}\right), \qquad j = 1, 2.$$

Therefore, by [11, Lemma 3.2], g_ε is a C^2-function with

$$\frac{\partial^2 g_\varepsilon}{\partial \lambda_i \partial \lambda_j} = \frac{1}{2}\tau\left(-h^{-1}\frac{\partial h}{\partial \lambda_i}h^{-1}\frac{\partial h}{\partial \lambda_j} + h^{-1}\frac{\partial^2 h}{\partial \lambda_i \partial \lambda_j}\right), \qquad i = 1, 2, \quad j = 1, 2. \tag{2.19}$$

Since g_ε is C^2, g_ε is subharmonic if and only if its Laplacian

$$\frac{\partial^2 g_\varepsilon}{\partial \lambda_1^2} + \frac{\partial^2 g_\varepsilon}{\partial \lambda_2^2}$$

is positive. Following standard notation, we let

$$\frac{\partial}{\partial \lambda} = \frac{1}{2}\left(\frac{\partial}{\partial \lambda_1} - i\frac{\partial}{\partial \lambda_2}\right) \qquad \text{and} \qquad \frac{\partial}{\partial \overline{\lambda}} = \frac{1}{2}\left(\frac{\partial}{\partial \lambda_1} + i\frac{\partial}{\partial \lambda_2}\right).$$

Then

$$\frac{\partial^2 g_\varepsilon}{\partial \lambda_1^2} + \frac{\partial^2 g_\varepsilon}{\partial \lambda_2^2} = 4\frac{\partial^2 g_\varepsilon}{\partial \overline{\lambda}\partial \lambda}$$

By application of (2.19), we find that

$$\frac{\partial^2 g_\varepsilon}{\partial \overline{\lambda}\partial \lambda} = \frac{1}{2}\tau\left(-h^{-1}\frac{\partial h}{\partial \overline{\lambda}}h^{-1}\frac{\partial h}{\partial \lambda} + h^{-1}\frac{\partial^2 h}{\partial \overline{\lambda}\partial \lambda}\right). \tag{2.20}$$

Since

$$h(\lambda) = a^*a - \lambda a^*b - \overline{\lambda}b^*a + |\lambda|^2 b^*b + \varepsilon\mathbf{1},$$

we have

$$\frac{\partial h}{\partial \lambda} = -a^*b + \overline{\lambda}b^*b = -(a-\lambda b)^*b,$$

$$\frac{\partial h}{\partial \overline{\lambda}} = -b^*a + \lambda b^*b = -b^*(a-\lambda b),$$

and

$$\frac{\partial^2 h}{\partial \overline{\lambda}\partial \lambda} = b^*b.$$

Applying the identity $x(x^*x+\varepsilon\mathbf{1})^{-1} = (xx^*+\varepsilon\mathbf{1})^{-1}x$ to $x = a-\lambda b$, we find that

$$\begin{aligned}\frac{\partial^2 h}{\partial \overline{\lambda}\partial \lambda} - \frac{\partial h}{\partial \overline{\lambda}}h^{-1}\frac{\partial h}{\partial \lambda} &= b^*b - b^*x(x^*x+\varepsilon\mathbf{1})^{-1}x^*b \\ &= b^*b - b^*(xx^*+\varepsilon\mathbf{1})^{-1}xx^*b \\ &= b^*b - b^*(\mathbf{1} - \varepsilon(xx^*+\varepsilon\mathbf{1})^{-1})b \\ &= \varepsilon b^*(xx^*+\varepsilon\mathbf{1})^{-1}b \\ &= \varepsilon b^*k^{-1}b.\end{aligned}$$

Then by (2.20),

$$\begin{aligned}\frac{\partial^2 g_\varepsilon}{\partial\bar{\lambda}\partial\lambda} &= \frac{\varepsilon}{2}\tau\big(h(\lambda)^{-1}b^*k(\lambda)^{-1}b\big)\\ &= \frac{\varepsilon}{2}\tau\big(h(\lambda)^{-\frac{1}{2}}b^*k(\lambda)^{-1}bh(\lambda)^{-\frac{1}{2}}\big) \geq 0,\end{aligned}$$

showing that g_ε is subharmonic.

Fix $\lambda \in \mathsf{C}$, and let $x = a - \lambda b$ as above. Then

$$g_\varepsilon(\lambda) = \frac{1}{2}\int_0^{\|x\|} \log(t^2 + \varepsilon)\, d\mu_{|x|}(t),$$

and

$$g(\lambda) = \frac{1}{2}\int_0^{\|x\|} \log(t^2)\, d\mu_{|x|}(t).$$

Hence, g_ε is a monotonically decreasing function of $\varepsilon > 0$, and

$$g(\lambda) = \lim_{\varepsilon\to 0^+} g_\varepsilon(\lambda).$$

According to [13], g is then either subharmonic or identically $-\infty$.

Proposition 2.9. *Let $T \in \mathcal{M}^\Delta$. Then the function $f : \mathsf{C} \to [-\infty, \infty[$ given by*

$$f(\lambda) = \log \Delta(T - \lambda\mathbf{1})$$

is subharmonic in C.

Proof. Define $T_1, T_2 \in \mathcal{M}$ by

$$T_1 = T(T^*T + \mathbf{1})^{-\frac{1}{2}} \tag{2.21}$$

and

$$T_2 = (T^*T + \mathbf{1})^{-\frac{1}{2}}. \tag{2.22}$$

Then for every $\lambda \in \mathsf{C}$,

$$T - \lambda\mathbf{1} = (T_1 - \lambda T_2)T_2^{-1},$$

where $\Delta(T_2) > 0$ (cf. (2.8)). Thus, $T - \lambda\mathbf{1} \in \mathcal{M}^\Delta$ with

$$\Delta(T - \lambda\mathbf{1}) = \Delta(T_1 - \lambda T_2)\Delta(T_2)^{-1},$$

i.e.

$$f(\lambda) = L(T - \lambda\mathbf{1}) = L(T_1 - \lambda T_2) - L(T_2). \tag{2.23}$$

Then by Lemma 2.8, f is either subharmonic or identically $-\infty$. With

$$h(\lambda) = L(T_2 - \lambda T_1) - L(T_2),$$

$h(0) = 0 > -\infty$, and it follows from Lemma 2.8 that h is subharmonic. In particular, $h(\lambda) > -\infty$ for almost every $\lambda \in \mathbb{C}$ w.r.t. Lebesgue measure. For $\lambda \in \mathbb{C} \setminus \{0\}$,

$$f(\lambda) = h\left(\frac{1}{\lambda}\right) + \log|\lambda|.$$

Hence, f is not identically $-\infty$.

Recall from [13, Section 3.5.4] that one can associate to every subharmonic function u the socalled *Riesz measure* μ_u, which is a positive Borel measure on $\mathbb{R}^2$ uniquely determined by

$$\forall \phi \in C_c^\infty(\mathbb{R}^2): \quad \frac{1}{2\pi}\int_{\mathbb{R}^2} u\nabla^2\phi\, dm = \int_{\mathbb{R}^2} \phi\, d\mu_u. \tag{2.24}$$

One uses the notation $d\mu_u = \frac{1}{2\pi}\nabla^2 u\, d\lambda$, and this is what is meant by (2.16).

In order to prove the rest of Theorem 2.7, we need some general lemmas on subharmonic functions:

LEMMA 2.10. *Let* $g : \mathbb{C} \to [-\infty, \infty[$ *be a subharmonic function, and for* $r > 0$ *define*

$$m(g, r) = \frac{1}{2\pi}\int_0^{2\pi} g(re^{i\theta})\, d\theta, \tag{2.25}$$

$$M(g, r) = \sup_{|z|=r} g(z). \tag{2.26}$$

Then

$$g(0) = \lim_{r\to 0} m(g, r) = \lim_{r\to 0} M(g, r). \tag{2.27}$$

PROOF. Clearly, $m(g, r) \le M(g, r)$ for every $r > 0$. Moreover, since g is subharmonic, $g(0) \le m(g, r)$, $(r > 0)$. It follows that

$$g(0) \le \begin{cases} \limsup_{r\to 0} m(g, r) \le \limsup_{r\to 0} M(g, r) \\ \liminf_{r\to 0} m(g, r) \le \liminf_{r\to 0} M(g, r) \end{cases} \tag{2.28}$$

Now, every upper semicontinuous function attains a maximum on every compact set. In particular, there exists for every $r > 0$ a complex number z_r of modulus r such that $g(z_r) = M(g, r)$. $z_r \to 0$ as $r \to 0$, and therefore

$$g(0) \ge \limsup_{r\to 0} g(z_r) = \limsup_{r\to 0} M(g, r). \tag{2.29}$$

It follows from (2.28) and (2.29) that

$$g(0) \le \liminf_{r\to 0} m(g,r) \le \left\{ \begin{matrix} \limsup_{r\to 0} m(g,r) \\ \liminf_{r\to 0} M(g,r) \end{matrix} \right\} \le \limsup_{r\to 0} M(g,r) \le g(0),$$

so the four inequalities above are in fact identities, and this proves (2.27).

LEMMA 2.11. *f given by (2.15) satisfies*

$$\lim_{r\to\infty} (M(f,r) - \log r) = \lim_{r\to\infty} (m(f,r) - \log r) = 0. \tag{2.30}$$

PROOF. Define $h : \mathsf{C} \to [-\infty, \infty[$ by

$$h(\lambda) = L(T_2 - \lambda T_1) - L(T_2), \qquad \lambda \in \mathsf{C}. \tag{2.31}$$

Then h is subharmonic with $h(0) = 0$, and it follows from Lemma 2.10 that

$$0 = \lim_{r\to 0} m(h,r) = \lim_{r\to 0} M(h,r). \tag{2.32}$$

Since

$$h(\lambda) = \log|\lambda| + f(\tfrac{1}{\lambda}), \qquad \lambda \ne 0, \tag{2.33}$$

we get that when $r > 0$,

$$M(f,r) = M\big(h, \tfrac{1}{r}\big) + \log r,$$
$$m(f,r) = m\big(h, \tfrac{1}{r}\big) + \log r,$$

and combining this with (2.32) we obtain the desired result.

LEMMA 2.12. *Let $R > r > 0$, and let g be subharmonic in C. Then with $d\mu = \frac{1}{2\pi}\nabla^2 g\, d\lambda$ and*

$$\psi(z) = \begin{cases} \log\left(\frac{R}{r}\right), & |z| \le r \\ \log\big(\frac{R}{|z|}\big), & r < |z| < R \\ 0, & |z| \ge R \end{cases}$$

one has that

$$m(g,R) - m(g,r) = \int_{\mathsf{C}} \psi(z)\, d\mu(z). \tag{2.34}$$

PROOF. Cf. [13, (3.5.7)].

PROOF OF THEOREM 2.7. When $R > 1 > 0$ define $\psi_R : \mathsf{C} \to \mathsf{R}$ by

$$\psi_R(z) = \begin{cases} \log R, & |z| \le 1 \\ \log\left(\frac{R}{|z|}\right), & 1 < |z| < R \\ 0, & |z| \ge R \end{cases}$$

Then, according to Lemma 2.12,

$$\int_{\mathsf{C}} \psi_R(z)\, d\mu_T(z) = m(f, R) - m(f, 1). \tag{2.35}$$

Now, $\frac{1}{\log R}\psi_R \nearrow 1$ as $R \to \infty$, so by the Monotone Convergence Theorem, (2.35) and Lemma 2.11,

$$\mu_T(\mathsf{C}) = \lim_{R\to\infty} \frac{m(f, R) - m(f, 1)}{\log R} = 1,$$

that is, μ_T is a probability measure.

When $R > 1$, let

$$\omega_R(z) = \log R - \psi_R(z), \qquad z \in \mathsf{C}. \tag{2.36}$$

Then $\omega_R(z) \nearrow \log^+ |z|$ as $R \to \infty$, and hence by one more application of Lemma 2.11,

$$\int_{\mathsf{C}} \log^+ |z|\, d\mu_T(z) = \lim_{R\to\infty} \int_{\mathsf{C}} \omega_R\, d\mu_T = \lim_{R\to\infty} (\log R - m(f, R) + m(f, 1))$$
$$= m(f, 1),$$

proving (2.18). Note that since f is subharmonic, (2.18) imlies that $\int_{\mathsf{C}} \log^+ |z|\, d\mu_T(z) < \infty$.

To see that (2.17) holds, it suffices to consider the case $\lambda = 0$. Indeed, for fixed $\lambda \in \mathsf{C}$ one easily sees that $\mu_{T-\lambda\mathbf{1}}$ is the push-forward measure of μ_T under the map $z \mapsto z - \lambda$ (cf. Lemma 2.14), and therefore

$$\int_{\mathsf{C}} \log |z - \lambda|\, d\mu_T(z) = \int_{\mathsf{C}} \log |z|\, d\mu_{T-\lambda\mathbf{1}}(z). \tag{2.37}$$

In the case $\lambda = 0$ one has to compute the integrals $\int_{\mathsf{C}} \log^{\pm} |z|\, d\mu_T(z)$. We have just seen that

$$\int_{\mathsf{C}} \log^+ |z|\, d\mu_T(z) = m(f, 1), \tag{2.38}$$

and with

$$\chi_r(z) = \begin{cases} \log \frac{1}{r}, & |z| \le r \\ \log \frac{1}{|z|}, & r < |z| \le 1 \\ 0, & |z| \ge 1 \end{cases}$$

$\chi_r(z) \nearrow \log^- |z|$ as $r \searrow 0$. Hence by Lemma 2.10 and Lemma 2.12,

$$\int_{\mathbf{C}} \log^- |z| \, d\mu_T(z) = \lim_{r\to 0} \int_{\mathbf{C}} \chi_r \, d\mu_T = \lim_{r\to 0}(m(f,1) - m(f,r))$$
$$= m(f,1) - f(0).$$

Combining this with (2.38) we get that

$$\int_{\mathbf{C}} \log |z| \, d\mu_T(z) = f(0) = L(T),$$

as desired.

In order to prove that μ_T is uniquely determined by (i) and (ii) of Theorem 2.7, suppose $\nu \in \mathrm{Prob}(\mathbf{C})$ satisfies

$$\int_{\mathbf{C}} \log^+ |z| \, d\nu(z) < \infty, \tag{2.39}$$

and

$$\forall \lambda \in \mathbf{C}: \quad \int_{\mathbf{C}} \log |z - \lambda| \, d\nu(z) = L(T - \lambda \mathbf{1}). \tag{2.40}$$

Note that (2.39) implies that $\int_{\mathbf{C}} \log |z - \lambda| \, d\nu(z)$ is well-defined, since

$$\log |z - \lambda| \le \log(|z| + |\lambda|),$$

and

$$|z| + |\lambda| \le (|\lambda| + 1) \cdot \max\{1, |z|\}.$$

Hence

$$\log |z - \lambda| \le \log(|\lambda| + 1) + \log^+ |z|. \tag{2.41}$$

Since μ and ν are both probability measures, it follows from a C^∞-version of Urysohn's Lemma (cf. [8, (8.18)]) that if

$$\int_{\mathbf{C}} \phi \, d\mu_T = \int_{\mathbf{C}} \phi \, d\nu$$

for every function $\phi \in C_c^\infty(\mathsf{R}^2)$, then $\mu_T = \nu$. Then consider an arbitrary function $\phi \in C_c^\infty(\mathsf{R}^2)$. Since the Laplacian of $w \mapsto \frac{1}{2\pi}\log|w-z|$ (in the distribution sense) is the Dirac measure δ_z at z, one has that

$$\begin{aligned}\int_{\mathsf{C}} \phi(z)\,d\nu(z) &= \int_{\mathsf{C}}\left(\int_{\mathsf{C}} \phi(\lambda)\,\delta_z(\lambda)\right) d\nu(z)\\ &= \frac{1}{2\pi}\int_{\mathsf{C}}\left(\int_{\mathsf{C}} (\nabla^2\phi)(\lambda)\log|z-\lambda|\,d\lambda\right) d\nu(z).\end{aligned} \tag{2.42}$$

At this place we would like to reverse the order of integration, but it is not entirely clear that this is a legal operation. Therefore we put $M = \|\nabla^2\phi\|_\infty$, and take $\chi \in C_c^\infty(\mathsf{R}^2)$ such that $0 \le \chi \le 1$ and $\chi|_{\mathrm{supp}(\nabla^2\phi)} = 1$. With

$$\psi_1 = \frac{1}{2}(M + \nabla^2\phi)\chi$$

and

$$\psi_2 = \frac{1}{2}(M - \nabla^2\phi)\chi$$

one has that $\psi_1, \psi_2 \in C_c^\infty(\mathsf{R}^2)^+$, and $\nabla^2\phi = \psi_1 - \psi_2$.

Also not that, according to (2.41),

$$h(\lambda, z) := \log(|\lambda|+1) + \log^+|z| - \log|z-\lambda| \ge 0.$$

Therefore by Tonelli's Theorem

$$\int_{\mathsf{C}} \psi_i(\lambda)\int_{\mathsf{C}} h(\lambda,z)\,d\nu(z)\,d\lambda = \int_{\mathsf{C}}\int_{\mathsf{C}} \psi_i(\lambda)h(\lambda,z)\,d\lambda\,d\nu(z), \qquad i = 1,2. \tag{2.43}$$

The map $\lambda \mapsto L(T-\lambda\mathbf{1})$ is subharmonic and therefore locally integrable. Since

$$\int_{\mathsf{C}} h(\lambda,z)\,d\nu(z) = \log(|\lambda|+1) + \int_{\mathsf{C}} \log^+|z|\,d\nu(z) - L(T-\lambda\mathbf{1}),$$

where $\lambda \mapsto L(T-\lambda\mathbf{1})$ is subharmonic and therefore locally integrable,

$$\int_{\mathsf{C}} \psi_i(\lambda)\int_{\mathsf{C}} h(\lambda,z)\,d\nu(z)\,d\lambda < \infty, \qquad i = 1,2.$$

It now follows from (2.43) that

$$\int_{\mathsf{C}} (\nabla^2\phi)(\lambda)\int_{\mathsf{C}} h(\lambda,z)\,d\nu(z)\,d\lambda = \int_{\mathsf{C}}\int_{\mathsf{C}} (\nabla^2\phi)(\lambda)h(\lambda,z)\,d\lambda\,d\nu(z),$$

and since

$$\int_{\mathbb{C}} |(\nabla^2\phi)(\lambda)| \int_{\mathbb{C}} \log(|\lambda| + 1)\, d\nu(z)\, d\lambda < \infty,$$

and

$$\int_{\mathbb{C}} |(\nabla^2\phi)(\lambda)| \int_{\mathbb{C}} \log^+ |z|\, d\nu(z)\, d\lambda < \infty,$$

we deduce that

$$\begin{aligned}\int_{\mathbb{C}} \phi(z)\, d\nu(z) &= \frac{1}{2\pi} \int_{\mathbb{C}}\left(\int_{\mathbb{C}} (\nabla^2\phi)(\lambda) \log|\lambda - z|\, d\lambda\right) d\nu(z) \\ &= \frac{1}{2\pi} \int_{\mathbb{C}} (\nabla^2\phi)(\lambda) \int_{\mathbb{C}} \log|\lambda - z|\, d\nu(z)\, d\lambda \\ &= \frac{1}{2\pi} \int_{\mathbb{C}} (\nabla^2\phi)(\lambda) L(T - \lambda\mathbf{1})\, d\lambda \\ &= \int_{\mathbb{C}} \phi(z)\, d\mu_T(z),\end{aligned}$$

and this is the desired identity.

It follows from Theorem 2.7 that one can associate to every operator $T \in \mathcal{M}^\Delta$ a probability measure μ_T on $(\mathbb{C}, \mathbb{B}_2)$, such that in the case where $T \in \mathcal{M}$, μ_T agrees with the Brown measure of T. Therefore we make the following definition:

DEFINITION 2.13. For $T \in \mathcal{M}^\Delta$ we shall say that the probability measure μ_T from Theorem 2.7 is the *Brown measure* of T.

In the remaining part of this section we will see that many of the properties of the Brown measure for bounded operators carry over to this more general setting.

PROPOSITION 2.14. *Let $T \in \mathcal{M}^\Delta$. Then for every $r > 0$ and every $\lambda \in \mathbb{C}$, the Brown measure of $rT + \lambda\mathbf{1}$, $\mu_{rT+\lambda\mathbf{1}}$, is the push-forward measure of μ_T via the map $z \mapsto rz + \lambda$.*

PROOF. Making use of Urysohn's Lemma for C^∞-functions on $\mathbb{R}^2$ (cf. [8, (8.18)]) and the fact that both of the measures considered here are probability measures, one easily sees that if

$$\int_{\mathbb{C}} \phi(z)\, d\mu_{rT+\lambda\mathbf{1}}(z) = \int_{\mathbb{C}} \phi(rz + \lambda)\, d\mu_T(z)$$

for every $\phi \in C_c^\infty(\mathbb{R}^2)$, then the two measures in speak agree on compact sets and hence on all of $\mathbb{B}_2$.

Let $\phi \in C_c^\infty(\mathbb{R}^2)$. Then by definition,

$$\begin{aligned}\int_{\mathbb{C}} \phi(rz+\lambda)\,d\mu_T(z) &= \frac{1}{2\pi}\int_{\mathbb{C}}\Big(\frac{\partial^2}{\partial z_1^2}+\frac{\partial^2}{\partial z_2^2}\Big)\phi(rz+\lambda)\,f(z)\,dz\\ &= \frac{1}{2\pi}\int_{\mathbb{C}} r^2\Big(\frac{\partial^2}{\partial w_1^2}+\frac{\partial^2}{\partial w_2^2}\Big)\phi(w)f\Big(\frac{1}{r}(w-\lambda)\Big)\frac{1}{r^2}\,dw\\ &= \frac{1}{2\pi}\int_{\mathbb{C}} \nabla^2\phi(w)f\Big(\frac{1}{r}(w-\lambda)\Big)\,dw\\ &= \frac{1}{2\pi}\int_{\mathbb{C}} \nabla^2\phi(w)\,[L(rT+\lambda\mathbf{1}-w\mathbf{1})-\log r]\,dw\\ &= \int_{\mathbb{C}}\phi(w)\,d\mu_{rT+\lambda\mathbf{1}}(w)-\log r\cdot\int_{\mathbb{C}}\nabla^2\phi(w)\,dw\\ &= \int_{\mathbb{C}}\phi(w)\,d\mu_{rT+\lambda\mathbf{1}}(w),\end{aligned}$$

where the last identity follows from Green's Theorem.

PROPOSITION 2.15. *For every $T \in \mathcal{M}^\Delta$ and every $m \in \mathbb{N}$, μ_{T^m} is the push-forward measure of μ_T via the map $z \mapsto z^m$.*

PROOF. Let $\nu \in \mathrm{Prob}(\mathbb{C})$ denote the push-forward measure of μ_T under the map $z \mapsto z^m$. According to Theorem 2.7 it suffices to prove that

$$\int_{\mathbb{C}} \log^+|z|\,d\nu(z) < \infty,$$

and

$$\forall\,\lambda \in \mathbb{C}: \qquad \int_{\mathbb{C}} \log|\lambda - z|\,d\nu(z) = L(T^m-\lambda\mathbf{1}).$$

Here

$$\int_{\mathbb{C}} \log^+|z|\,d\nu(z) = \int_{\mathbb{C}}\log^+|z^m|\,d\mu_T(z) = m\int_{\mathbb{C}}\log^+|z|\,d\mu_T(z) < \infty,$$

and if we let $\theta_1,\dots,\theta_m$ denote the m complex roots of $Q(z) = z^m - 1$, then for every $\lambda \in \mathbb{C}$,

$$|\lambda - z^m| = \prod_{k=1}^m \big|\theta_k\lambda^{\frac{1}{m}} - z\big|.$$

Hence

$$\begin{aligned}\int_{\mathbf{C}} \log|\lambda - z|\, d\nu(z) &= \int_{\mathbf{C}} \log|\lambda - z^m|\, d\mu_T(z)\\ &= \int_{\mathbf{C}} \sum_{k=1}^{m} \log\left|\theta_k \lambda^{\frac{1}{m}} - z\right| d\mu_T(z)\\ &= \sum_{k=1}^{m} L\big(T - \theta_k \lambda^{\frac{1}{m}} \mathbf{1}\big)\\ &= L\left(\prod_{k=1}^{m} \big(T - \theta_k \lambda^{\frac{1}{m}} \mathbf{1}\big)\right)\\ &= L(T^m - \lambda \mathbf{1}),\end{aligned}$$

as desired.

PROPOSITION 2.16. *If $T \in \mathcal{M}^{\Delta}$ with*

$$\int_0^1 \log t\, d\mu_{|T|}(t) > -\infty, \tag{2.44}$$

then $\mu_T(\{0\}) = \mu_{|T|}(\{0\}) = 0$, and T has an inverse $T^{-1} \in \mathcal{M}^{\Delta}$. Moreover, $\mu_{T^{-1}}$ is the push-forward measure of μ_T via the map $z \mapsto z^{-1}$.

PROOF. According to Theorem 2.7,

$$\int_{\mathbf{C}} \log|z|\, d\mu_T(z) = L(T) = \int_0^{\infty} \log t\, d\mu_{|T|}(t). \tag{2.45}$$

Hence, if (2.44) holds, then

$$-\infty < \int_{\mathbf{C}} \log|z|\, d\mu_T(z) < \infty, \tag{2.46}$$

and therefore $\mu_T(\{0\}) = \mu_{|T|}(\{0\}) = 0$. Moreover, $|T|$ has an inverse $|T|^{-1} \in \tilde{\mathcal{M}}$ with

$$\begin{aligned}\int_0^{\infty} \log^+(t)\, d\mu_{|T|^{-1}}(t) &= \int_0^{\infty} \log^+\left(\frac{1}{t}\right) d\mu_{|T|}(t)\\ &= -\int_0^1 \log t\, d\mu_{|T|}(t) < \infty,\end{aligned}$$

so $|T|^{-1} \in \mathcal{M}^{\Delta}$. Take $U \in \mathcal{U}(\mathcal{M})$ such that $T = U|T|$. Then $T^{-1} = |T|^{-1}U^* \in \mathcal{M}^{\Delta}$.

Now, let ν denote the push-forward measure of μ_T under the map $z \mapsto z^{-1}$. According to Theorem 2.7, if

$$\int_{\mathbf{C}} \log^+ |z| \, d\nu(z) < \infty, \tag{2.47}$$

and

$$\forall \lambda \in \mathbf{C}: \quad \int_{\mathbf{C}} \log |z - \lambda| \, d\nu(z) = L(T^{-1} - \lambda \mathbf{1}), \tag{2.48}$$

then $\nu = \mu_{T^{-1}}$. Applying (2.46) we find that

$$\int_{\mathbf{C}} \log^+ |z| \, d\nu(z) = \int_{\mathbf{C}} \log^+ \left| \frac{1}{z} \right| d\mu_T(z) = -\int_{(|z| \leq 1)} \log |z| \, d\mu_T(z) < \infty.$$

In order to prove that (2.48) holds, let $\lambda \in \mathbf{C}$. If $\lambda \neq 0$, then, using the multiplicativity of Δ on $\mathscr{M}^\Delta$, we find that

$$\begin{aligned}
\int_{\mathbf{C}} \log |z - \lambda| \, d\nu(z) &= \int_{\mathbf{C}} \log \left| \frac{1}{z} - \lambda \right| d\mu_T(z) \\
&= \int_{\mathbf{C}} \log \left| \frac{1}{z} \left(\frac{1}{\lambda} - z \right) \lambda \right| d\mu_T(z) \\
&= \int_{\mathbf{C}} \left(\log |\lambda| + \log \left| \frac{1}{\lambda} - z \right| - \log |z| \right) d\mu_T(z) \\
&= L(\lambda \mathbf{1}) + L\left(T - \frac{1}{\lambda} \mathbf{1} \right) - L(T) \\
&= L\left(\lambda \mathbf{1} \left(T - \frac{1}{\lambda} \mathbf{1} \right) T^{-1} \right) \\
&= L(T^{-1} - \lambda \mathbf{1}).
\end{aligned}$$

In the case $\lambda = 0$ we have:

$$\int_{\mathbf{C}} \log |z| \, d\nu(z) = -\int_{\mathbf{C}} \log |z| \, d\mu_T(z) = -L(T) = L(T^{-1}).$$

Hence (2.48) holds, and $\nu = \mu_{T^{-1}}$.

PROPOSITION 2.17. *Let $T \in \mathscr{M}^\Delta$. Then* $\operatorname{supp}(\mu_T) \subseteq \sigma(T)$.

PROOF. Let $\lambda \in \mathbf{C} \setminus \sigma(T)$. Then $T - \lambda \mathbf{1}$ is invertible with bounded inverse. Moreover, according to Proposition 2.16, $\mu_{(T-\lambda\mathbf{1})^{-1}}$ is the push-forward measure of $\mu_{T-\lambda\mathbf{1}}$ via the map $z \mapsto z^{-1}$, $z \in \mathbf{C} \setminus \{0\}$. Since $(T - \lambda \mathbf{1})^{-1}$ is bounded, we have from [3] that

$$\operatorname{supp}(\mu_{(T-\lambda\mathbf{1})^{-1}}) \subseteq \sigma((T-\lambda\mathbf{1})^{-1}) \subseteq \overline{B(0,r)},$$

where $r = \|(T-\lambda\mathbf{1})^{-1}\|$. Hence,

$$\operatorname{supp}(\mu_{T-\lambda\mathbf{1}}) \subseteq \left\{z \in \mathsf{C} \mid |z| \geq \tfrac{1}{r}\right\}.$$

In particular, $0 \notin \operatorname{supp}(\mu_{T-\lambda\mathbf{1}})$, which by Proposition 2.14 is equivalent to $\lambda \notin \operatorname{supp}(\mu_T)$. Hence, $\operatorname{supp}(\mu_T) \subseteq \sigma(T)$.

LEMMA 2.18. *For every $p \in (0,\infty)$ and every $t \in [0,\infty[$,*

$$t^p = p^2 \int_0^\infty \log^+(at)a^{-p-1}\,da. \tag{2.49}$$

PROOF. For $t = 0$ this is trivial. For $t > 0$ we find that

$$\begin{aligned}
\int_0^\infty \log^+(at)a^{-p-1}\,da &= \int_{\frac{1}{t}}^\infty \log(at)a^{-p-1}\,da \\
&= \left[-\frac{1}{p}\log(at)\,a^{-p}\right]_{\frac{1}{t}}^\infty - \int_{\frac{1}{t}}^\infty -\frac{1}{p\,a}\,a^{-p}\,da \\
&= 0 - \left[-\frac{1}{p^2}a^{-p}\right]_{\frac{1}{t}}^\infty \\
&= \frac{1}{p^2}t^p.
\end{aligned}$$

We will now prove Weil's inequality for operators T in $L^p(\mathcal{M})$ (cf. [3, corollary 3.8] for the case $T \in \mathcal{M}$):

THEOREM 2.19. *Let $p \in (0,\infty)$ and let $T \in L^p(\mathcal{M})$. Then*

$$\int_{\mathsf{C}} |z|^p\,d\mu_T(z) \leq \|T\|_p^p. \tag{2.50}$$

In the proof of this theorem we shall need the following lemma, the proof of which we postpone for a while:

LEMMA 2.20. *Let $T \in \mathcal{M}^\Delta$. Then*

$$\int_{\mathsf{C}} \log^+|z|\,d\mu_T(z) \leq \tau(\log^+|T|). \tag{2.51}$$

PROOF OF THEOREM 2.19. Let $a \geq 0$. Then, according to Lemma 2.14 and Lemma 2.20,

$$\begin{aligned}\int_{\mathbb{C}} \log^+(a|z|)\, d\mu_T(z) &= \int_{\mathbb{C}} \log^+|z|\, d\mu_{aT}(z) \\ &\leq \int_0^\infty \log^+ t\, d\mu_{|aT|}(t) \\ &= \int_0^\infty \log^+(at) d\mu_{|T|}(t).\end{aligned}$$

Hence by Lemma 2.18 and Tonelli's Theorem,

$$\begin{aligned}\int_{\mathbb{C}} |z|^p d\mu_T(z) &= p^2 \int_0^\infty \Big(\int_{\mathbb{C}} \log^+(a|z|)\, d\mu_T(z)\Big) a^{-p-1}\, da \\ &\leq p^2 \int_0^\infty \Big(\int_0^\infty \log^+(at)\, d\mu_{|T|}(t)\Big) a^{-p-1}\, da \\ &= \int_0^\infty t^p\, d\mu_{|T|}(t) = \tau(|T|^p).\end{aligned}$$

In order to prove Lemma 2.20 we shall need some additional results:

LEMMA 2.21. *Suppose $A, B, C \in \mathcal{M}^\Delta$ with A and B invertible in $\mathcal{M}^\Delta$ and*

$$\begin{pmatrix} A & C^* \\ C & B \end{pmatrix} \geq 0.$$

Then

$$\Delta(C) \leq \Delta(A)^{\frac{1}{2}} \Delta(B)^{\frac{1}{2}}. \tag{2.52}$$

PROOF. Note that $A, B \geq 0$ and that

$$\begin{pmatrix} \mathbf{1} & A^{-\frac{1}{2}} C^* B^{-\frac{1}{2}} \\ B^{-\frac{1}{2}} C A^{-\frac{1}{2}} & \mathbf{1} \end{pmatrix} = \begin{pmatrix} A^{-\frac{1}{2}} & 0 \\ 0 & B^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} A & C^* \\ C & B \end{pmatrix} \begin{pmatrix} A^{-\frac{1}{2}} & 0 \\ 0 & B^{-\frac{1}{2}} \end{pmatrix} \geq 0,$$

which is equivalent to saying that $\|B^{-\frac{1}{2}} C A^{-\frac{1}{2}}\| \leq 1$, and this clearly implies that

$$\Delta\big(B^{-\frac{1}{2}} C A^{-\frac{1}{2}}\big) \leq 1.$$

LEMMA 2.22. *For every $S \in \mathcal{M}^\Delta$,*

$$\Delta(\mathbf{1} + S) \leq \Delta(\mathbf{1} + |S|). \tag{2.53}$$

PROOF. Take a unitary $U \in \mathcal{M}$ such that $S = U|S|$. Then

$$\begin{pmatrix} |S| & |S| \\ |S| & |S| \end{pmatrix} \geq 0,$$

and

$$\begin{pmatrix} \mathbf{1} & U^* \\ U & \mathbf{1} \end{pmatrix} \geq 0,$$

whence

$$\begin{pmatrix} |S| + \mathbf{1} & |S| + U^* \\ |S| + U & |S| + \mathbf{1} \end{pmatrix} \geq 0.$$

Now Lemma 2.21 implies that

$$\begin{aligned} \Delta(S + \mathbf{1}) &= \Delta(U^*(S + \mathbf{1})) = \Delta(U^*(U|S| + \mathbf{1})) \\ &= \Delta(|S| + U^*) \leq \Delta(|S| + \mathbf{1})^{\frac{1}{2}} \Delta(|S| + \mathbf{1})^{\frac{1}{2}} = \Delta(|S| + \mathbf{1}), \end{aligned}$$

as desired.

LEMMA 2.23. *Every $S \in \mathcal{M}^\Delta$ satisfies*

$$\Delta(\mathbf{1} + |S^2|) \leq \Delta(\mathbf{1} + |S|^2), \tag{2.54}$$

implying that for arbitrary $n \in \mathsf{N}$,

$$\Delta(\mathbf{1} + |S^{2^n}|) \leq \Delta(\mathbf{1} + |S|^{2^n}). \tag{2.55}$$

PROOF. Take a unitary $U \in \mathcal{M}$ such that $S^2 = U|S^2|$. Since

$$\begin{pmatrix} SS^* & S^2 \\ (S^*)^2 & S^*S \end{pmatrix} = \begin{pmatrix} S \\ S^* \end{pmatrix} \begin{pmatrix} S^* & S \end{pmatrix} \geq 0,$$

we find as in the foregoing proof that

$$\begin{pmatrix} \mathbf{1} + SS^* & U^* + S^2 \\ U + (S^*)^2 & \mathbf{1} + S^*S \end{pmatrix} \geq 0.$$

Again this implies that

$$\Delta(\mathbf{1} + |S^2|) = \Delta(S^2 + U^*) \leq \Delta(\mathbf{1} + S^*S)^{\frac{1}{2}} \Delta(\mathbf{1} + SS^*)^{\frac{1}{2}} = \Delta(\mathbf{1} + S^*S),$$

where the last identity follows from the fact that S^*S and SS^* have the same distribution w.r.t. τ.

PROOF OF LEMMA 2.20. According to (2.38) we have:

$$\int_{\mathbb{C}} \log^+ |z| \, d\mu_T(z) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \, d\theta, \tag{2.56}$$

where

$$f(\lambda) = \tau(\log |T - \lambda \mathbf{1}|) = \log \Delta(T - \lambda \mathbf{1}), \qquad \lambda \in \mathbb{C}. \tag{2.57}$$

For every positive integer n define f_n by

$$f_n(z) = \sum_{k=0}^{2^n-1} f\big(e^{\frac{2\pi k}{2^n} i} z\big), \qquad z \in \mathbb{C}. \tag{2.58}$$

Then clearly,

$$\frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \, d\theta = \frac{1}{2\pi 2^n} \int_0^{2\pi} f_n(e^{i\theta}) \, d\theta. \tag{2.59}$$

Applying Lemma 2.22 and Lemma 2.23 we obtain an estimate of $f_n(e^{i\theta})$:

$$\begin{aligned} f_n(e^{i\theta}) &= \sum_{k=0}^{2^n-1} \log \Delta\big(e^{-i\theta} e^{-\frac{2\pi k}{2^n} i} T - \mathbf{1}\big) = \log \Delta\Big(\prod_{k=0}^{2^n-1} \big(e^{-i\theta} e^{-\frac{2\pi k}{2^n} i} T - \mathbf{1}\big)\Big) \\ &= \log \Delta\big(\mathbf{1} - e^{-i2^n\theta} T^{2^n}\big) \le \log \Delta\big(\mathbf{1} + |T^{2^n}|\big) \le \log \Delta\big(\mathbf{1} + |T|^{2^n}\big) \\ &= \tau\big(\log(\mathbf{1} + |T|^{2^n})\big). \end{aligned}$$

Combining (2.56) and (2.59) with the above estimate we see that

$$\begin{aligned} \int_{\mathbb{C}} \log^+ |z| \, d\mu_T(z) &\le \frac{1}{2^n} \tau\big(\log\big(\mathbf{1} + |T|^{2^n}\big)\big) \\ &= \frac{1}{2^n} \int_{[0,\infty[} \log(1 + t^{2^n}) \, d\mu_{|T|}(t) \\ &\le \frac{1}{2^n} \int_{[0,1[} \log 2 \, d\mu_{|T|}(t) \\ &\qquad + \frac{1}{2^n} \int_{[1,\infty[} (\log 2 + 2^n \log t) \, d\mu_{|T|}(t) \\ &\le \frac{2 \log 2}{2^n} + \int_{[0,\infty[} \log^+ t \, d\mu_{|T|}(t). \end{aligned}$$

Finally, let $n \to \infty$, and conclude that

$$\int_{\mathbb{C}} \log^+ |z| \, d\mu_T(z) \leq \int_{[0,\infty[} \log^+ t \, d\mu_{|T|}(t).$$

PROPOSITION 2.24. *Let $T \in \mathcal{M}^\Delta$, and suppose $P \in \mathcal{M}$ is a non-trivial T-invariant projection, i.e. $PTP = TP$. Then*

$$\Delta(T) = \Delta_{P\mathcal{M}P}(PTP)^{\tau(P)} \Delta_{P^\perp \mathcal{M} P^\perp}(P^\perp T P^\perp)^{1-\tau(P)}, \tag{2.60}$$

where $\Delta_{P\mathcal{M}P}$ and $\Delta_{P^\perp\mathcal{M}P^\perp}$ refer to the Fuglede-Kadison determinant computed relative to the normalized traces $\frac{1}{\tau(P)}\tau\big|_{P\mathcal{M}P}$ and $\frac{1}{\tau(P^\perp)}\tau\big|_{P^\perp\mathcal{M}P^\perp}$ on $P\mathcal{M}P$ and $P^\perp\mathcal{M}P^\perp$, respectively.

PROOF. Put $T_{11} = PTP$, $T_{12} = PTP^\perp$ and $T_{22} = P^\perp TP^\perp$. Then, w.r.t. to the decomposition $\mathcal{H} = P(\mathcal{H}) \oplus P(\mathcal{H})^\perp$, we may write

$$T = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix} = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & T_{22} \end{pmatrix} \begin{pmatrix} \mathbf{1} & T_{12} \\ 0 & \mathbf{1} \end{pmatrix} \begin{pmatrix} T_{11} & 0 \\ 0 & \mathbf{1} \end{pmatrix},$$

where

$$\Delta\left(\begin{pmatrix} \mathbf{1} & 0 \\ 0 & T_{22} \end{pmatrix}\right) = \Delta_{P^\perp\mathcal{M}P^\perp}(P^\perp TP^\perp)^{1-\tau(P)},$$

and

$$\Delta\left(\begin{pmatrix} T_{11} & 0 \\ 0 & \mathbf{1} \end{pmatrix}\right) = \Delta_{P\mathcal{M}P}(PTP)^{\tau(P)}.$$

Thus, (2.60) holds if

$$\Delta\left(\begin{pmatrix} \mathbf{1} & T_{12} \\ 0 & \mathbf{1} \end{pmatrix}\right) = 1. \tag{2.61}$$

To that (2.60) holds, note that

$$\begin{pmatrix} \mathbf{1} & T_{12} \\ 0 & \mathbf{1} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{1} & -T_{12} \\ 0 & \mathbf{1} \end{pmatrix},$$

and hence

$$\Delta\left(\begin{pmatrix} \mathbf{1} & T_{12} \\ 0 & \mathbf{1} \end{pmatrix}\right) \Delta\left(\begin{pmatrix} \mathbf{1} & -T_{12} \\ 0 & \mathbf{1} \end{pmatrix}\right) = 1. \tag{2.62}$$

Also,

$$\begin{pmatrix} \mathbf{1} & -T_{12} \\ 0 & \mathbf{1} \end{pmatrix} = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{1} & T_{12} \\ 0 & \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix},$$

so that

$$\Delta\left(\begin{pmatrix} \mathbf{1} & -T_{12} \\ 0 & \mathbf{1} \end{pmatrix}\right) = \Delta\left(\begin{pmatrix} \mathbf{1} & T_{12} \\ 0 & \mathbf{1} \end{pmatrix}\right),$$

and then by (2.62),

$$\Delta\left(\begin{pmatrix} \mathbf{1} & T_{12} \\ 0 & \mathbf{1} \end{pmatrix}\right) = 1,$$

as desired.

LEMMA 2.25. *Let $p \in (0, \infty)$, and let $\varepsilon > 0$. Then the map $L_\varepsilon : L^p(\mathcal{M}, \tau) \to \mathsf{R}$ given by*

$$L_\varepsilon(T) = \tfrac{1}{2}\,\tau(\log(T^*T + \varepsilon\mathbf{1})), \qquad T \in L^p(\mathcal{M}, \tau), \tag{2.63}$$

is continuous w.r.t. $\|\cdot\|_p$.

PROOF. Suppose $T, T_n \in L^p(\mathcal{M}, \tau)$ with

$$\lim_{n\to\infty} \|T - T_n\|_p = 0.$$

Then $T_n \to T$ in the measure topology (cf. [6]). Therefore, $T_n^*T_n \to T^*T$ in measure, and then with respect to the weak topology on Prob(R),

$$\mu_{T^*T} = \lim_{n\to\infty} \mu_{T_n^*T_n}. \tag{2.64}$$

Define a sequence $(\nu_n)_{n=1}^\infty$ of (finite) measures on (R, B) by

$$d\nu_n(t) = \big(1 + t^{\frac{p}{2}}\big)\, d\mu_{T_n^*T_n}(t), \tag{2.65}$$

and note that since $\lim_{n\to\infty} \|T_n\|_p = \|T\|_p$,

$$\sup_{n\in\mathsf{N}} \nu_n(\mathsf{R}) < \infty. \tag{2.66}$$

Similarly, define a finite measure ν on (R, B) by

$$d\nu(t) = \big(1 + t^{\frac{p}{2}}\big)\, d\mu_{T^*T}(t). \tag{2.67}$$

Because of (2.64), we have that for every $\phi \in C_c(\mathsf{R})$,

$$\int_0^\infty \phi(t)\, d\nu(t) = \lim_{n\to\infty} \int_0^\infty \phi(t)\, d\nu_n(t). \tag{2.68}$$

When $\phi \in C_0(\mathsf{R})$, ϕ may be approximated (uniformly) by functions from $C_c(\mathsf{R})$. Thus, taking (2.66) and (2.68) into account, one easily sees that

$$\int_0^\infty \phi(t)\, d\nu(t) = \lim_{n\to\infty} \int_0^\infty \phi(t)\, d\nu_n(t). \tag{2.69}$$

In particular, with

$$\phi(t) = \frac{\log(t+\varepsilon)}{1+t^{\frac{p}{2}}}, \qquad t \geq 0, \tag{2.70}$$

(2.69) implies that

$$L_\varepsilon(T) = \int_0^\infty \phi(t)\,d\nu(t) = \lim_{n\to\infty} \int_0^\infty \phi(t)\,d\nu_n(t) = \lim_{n\to\infty} L_\varepsilon(T_n).$$

Corollary 2.26. *For $p \in (0,\infty)$ the map $L : L^p(\mathcal{M},\tau) \to [-\infty,\infty[$ given by*

$$L(T) = \log \Delta(T), \qquad T \in L^p(\mathcal{M},\tau), \tag{2.71}$$

is upper semicontinuous w.r.t. $\|\cdot\|_p$.

Proof. Indeed, this follows from Lemma 2.25, since for every $T \in L^p(\mathcal{M},\tau)$ we have that

$$L(T) = \inf_{\varepsilon>0} L_\varepsilon(T).$$

3. Unbounded *R*-diagonal operators

Consider a von Neumann algebra $\mathcal{M}$ equipped with a faithful, normal, tracial state τ.

Definition 3.1. For $T \in \tilde{\mathcal{M}}$ with polar decomposition $T = U|T|$, we denote by $W^*(T)$ the von Neumann algebra generated by U and all the spectral projections of $|T|$.

Note that T is affiliated with $W^*(T)$ and that $W^*(T)$ is the smallest von Neumann subalgebra of $\mathcal{M}$ with this property.

If $\mathcal{M}_1$ and $\mathcal{M}_2$ are finite von Neumann algebras with faithful, normal, tracial states τ_1 and τ_2, respectively, then any $*$-isomorphism $\phi : \mathcal{M}_1 \to \mathcal{M}_2$ with $\tau_1 = \tau_2 \circ \phi$ is continuous w.r.t. the measure topologies on the two von Neumann algebras and thus has a unique extension to a (surjective) $*$-isomorphism $\tilde{\phi} : \tilde{\mathcal{M}}_1 \to \tilde{\mathcal{M}}_2$.

Definition 3.2. Let $S, T \in \tilde{\mathcal{M}}$.

(a) We say that S and T have the same $*$-distribution, in symbols $S \underset{*\mathcal{D}}{\sim} T$, if there exists a trace-preserving $*$-isomporphism ϕ from $W^*(S)$ onto $W^*(T)$ with $\tilde{\phi}(S) = T$.

(b) We say that S and T are $*$-free if $W^*(S)$ and $W^*(T)$ are $*$-free.

Note that in case S and T are bounded, the two definitions (a) and (b) given above agree with the ones given in [18] .

Recall from [16, p. 155ff.] that if $U, H \in \mathcal{M}$ are $*$-free elements with U Haar unitary, then UH is R-diagonal in the sense of Nica and Speicher (cf. [16]). Conversely, if $T \in \mathcal{M}$ is R-diagonal, then T has the same $*$-distribution as a product UH, where U and H are $*$-free elements in some tracial C^*-probability space, U is a Haar unitary, and $H \geq 0$. We therefore define R-diagonality for operators in $\tilde{\mathcal{M}}$ as follows:

DEFINITION 3.3. $T \in \tilde{\mathcal{M}}$ is said to be *R-diagonal* if there exist a von Neumann algebra $\mathcal{N}$, with a faithful, normal, tracial state, and $*$-free elements U and H in $\tilde{\mathcal{N}}$, such that U is Haar unitary, $H \geq 0$, and such that T has the same $*$-distribution as UH.

REMARK 3.4. Note that if $T \in \tilde{\mathcal{M}}$ is R-diagonal with $\ker(T) = 0$, then the partial isometry V in the polar decomposition of T, $T = V|T|$, is a unitary ($\mathcal{M}$ is finite). It follows from Definition 3.3 and Definition 3.2 that V is in fact a Haar unitary which is $*$-free from $|T|$.

In this section we will see that certain algebraic operations on (sets of $*$-free) R-diagonal operators preserve R-diagonality, exactly as in the bounded case (cf. [9]). Our proofs are to a large extent inspired by the techniques used in [9] and in [14]. In particular, we will repeatedly make use of [9, Lemma 3.7] which we state here for the convenience of the reader:

LEMMA 3.5. *[9] Let $U \in \mathcal{M}$ be a Haar unitary, and suppose $\mathcal{S} \subset \mathcal{M}$ is a set which is $*$-free from U. Then for any $n \in \mathbb{N}$,*

(i) *the sets $\mathcal{S}, U\mathcal{S}U^*, U^2\mathcal{S}(U^*)^2, \ldots$ are $*$-free,*

(ii) *the sets $\mathcal{S}, U\mathcal{S}U^*, \ldots, U^{n-1}\mathcal{S}(U^*)^{n-1}, \{U^n\}$ are $*$-free,*

(iii) *the sets $U\mathcal{S}U^*, \ldots, U^n\mathcal{S}(U^*)^n, \{U^n\}$ are $*$-free.*

PROPOSITION 3.6. *If $T \in \tilde{\mathcal{M}}$ is R-diagonal with* $\ker(T) = 0$*, then T has an inverse $T^{-1} \in \tilde{\mathcal{M}}$, and T^{-1} is R-diagonal as well.*

PROOF. Let $T = V|T|$ be the polar decomposition of T with $V \in \mathcal{M}$ Haar unitary and $*$-free from $|T|$. Since $\ker(T) = 0$, T has an inverse $T^{-1} \in \tilde{\mathcal{M}}$:

$$T^{-1} = V^*V|T|^{-1}V^* = V^*(V|T|V^*)^{-1},$$

where V^* is Haar unitary and, according to Lemma 3.5, it is $*$-free from $V|T|V^*$ and thus from $(V|T|V^*)^{-1}$. This shows that T^{-1} is R-diagonal.

LEMMA 3.7. *Let $S, T \in \tilde{\mathcal{M}}$, and let $V \in \mathcal{M}$ be a Haar unitary. If S, T and V are $*$-free, then VS and TVS are R-diagonal.*

PROOF. The case where S and T are bounded was treated by F. Larsen (cf. [14, Lemma 3.6]). Our proof resembles the one given by F. Larsen.

Enlarging the algebra if necessary, we may assume that there are Haar unitaries $V_1, V_2 \in \mathscr{M}$, such that V_1, V_2 and S are $*$-free and $V = V_1 V_2$.

Since $W^*(S) \subseteq \mathscr{M}$ is finite, there is a unitary $U_1 \in W^*(S)$ such that $S = U_1|S|$. Then $VS = V_1(V_2U_1)|S|$, where

(i) V_1 is $*$-free from $|S|$ and V_2U_1,

(ii) $\tau(V_1) = \tau(V_1^*) = 0$ and $\tau(V_2U_1) = \tau((V_2U_1)^*) = 0$,

(ii) for all $A \in W^*(|S|)$ with $\tau(A) = 0$, $\tau(V_2U_1A) = \tau(V_2)\tau(U_1A) = 0$, $\tau(AU_1^*V_2^*) = \tau(AU_1^*)\tau(V_2^*) = 0$ and $\tau(V_2U_1A(V_2U_1)^{-1}) = \tau(A) = 0$.

It follows now from [17, Lemma 2.4] that $V_1(V_2U_1)$ is $*$-free from $|S|$. Thus, if $V_1(V_2U_1)$ is Haar unitary, then VS is R-diagonal. By (ii) and the freeness of V_1 and V_2U, we have that for all $n \in \mathsf{N}$,

$$\tau((V_1V_2U_1)^n) = \tau(V_1(V_2U_1)V_1(V_2U_1)\cdots V_1(V_2U_1)) = 0.$$

Then $\tau((V_1V_2U_1)^{-n}) = \overline{\tau((V_1V_2U_1)^n)} = 0$, $n \in \mathsf{N}$. That is, $V_1V_2U_1$ is Haar unitary, and it follows that $VS = V_1V_2U_1|S|$ is R-diagonal.

Now, $TVS = V(V^*TVS)$. Put

$$\mathscr{B}_1 = W^*(V), \quad \mathscr{B}_2 = W^*(T), \quad \text{and} \quad \mathscr{B}_3 = W^*(S).$$

Then $\mathscr{B}_1$, $\mathscr{B}_2$ and $\mathscr{B}_3$ are $*$-free. We may write T as $T = U_2|T|$ for a unitary $U_2 \in \mathscr{B}_2$. Then

$$V^*TV = (V^*U_2V)V^*|T|V, \tag{3.1}$$

where $V^*|T|V$ is affiliated with $V^*\mathscr{B}_2V$.

$\mathscr{B}_3$ and $V^*\mathscr{B}_2V$ are $*$-free, and according to Lemma 3.5, $\mathscr{B}_1$ and $V^*\mathscr{B}_2V$ are $*$-free. But then V is $*$-free from $\mathscr{B}_4 = \mathscr{B}_3 \vee V^*\mathscr{B}_2V$.

Since S and V^*TV are both affiliated with $\mathscr{B}_4$, their product, V^*TVS, is affiliated with $\mathscr{B}_4$, so V is $*$-free from V^*TVS. It follows now from the first part of the proof that $TVS = V(V^*TVS)$ is R-diagonal.

PROPOSITION 3.8. *If $S, T \in \tilde{\mathscr{M}}$ are $*$-free R-diagonal elements, then ST is R-diagonal as well. Moreover,*

$$\mu_{(ST)^*ST} = \mu_{S^*S} \boxtimes \mu_{T^*T}. \tag{3.2}$$

PROOF. Taking a free product of tracial von Neumann algebras if necessary, we can find a von Neumann algebra $\mathscr{N}$ with faithful, normal, tracial state ω

and $*$-free elements $U_1, H_1, U_2, H_2 \in \tilde{\mathcal{N}}$ such that U_1, U_2 are Haar unitaries, $H_1, H_2 \geq 0$, and $S \underset{*\mathcal{D}}{\sim} U_1 H_1$ and $T \underset{*\mathcal{D}}{\sim} U_2 H_2$.

Choose trace-preserving $*$-isomorphisms

$$\phi_1 : W^*(S) \to W^*(U_1 H_1),$$
$$\phi_2 : W^*(T) \to W^*(U_2 H_2),$$

with $\tilde{\phi}_1(S) = U_1 H_1$ and $\tilde{\phi}_2(T) = U_2 H_2$. ϕ_1 and ϕ_2 give rise to a trace-preserving $*$-isomorphism

$$\phi = \phi_1 * \phi_2 : W^*(S) * W^*(T) \to W^*(U_1 H_1) * W^*(U_2 H_2)$$

(the free products are taken within the category of tracial von Neumann algebras) with

$$\tilde{\phi}(ST) = \tilde{\phi}_1(S)\tilde{\phi}_2(T) = U_1 H_1 U_2 H_2.$$

Thus, $\psi := \phi|_{W^*(ST)}$ is a trace-preserving $*$-isomorphism onto $W^*(U_1 H_1 U_2 H_2)$ with $\tilde{\psi}(ST) = U_1 H_1 U_2 H_2$. According to Lemma 3.7, $U_1(H_1 U_2 H_2)$ is R-diagonal, and hence ST is R-diagonal.

In order to prove (3.2), note that if $S = 0$, then $\mu_{S^*S} = \delta_0$, so that by the definition of multiplicative free convolution given on p. 744 in [4],

$$\mu_{S^*S} \boxtimes \mu_{T^*T} = \delta_0 \boxtimes \mu_{T^*T} = \delta_0.$$

This shows that $\mu_{S^*S} \boxtimes \mu_{T^*T} = \mu_{(ST)^*ST}$ if $S = 0$. The same holds if $T = 0$.

Now assume that $S, T \neq 0$. Note that

$$S^*S \underset{*\mathcal{D}}{\sim} H_1^2,$$
$$T^*T \underset{*\mathcal{D}}{\sim} H_2^2,$$
$$(ST)^*ST \underset{*\mathcal{D}}{\sim} H_2 U_2^* H_1^2 U_2 H_2.$$

Thus, (3.2) holds if

$$\mu_{H_2 U_2^* H_1^2 U_2 H_2} = \mu_{H_1^2} \boxtimes \mu_{H_2^2}.$$

For every $n \in \mathbf{N}$, the bounded operators

$$S_n = U_1 H_1 1_{[0,n]}(H_1) \qquad \text{and} \qquad T_n = U_2 H_2 1_{[0,n]}(H_2)$$

are $*$-free. According to [9, Lemma 3.9] they are both R-diagonal in the sense of Nica and Speicher (cf. [16]). Then, by [9, Proposition 3.6],

$$\mu_{(S_n T_n)^* S_n T_n} = \mu_{S_n^* S_n} \boxtimes \mu_{T_n^* T_n} \tag{3.3}$$

Since $S_n \to U_1 H_1$ and $T_n \to U_2 H_2$ in the measure topology, $(S_n T_n)^* S_n T_n \to H_2 U_2^* H_1^2 U_2 H_2$ in measure as well. These facts imply that $\mu_{S_n^* S_n} \xrightarrow{w^*} \mu_{H_1^2}$, $\mu_{T_n^* T_n} \xrightarrow{w^*} \mu_{H_2^2}$ and $\mu_{(S_n T_n)^* S_n T_n} \xrightarrow{w^*} \mu_{H_2 U_2^* H_1^2 U_2 H_2}$. Moreover, $\mu_{H_1^2} \neq \delta_0$ and $\mu_{H_2^2} \neq \delta_0$, because $S^* S$ and $T^* T$ are non-zero. Hence, by [4, Corollary 6.7] and by (3.3),

$$\mu_{H_2 U_2^* H_1^2 U_2 H_2} = w^* - \lim_{n\to\infty} \mu_{S_n^* S_n} \boxtimes \mu_{T_n^* T_n} = \mu_{H_1^2} \boxtimes \mu_{H_2^2}.$$

Proposition 3.9. *Let $S \in \tilde{\mathcal{M}}$ be R-diagonal, and let $n \in \mathbb{N}$. Then S^n is R-diagonal. Moreover,*

$$\mu_{(S^n)^* S^n} = \mu_{S^* S}^{\boxtimes n}. \tag{3.4}$$

Proof. Choose a von Neumann algebra $\mathcal{N}$ with faithful, normal, tracial state ω and with $*$-free elements $U, H \in \tilde{\mathcal{N}}$ such that U is Haar unitary, $H \geq 0$, and $S \underset{*\mathcal{D}}{\sim} UH$. Then $S^n \underset{*\mathcal{D}}{\sim} (UH)^n$. Since

$$(UH)^n = U^n [U^{1-n} H U^{n-1}][U^{2-n} H U^{n-2}] \cdots [U^{-1} H U] H,$$

where

$$U^n,\ U^{1-n} H U^{n-1},\ U^{2-n} H U^{n-2},\ \ldots,\ U^{-1} H U,\ H$$

are $*$-free (cf. Lemma 3.5 (ii)), and U^n is Haar unitary, Lemma 3.7 gives us that $(UH)^n$ is R-diagonal, and hence S^n is.

In order to prove (3.4), note that if $\mu_{S^* S} = \delta_0$, then $S = S^n = 0$ and (3.4) trivially holds.

Now assume that $\mu_{S^* S} \neq \delta_0$. For $k \in \mathbb{N}$ define $S_k \in \mathcal{M}$ and $T_k \in \mathcal{N}$ by

$$S_k = S\, 1_{[0,k]}(|S|) \qquad \text{and} \qquad T_k = U\, H\, 1_{[0,k]}(H).$$

Then $T_k \underset{*\mathcal{D}}{\sim} S_k$. Moreover, by Lemma 3.7, T_k is R-diagonal in the sense of Nica and Speicher, so S_k is R-diagonal. It now follows from [9, Proposition 3.10] that

$$\mu_{[(S_k)^n]^* (S_k)^n} = \mu_{[(T_k)^n]^* (T_k)^n} = \mu_{T_k^* T_k}^{\boxtimes n} = \mu_{S_k^* S_k}^{\boxtimes n}. \tag{3.5}$$

As k tends to infinity, $S_k^* S_k \to S^* S$ and $[(S_k)^n]^* (S_k)^n \to (S^n)^* S^n$ in the measure topology. Since $\mu_{S^* S} \neq \delta_0$, we infer from [4, Corollary 6.7] and from (3.5) that

$$\mu_{(S^n)^* S^n} = w^* - \lim_{k\to\infty} \mu_{[(S_k)^n]^* (S_k)^n} = w^* - \lim_{k\to\infty} \mu_{S_k^* S_k}^{\boxtimes n} = \mu_{S^* S}^{\boxtimes n}.$$

DEFINITION 3.10. For $\mu \in \operatorname{Prob}(\mathsf{R}, \mathsf{B})$ let $\tilde{\mu}$ denote the *symmetrization* of μ. That is, $\tilde{\mu} \in \operatorname{Prob}(\mathsf{R}, \mathsf{B})$ is given by

$$\tilde{\mu}(B) = \tfrac{1}{2}(\mu(B) + \mu(-B)), \qquad (B \in \mathsf{B}).$$

PROPOSITION 3.11. *Let $S, T \in \tilde{\mathscr{M}}$ be $*$-free R-diagonal elements. Then*

$$\tilde{\mu}_{|S+T|} = \tilde{\mu}_{|S|} \boxplus \tilde{\mu}_{|T|}. \tag{3.6}$$

PROOF. As in the proof of Proposition 3.8, choose $(\mathscr{N}, \omega)$ and $*$-free elements $U_1, H_1, U_2, H_2 \in \tilde{\mathscr{N}}$ such that U_1, U_2 are Haar unitaries, $H_1, H_2 \geq 0$, and $S \underset{*\mathscr{D}}{\sim} U_1 H_1$ and $T \underset{*\mathscr{D}}{\sim} U_2 H_2$.

Again, for $n \in \mathsf{N}$, let

$$S_n = U_1\, H_1\, 1_{[0,n]}(H_1)$$

and

$$T_n = U_2\, H_2\, 1_{[0,n]}(H_2).$$

Then S_n and T_n are $*$-free and R-diagonal and therefore, according to [9, Proposition 3.5],

$$\tilde{\mu}_{|S_n+T_n|} = \tilde{\mu}_{|S_n|} \boxplus \tilde{\mu}_{|T_n|}. \tag{3.7}$$

$|S_n| \to H_1$ and $|T_n| \to H_2$ in measure, implying that $\mu_{|S_n|} \xrightarrow{w^*} \mu_{H_1} = \mu_{|S|}$ and $\mu_{|T_n|} \xrightarrow{w^*} \mu_{H_2} = \mu_{|T|}$. Then we also have weak convergence of the symmetrized measures:

$$\tilde{\mu}_{|S_n|} \xrightarrow{w^*} \tilde{\mu}_{|S|}$$

and

$$\tilde{\mu}_{|T_n|} \xrightarrow{w^*} \tilde{\mu}_{|T|}.$$

Let d denote the Lévy metric on $\operatorname{Prob}(\mathsf{R}, \mathsf{B})$ (cf. [4, p. 743]). Then d induces the topology of weak convergence, and according to [4, Proposition 4.13] and the above observations,

$$d(\tilde{\mu}_{|S|} \boxplus \tilde{\mu}_{|T|}, \tilde{\mu}_{|S_n|} \boxplus \tilde{\mu}_{|T_n|}) \leq d(\tilde{\mu}_{|S|}, \tilde{\mu}_{|S_n|}) + d(\tilde{\mu}_{|T|}, \tilde{\mu}_{|T_n|}) \to 0 \quad \text{as } n \to \infty.$$

It follows that

$$\begin{aligned} \tilde{\mu}_{|S|} \boxplus \tilde{\mu}_{|T|} &= w^* - \lim_{n\to\infty} \tilde{\mu}_{|S_n|} \boxplus \tilde{\mu}_{|T_n|} \\ &= w^* - \lim_{n\to\infty} \tilde{\mu}_{|S_n+T_n|}. \end{aligned} \tag{3.8}$$

Since S and T (U_1H_1 and U_2H_2, resp.) are $*$-free with $S \underset{*\mathcal{D}}{\sim} U_1H_1$ and $T \underset{*\mathcal{D}}{\sim} U_2H_2$, it follows that $S + T \underset{*\mathcal{D}}{\sim} U_1H_1 + U_2H_2$. Moreover, $|S_n + T_n| \to |U_1H_1 + U_2H_2| \underset{*\mathcal{D}}{\sim} |S+T|$ in measure, and thus $\tilde{\mu}_{|S_n+T_n|} \xrightarrow{w^*} \tilde{\mu}_{|S+T|}$. Finally, this implies that

$$\tilde{\mu}_{|S|} \boxplus \tilde{\mu}_{|T|} = \tilde{\mu}_{|S+T|}.$$

We close this section by proving two simple results on the S-transform of probability measures on $(0, \infty)$ (cf. [4]).

For $\mu \in \mathrm{Prob}((0,\infty), \mathsf{B})$ define $\psi_\mu : \mathsf{C} \setminus (0,\infty) \to \mathsf{C}$ by

$$\psi_\mu(z) = \int_0^\infty \frac{1}{1-zt}\, d\mu(t) - 1, \qquad z \in \mathsf{C} \setminus (0,\infty). \tag{3.9}$$

Then ψ_μ is analytic and satisfies

(i) $\psi'_\mu(t) > 0$, $t \in (-\infty, 0)$,
(ii) $\psi_\mu(z) \to -1$ as $z \to -\infty$,
(iii) $\psi_\mu(z) \to 0$ as $z \to 0$.

Hence, ψ_μ maps a (connected) neighbourhood $\mathcal{U}_\mu$ of $(-\infty, 0)$ injectively onto a neighbourhood $\mathcal{V}_\mu$ of $(-1, 0)$. Define $\chi_\mu, \mathcal{S}_\mu : \mathcal{V}_\mu \to \mathsf{C}$ by

$$\chi_\mu(z) = \psi_\mu^{-1}(z), \qquad z \in \mathcal{V}_\mu, \tag{3.10}$$

$$\mathcal{S}_\mu(z) = \frac{z+1}{z}\chi_\mu(z), \qquad z \in \mathcal{V}_\mu. \tag{3.11}$$

PROPOSITION 3.12. *The map* $\mu \mapsto \mathcal{S}_\mu$ *is one-to-one on* $\mathrm{Prob}((0,\infty), \mathsf{B})$.

PROOF. Suppose $\mu, \nu \in \mathrm{Prob}((0,\infty), \mathsf{B})$ with $\mathcal{S}_\mu = \mathcal{S}_\nu$. That is, in a neighbourhood $\mathcal{V} = \mathcal{V}_\mu \cap \mathcal{V}_\nu$ of $(-1, 0)$, χ_μ agrees with χ_ν. It follows that on $(-\infty, 0)$, ψ_μ agrees with ψ_ν, and then, by uniqueness of analytic continuation,

$$\psi_\mu\left(\frac{1}{\lambda}\right) = \psi_\nu\left(\frac{1}{\lambda}\right), \qquad \lambda \in \mathsf{C} \setminus [0, \infty[. \tag{3.12}$$

That is, the Stieltjes-transforms G_μ and G_ν agree on $\mathsf{C} \setminus [0,\infty[$. Recall that

$$d\mu(x) = -\frac{1}{\pi} \lim_{y \to 0^+} G_\mu(x+iy)\, dx \tag{3.13}$$

(weak convergence of measures), and similarly,

$$d\nu(x) = -\frac{1}{\pi} \lim_{y \to 0^+} G_\nu(x+iy)\, dx. \tag{3.14}$$

Thus $\mu = \nu$.

PROPOSITION 3.13. *Let $\mathcal{M}$ be a II_1-factor with tracial state τ, and let $a \in \tilde{\mathcal{M}}_+$ with* $\ker(a) = \{0\}$. *Then for all z in a neighbourhood of* $(-1, 0)$,

$$\mathcal{S}_{\mu_{a^{-1}}}(z) = \frac{1}{\mathcal{S}_{\mu_a}(-1-z)}. \tag{3.15}$$

PROOF. Let $z \in \mathbf{C} \setminus [0, \infty[$. Then

$$\begin{aligned}\psi_{a^{-1}}(z) &= \int_0^\infty \frac{1}{1-zt}\, d\mu_{a^{-1}}(t) - 1 \\ &= \int_0^\infty \frac{1}{1-\frac{z}{t}}\, d\mu_a(t) - 1 \\ &= \int_0^\infty \frac{z}{t-z}\, d\mu_a(t),\end{aligned}$$

and hence

$$\psi_{a^{-1}}\left(\frac{1}{z}\right) = -\int_0^\infty \frac{1}{1-zt}\, d\mu_a(t) = -(\psi_a(z)+1). \tag{3.16}$$

It follows that for all $z \in \mathbf{C} \setminus [0, \infty[$,

$$z = \chi_a(\psi_a(z)) = \chi_a\left(-1 - \psi_{a^{-1}}\left(\frac{1}{z}\right)\right), \tag{3.17}$$

implying that $w = \psi_{a^{-1}}\left(\frac{1}{z}\right)$ satisfies

$$\chi_{a^{-1}}(w) = \frac{1}{z} = \frac{1}{\chi_a(-1-w)}, \tag{3.18}$$

and thus

$$\mathcal{S}_{\mu_{a^{-1}}}(w) \cdot \mathcal{S}_{\mu_a}(-1-w) = 1. \tag{3.19}$$

(3.19) holds for all $w \in \psi_{a^{-1}}(\mathbf{C} \setminus [0, \infty[)$ and in particular for all w in a neighbourhood of $(-1, 0)$.

4. The Brown measure of an unbounded *R*-diagonal operator

The Brown measure of a general bounded R-diagonal operator was computed in [9, Theorem 4.4]. We will generalize this result to unbounded R-diagonal elements in $\mathcal{M}^{\Delta}$. Our proof will take a different route than the one in [9]. This new approach will enable us to obtain an estimate of the p-norm of the resolvent $(T-\lambda\mathbf{1})^{-1}$, $0<p<1$, for special R-diagonal elements T (cf. Section 5).

LEMMA 4.1. *Let $T \in \tilde{\mathcal{M}}$ be an R-diagonal element, and let $U \in \mathcal{M}$ be a Haar unitary which is $*$-free from T. Then for every $\lambda \in \mathbb{C}$,*

$$|T-\lambda\mathbf{1}| \underset{*\mathcal{D}}{\sim} |T+|\lambda|U|. \tag{4.1}$$

PROOF. By passing to a larger algebra, we may assume that $T = V|T|$ where $V \in \mathcal{M}$ is a Haar unitary and U, V and $|T|$ are $*$-free. The case $\lambda = 0$ is trivial. For $\lambda \neq 0$, let $\alpha = -\frac{\lambda}{|\lambda|}$. Then $\alpha U^* V$ is a Haar unitary which is $*$-free from T. Hence,

$$\alpha U^* V|T| \underset{*\mathcal{D}}{\sim} T.$$

Therefore,

$$|T-\lambda\mathbf{1}| \underset{*\mathcal{D}}{\sim} |\alpha U^* V|T| - \lambda\mathbf{1}| = |T - \overline{\alpha}\lambda U| = |T+|\lambda|U|.$$

LEMMA 4.2. *Let $T \in \tilde{\mathcal{M}}$ be an R-diagonal operator, and define*

$$h(s) = s\,\tau\big((T^*T+s^2\mathbf{1})^{-1}\big), \qquad s>0.$$

Moreover, for $\lambda \in \mathbb{C}\setminus\{0\}$, set

$$h_\lambda(s) = s\,\tau\big([(T-\lambda\mathbf{1})^*(T-\lambda\mathbf{1})+s^2\mathbf{1}]^{-1}\big).$$

Then there exists an $s_\lambda > 0$ such that for $s > s_\lambda$,

$$h(s) = h_\lambda\left(s + \frac{\sqrt{1-4|\lambda|^2 h(s)^2}-1}{2h(s)}\right).$$

PROOF. By passing to a larger algebra, we may assume that there exists a Haar unitary $U \in \mathcal{M}$ which is $*$-free from T. Then, according to Lemma 4.1,

$$|T-\lambda\mathbf{1}| \underset{*\mathcal{D}}{\sim} |T+|\lambda|U|.$$

It follows now from Proposition 3.11 that

$$\tilde{\mu}_{|T-\lambda\mathbf{1}|} = \tilde{\mu}_{|T|} \boxplus \tilde{\mu}_{|\lambda|\mathbf{1}} = \tilde{\mu}_{|T|} \boxplus \nu,$$

where $\nu = \frac{1}{2}(\delta_{-|\lambda|} + \delta_{|\lambda|})$.

For $\beta > 0$ define

$$\Omega_\beta = \left\{ w \in \mathsf{C} \;\middle|\; 0 < |w| < \beta,\ \tfrac{5\pi}{4} < \arg(w) < \tfrac{7\pi}{4} \right\}.$$

According to [4, Corollary 5.8], there is a $\beta > 0$ such that for every $w \in \Omega_\beta$,

$$\mathcal{R}_{\tilde{\mu}_{|T-\lambda\mathbf{1}|}}(w) = \mathcal{R}_{\tilde{\mu}_{|T|}}(w) + \mathcal{R}_\nu(w),$$

where

$$\mathcal{R}_\nu(w) = \frac{\sqrt{1+4|\lambda|^2 w^2} - 1}{2w},$$

and

$$G_{\tilde{\mu}_{|T|}}(is) = -ih(s), \qquad s > 0,$$

whence

$$\mathcal{R}_{\tilde{\mu}_{|T|}}(-ih(s)) + \frac{1}{-ih(s)} = G^{\langle -1 \rangle}_{\tilde{\mu}_{|T|}}(-ih(s)) = is, \qquad s > 0.$$

Take $s_\lambda > 0$ such that for every $s > s_\lambda$, $-ih(s) \in \Omega_\beta$. Then, when $s > s_\lambda$,

$$\mathcal{R}_{\tilde{\mu}_{|T-\lambda\mathbf{1}|}}(-ih(s)) = is + \frac{1}{ih(s)} + \frac{\sqrt{1-4|\lambda|^2 h(s)^2} - 1}{-2ih(s)},$$

implying that

$$h(s) = h_\lambda\left(s + \frac{\sqrt{1-4|\lambda|^2 h(s)^2} - 1}{2h(s)} \right).$$

That is, when $s > s_\lambda$ and

$$t = s + \frac{\sqrt{1-4|\lambda|^2 h(s)^2} - 1}{2h(s)},$$

then $h(s) = h_\lambda(t)$.

Note that if

$$t = s + \frac{\sqrt{1-4|\lambda|^2 h(s)^2} - 1}{2h(s)},$$

then (s, t) satisfies the following equation:

$$(s-t)\left(\frac{1}{h(s)} - s + t \right) = |\lambda|^2. \tag{4.2}$$

In the following we will investigate this equation further.

DEFINITION 4.3. Let $m, n \in \mathsf{N}$, and let U be an open set in R^m. A map $f : U \to \mathsf{R}^n$ is said to be *analytic* if it has a power series expansion in m variables in a neighborhood of every $x \in U$.

We shall need the following two well-known lemmas about analytic functions of several variables:

LEMMA 4.4. *Let U be a connected, open subset of R^m. If $f, g : U \to \mathsf{R}^n$ are two analytic functions which coincide on a non-empty, open subset V of U, then $f = g$.*

LEMMA 4.5. *Let $U \subseteq \mathsf{R}^m$ be open and let $f : U \to \mathsf{R}^m$ be an analytic function for which the Jacobian $\mathscr{J}(x_0) = \det f'(x_0)$ is non-zero for some $x_0 \in U$. Then f is one-to-one in some neighborhood V of x_0, and the inverse of $f|_V$ is analytic in a neighborhood of $f(x_0)$.*

LEMMA 4.6. *Let μ be a probability measure on* $[0, \infty)$, *and define*

$$h(s) = \int_0^\infty \frac{s}{s^2 + u^2}\, d\mu(u), \qquad s > 0. \tag{4.3}$$

Then h is analytic on $(0, \infty)$. *Moreover, if μ is not a Dirac measure, then for all $s > 0$,*

$$0 < h(s) < \frac{1}{s} \qquad \textit{and} \qquad h'(s) < \frac{h(s)}{s} - 2h(s)^2.$$

PROOF. Since

$$h(s) = \frac{1}{2}\int_0^\infty \left(\frac{1}{s + iu} + \frac{1}{s - iu}\right) d\mu, \qquad s > 0,$$

h has a complex analytic extension

$$\tilde{h} : \{z \in \mathsf{C} \mid \operatorname{Im} z > 0\} \to \mathsf{C}$$

given by the same formula. In particular, h is an analytic function of $s \in (0, \infty)$. If μ is not a Dirac measure, then $\mu \neq \delta_0$, and so $h(s) > 0$ for all $s > 0$. Moreover,

$$sh(s) = \int_0^\infty \frac{s^2}{s^2 + u^2}\, d\mu(u) < 1, \qquad s > 0.$$

Finally, for $s > 0$,

$$\begin{aligned} h(s)^2 &= \int_0^\infty \int_0^\infty \frac{s}{s^2+u^2} \frac{s}{s^2+v^2} \, d\mu(u)\, d\mu(v) \\ &\le \int_0^\infty \int_0^\infty \frac{1}{2}\Big(\Big(\frac{s}{s^2+u^2}\Big)^2 + \Big(\frac{s}{s^2+v^2}\Big)^2\Big)\, d\mu(u)\, d\mu(v) \\ &= \int_0^\infty \frac{s^2}{(s^2+u^2)^2}\, d\mu(u) \\ &= \frac{1}{2}\Big(\int_0^\infty \frac{s^2+u^2}{(s^2+u^2)^2}\, d\mu(u) + \int_0^\infty \frac{s^2-u^2}{(s^2+u^2)^2}\, d\mu(u)\Big) \\ &= \frac{1}{2}\Big(\frac{h(s)}{s} - h'(s)\Big). \end{aligned}$$

Hence,

$$h'(s) \le \frac{h(s)}{s} - 2h(s)^2,$$

and equality holds if and only if the product measure $\mu \otimes \mu$ is concentrated on the diagonal $\{(u,u) \mid u > 0\}$. But this would imply that μ is a Dirac measure. Thus, if μ is not a Dirac measure, then

$$h'(s) < \frac{h(s)}{s} - 2h(s)^2, \qquad s > 0$$

LEMMA 4.7. *Let μ be a probability measure on $[0,\infty)$ which is not a Dirac measure, and put*

$$\lambda_1(\mu) = \Big(\int_0^\infty \frac{1}{u^2}\, d\mu(u)\Big)^{-\frac{1}{2}} \qquad \textit{and} \qquad \lambda_2(\mu) = \Big(\int_0^\infty u^2\, d\mu(u)\Big)^{\frac{1}{2}},$$

with the convention that $\infty^{-\frac{1}{2}} = 0$. Then $0 \le \lambda_1(\mu) < \lambda_2(\mu) \le \infty$.

PROOF. Clearly, $\lambda_1(\mu) < \infty$, and since $\mu \ne \delta_0$, $\lambda_2(\mu) > 0$. The lemma is then trivially true if $\lambda_1(\mu) = 0$ or $\lambda_2(\mu) = +\infty$. Thus, we can assume that $\lambda_1(\mu), \lambda_2(\mu) \in (0,\infty)$. Then, by the Schwartz inequality,

$$\begin{aligned} \frac{\lambda_2(\mu)}{\lambda_1(\mu)} &= \Big(\int_0^\infty u^2\, d\mu(u)\Big)^{\frac{1}{2}} \Big(\int_0^\infty \frac{1}{u^2}\, d\mu(u)\Big)^{\frac{1}{2}} \\ &\ge \int_0^\infty u\, \frac{1}{u}\, d\mu(u) = 1, \end{aligned}$$

and equality holds if and only if for some $c \in (0, \infty)$, $\frac{1}{u} = cu$ holds for μ-a.e. $u \in [0, \infty)$. However, this can not be the case when μ is not a Dirac measure.

LEMMA 4.8. *Let μ, $\lambda_1(u)$ and $\lambda_2(\mu)$ be as in Lemma 4.7, and let h be as in Lemma 4.6. Then put*

$$k(s,t) = (s-t)\left(\frac{1}{h(s)} - s + t\right), \qquad s > 0,\ t \in \mathsf{R}.$$

Then k is an analytic function on $(0,\infty) \times \mathsf{R}$. Moreover, for $t > 0$ the map $s \mapsto k(s,t)$ is a strictly increasing bijection of (t,∞) onto $(0,\infty)$, and for $t = 0$ the map $s \mapsto k(s,t)$ is a strictly increasing bijection of $(0,\infty)$ onto $(\lambda_1(\mu)^2, \lambda_2(\mu)^2)$.

PROOF. Clearly, k is analytic. Moreover,

$$\frac{\partial k}{\partial s}(s,t) = \frac{1}{h(s)} - (s-t)\left(2 + \frac{h'(s)}{h(s)^2}\right). \tag{4.4}$$

For $s \in (0,\infty)$, we get from Lemma 4.6 that

$$\frac{\partial k}{\partial s}(s,0) = \frac{s}{h(s)^2}\left(\frac{h(s)}{s} - 2h(s)^2 - h'(s)\right) > 0,$$

and

$$\frac{\partial k}{\partial s}(s,s) = \frac{1}{h(s)} > s.$$

Since the right-hand side of (4.4) is an affine function of $t \in \mathsf{R}$, it follows that

$$\frac{\partial k}{\partial s}(s,t) > t, \qquad s > 0,\ \ t \in [0,s]. \tag{4.5}$$

Hence, $s \mapsto k(s,t)$ is a strictly increasing function of $s \in (t,\infty)$ for every $t \in [0,\infty)$. For $s > t > 0$,

$$k(s,t) = \int_t^s \frac{\partial k}{\partial s'}(s',t)\,ds' > \int_t^s t\,ds' = t(s-t). \tag{4.6}$$

Hence, when $t > 0$,

$$\lim_{s\to\infty} k(s,t) = \infty,$$

and

$$\lim_{s\to t+} k(s,t) = k(t,t) = 0.$$

Thus, $s \mapsto k(s,t)$ is a bijection of (t,∞) onto $(0,\infty)$.

Next, consider the case $t = 0$. We have already seen that $s \mapsto k(s, 0)$ is strictly increasing on $(0, \infty)$. Note that for $s > 0$,

$$k(s, 0) = \frac{1 - sh(s)}{h(s)/s} = \frac{n(s)}{d(s)}$$

where

$$n(s) = \int_0^\infty \frac{u^2}{s^2 + u^2}\, d\mu(u) \qquad \text{and} \qquad d(s) = \int_0^\infty \frac{1}{s^2 + u^2}\, d\mu(u).$$

By the monotone convergence theorem,

$$\lim_{s \to 0+} n(s) = 1,$$

$$\lim_{s \to 0+} d(s) = \int_0^\infty \frac{1}{u^2}\, d\mu(u) = \frac{1}{\lambda_1(\mu)^2},$$

$$\lim_{s \to \infty} s^2 n(s) = \int_0^\infty u^2\, d\mu(u) = \lambda_2(\mu)^2,$$

and

$$\lim_{s \to \infty} s^2 d(s) = 1.$$

Hence,

$$\lim_{s \to 0+} k(s, 0) = \lambda_1(\mu)^2,$$

and

$$\lim_{s \to \infty} k(s, 0) = \lambda_2(\mu)^2.$$

This shows that $s \mapsto k(s, 0)$ is a bijection of $(0, \infty)$ onto $\big(\lambda_1(\mu)^2, \lambda_2(\mu)^2\big)$.

Definition 4.9. Let μ, $\lambda_1(u)$ and $\lambda_2(\mu)$ be as in Lemma 4.7, let h be as in Lemma 4.6, and let k be as in Lemma 4.8. For $\lambda, t \in (0, \infty)$, let $s(\lambda, t)$ denote the unique solution $s \in (t, \infty)$ to the equation $k(s, t) = \lambda^2$ (cf. Lemma 4.8), and for $\lambda \in (\lambda_1(\mu), \lambda_2(\mu))$, let $s(\lambda, 0)$ denote the unique solution $s \in (0, \infty)$ to the equation $k(s, 0) = \lambda^2$.

Lemma 4.10. *The function* $(\lambda, t) \mapsto s(\lambda, t)$ *is analytic in* $(0, \infty) \times (0, \infty)$. *Moreover, for* $\lambda \in (\lambda_1(\mu), \lambda_2(\mu))$,

$$\lim_{t \to 0+} s(\lambda, t) = s(\lambda, 0). \tag{4.7}$$

Proof. Let

$$\Omega = \{(s, t) \in \mathsf{R}^2 \mid 0 < t < s\}.$$

According to Lemma 4.8, k is a strictly positive, analytic function in Ω. Let

$$F(s,t) = \big(\sqrt{k(s,t)}, t\big), \qquad (s,t) \in \Omega.$$

Then F is analytic in Ω, and by Lemma 4.8, F is a one-to-one map of Ω onto $(0,\infty) \times (0,\infty)$. Moreover, its inverse $F^{-1} : (0,\infty) \times (0,\infty) \to \Omega$ is given by

$$F^{-1}(\lambda, t) = (s(\lambda,t), t), \qquad s, t > 0.$$

The Jacobian of F is

$$\mathscr{J}(F)(s,t) = \frac{\partial}{\partial s}\sqrt{k(s,t)} = \frac{1}{2\sqrt{k(s,t)}}\frac{\partial k}{\partial s}(s,t),$$

which by (4.5) is strictly positive for all $(s,t) \in \Omega$. Hence, by Lemma 4.5, F^{-1} is analytic in $(0,\infty) \times (0,\infty)$. In particular, $s(\lambda,t)$ is analytic in $(0,\infty) \times (0,\infty)$.

Now, let $\lambda_0 \in (\lambda_1(\mu), \lambda_2(\mu))$ and put $s_0 = s(\lambda_0, 0)$. Then $k(s_0, 0) = \lambda_0^2$, and by the proof of Lemma 4.8, $\frac{\partial k}{\partial s}(s_0, 0) > 0$. Let

$$F_0(s,t) := \big(\sqrt{k(s,t)}, t\big).$$

F_0 is then analytic in some neighborhood U_0 of $(s_0, 0)$. Moreover, $\mathscr{J}(F_0)(s_0, 0) \neq 0$, and therefore, by Lemma 4.5, F_0 has an analytic inverse F_0^{-1} in a neighborhood V_0 of $F_0(s_0, 0) = (\lambda_0, 0)$. Clearly, $F_0^{-1}(\lambda, t) = F^{-1}(\lambda, t)$, whenever $(\lambda, t) \in V_0 \cap [(0,\infty) \times (0,\infty)]$, and $F_0^{-1}(\lambda, t) \in \Omega$.

Note that

$$\lim_{t \to 0+} F_0^{-1}(\lambda_0, t) = F_0^{-1}(\lambda_0, 0) = (s_0, 0), \tag{4.8}$$

and since the second coordinate of $F_0^{-1}(\lambda_0, t)$ is t, we conclude that $F_0^{-1}(\lambda_0, t) \in \Omega$, eventually as $t \to 0+$. Hence,

$$(s_0, 0) = \lim_{t \to 0+} F_0^{-1}(\lambda_0, t) = \lim_{t \to 0+} F^{-1}(\lambda_0, t) = \lim_{t \to 0+} (s(\lambda_0, t), t),$$

and therefore,

$$\lim_{t \to 0+} s(\lambda_0, t) = s_0 = s(\lambda_0, 0).$$

REMARK 4.11. We get from Lemma 4.10 that

$$\lim_{t \to 0+} s(\lambda, t) = 0, \qquad 0 < \lambda \le \lambda_1(\mu), \tag{4.9}$$

and

$$\lim_{t\to 0+} s(\lambda, t) = +\infty, \qquad \lambda \geq \lambda_2(\mu). \tag{4.10}$$

Indeed, for fixed $t > 0$, $\lambda \mapsto s(\lambda, t)$ is a monotonically increasing function of λ. Hence, if $0 < \lambda \leq \lambda_1(\mu)$, then

$$\limsup_{t\to 0+} s(\lambda, t) \leq \limsup_{t\to 0+} s(\lambda', t) = s(\lambda', 0),$$

for all $\lambda' \in (\lambda_1(\mu), \lambda_2(\mu))$.

But $\lambda' \mapsto s(\lambda', 0)$ is the inverse function of $s \mapsto \sqrt{k(s,0)}$, and hence $\lambda' \mapsto s(\lambda', 0)$ is a bijection of $(\lambda_1(\mu), \lambda_2(\mu))$ onto $(0, \infty)$. It follows that $\limsup_{t\to 0+} s(\lambda, t) = 0$, and this proves (4.9).

For $\lambda \geq \lambda_2(\mu)$, a similar argument shows that $\liminf_{t\to 0+} s(\lambda, t) = +\infty$, and this proves (4.10).

LEMMA 4.12. *Let $\lambda > 0$. Then*

(i) $\lim_{t\to\infty}(s(\lambda, t) - t) = 0$*, and*

(ii) *there exists a $t_\lambda > 0$ such that when $t > t_\lambda$ and $s = s(\lambda, t)$, then*

$$t = s + \frac{\sqrt{1 + 4\lambda^2 h(s)^2} - 1}{2h(s)}.$$

PROOF. Fix $t > 0$, and put $s = s(\lambda, t)$. Then by Definition 4.9, $s > t$ and $k(s, t) = \lambda^2$. According to (4.6), $k(s, t) > t(s - t)$. Hence,

$$0 < s - t < \frac{\lambda^2}{t}.$$

This proves (i). With s and t as above,

$$\lambda^2 = k(s, t) = (s - t)\left(\frac{1}{h(s)} - s + t\right).$$

Solving this equation for t, we get that t is one of the two numbers

$$t_\pm = s - \frac{1}{h(s)} \pm \frac{\sqrt{1 + 4\lambda^2 h(s)^2}}{2h(s)}.$$

If $t = t_-$, then

$$s - t > \frac{1}{2h(s)},$$

and since $\frac{1}{h(s)} \to \infty$ as $s \to \infty$, this can not hold for large t because of (i). Hence, $t = t_+$ for t sufficiently large.

Combining the previous lemmas we get:

PROPOSITION 4.13. *Let $T \in \tilde{\mathcal{M}}$ be an R-diagonal element, let $\lambda \in \mathsf{C} \setminus \{0\}$, and define $h(s)$ and $h_\lambda(s)$ as in Lemma 4.2. Let $\mu = \mu_{|T|}$, and let $s(|\lambda|, t)$ be as in Definition 4.9. Then*

$$h_\lambda(s(|\lambda|, t)) = h(t), \qquad t > 0.$$

PROOF. According to Lemma 4.12, if $t > t_{|\lambda|}$ and $s = s(|\lambda|, t)$, then

$$t = s + \frac{\sqrt{1 + 4\lambda^2 h(s)^2} - 1}{2h(s)}.$$

Since $s(|\lambda|, t) > t$, we infer from Lemma 4.2 that for t sufficiently large,

$$h_\lambda(t) = h(s(|\lambda|, t)).$$

Hence, by Lemma 4.4 and Lemma 4.10, the same formula holds for all $t > 0$.

LEMMA 4.14. *Let T be an unbounded R-diagonal element in $\mathcal{M}^\Delta$, let $\lambda \in \mathsf{C} \setminus \{0\}$, and let $t > 0$. With $\mu = \mu_{|T|}$ and $s(|\lambda|, t)$ as in Definition 4.9 we then have:*

(4.11)

$$\Delta\big((T - \lambda\mathbf{1})^*(T - \lambda\mathbf{1}) + t^2\mathbf{1}\big) = \frac{|\lambda|^2}{|\lambda|^2 + (s(|\lambda|, t) - t)^2}\Delta\big(T^*T + s(|\lambda|, t)^2\mathbf{1}\big).$$

PROOF. Since T is R-diagonal, $T \underset{*\mathcal{D}}{\sim} cT$ for all $c \in \mathsf{T}$. Hence, the left-hand side of (4.11) depends only on $|\lambda|$. It therefore suffices to consider only the case $\lambda > 0$. For $\lambda, t > 0$, let

$$H(t) = \tfrac{1}{2} \log \Delta(T^*T + t^2\mathbf{1})$$

and

$$H_\lambda(t) = \tfrac{1}{2} \log \Delta\big((T - \lambda\mathbf{1})^*(T - \lambda\mathbf{1}) + t^2\mathbf{1}\big).$$

Then with $\mu_\lambda = \mu_{|T - \lambda\mathbf{1}|}$,

$$H(t) = \frac{1}{2} \int_0^\infty \log(u^2 + t^2)\, d\mu(u),$$

and

$$H_\lambda(t) = \frac{1}{2} \int_0^\infty \log(u^2 + t^2)\, d\mu_\lambda(u).$$

Since T and $T - \lambda\mathbf{1}$ belong to $\mathcal{M}^{\Delta}$, H and H_λ take values in R. Moreover, H and H_λ are differentiable with derivatives $H'(t) = h(t)$ and $H'_\lambda(t) = h_\lambda(t)$. Also, since $T \in \mathcal{M}^{\Delta}$,

$$\lim_{t\to\infty}(H(t) - \log t) = \frac{1}{2}\lim_{t\to\infty}\int_0^\infty \log\Big(1 + \frac{u^2}{t^2}\Big)\,d\mu(u) = 0, \tag{4.12}$$

and similarly

$$\lim_{t\to\infty}(H_\lambda(t) - \log t) = 0. \tag{4.13}$$

Fix $\lambda > 0$ and $t_0 > 0$. There is a constant C such that

$$H_\lambda(t) = \int_{t_0}^t h_\lambda(t')\,dt' + C.$$

Moreover, according to Proposition 4.13,

$$h_\lambda(t) = h(s(\lambda, t)), \qquad t > 0.$$

Put $s(t) = s(\lambda, t)$ and $u(t) = t - s(t)$. Then $s(t) + u(t) = t$ and $s'(t) + u'(t) = 1$. Moreover, by Definition 4.9,

$$(s(t) - t)\Big(\frac{1}{h(s(t))} - s(t) + t\Big) = \lambda^2.$$

Hence,

$$u(t)\Big(\frac{1}{h(s(t))} - u(t)\Big) = \lambda^2,$$

implying that

$$h(s(t)) = \frac{u(t)}{\lambda^2 + u(t)^2}.$$

It follows that

$$\begin{aligned}
&\int_{t_0}^t h_\lambda(v)\,dv \\
&\qquad = \int_{t_0}^t h(s(v))(s'(v) + u'(v))\,dv \\
&\qquad = \int_{t_0}^t \Big(h(s(v))s'(v) + \frac{u(v)}{\lambda^2 + u(v)^2}u'(v)\Big)\,dv \\
&\qquad = H(s(t)) - H(s(t_0)) + \frac{1}{2}\log\Big(\frac{\lambda^2}{\lambda^2 + u(t)^2}\Big) + \frac{1}{2}\log\Big(\frac{\lambda^2 + u(t_0)^2}{\lambda^2}\Big).
\end{aligned}$$

Hence,

$$H_\lambda(t) = H(s(t)) + \frac{1}{2}\log\left(\frac{\lambda^2}{\lambda^2 + (s(t)-t)^2}\right) + C',$$

for a constant C'. Recall that $s(t) - t \to 0$ as $t \to \infty$ (cf. Lemma 4.12). It then follows from (4.12) and (4.13) that C' must be 0. This finally shows us that

$$\exp(2H_\lambda(t)) = \frac{\lambda^2}{\lambda^2 + (s(t)-t)^2}\,\exp(2H(t)),$$

and this proves (4.11).

THEOREM 4.15. *Let $T \in \mathcal{M}^\Delta$ be R-diagonal, let $\mu = \mu_{|T|}$, and let $s(|\lambda|, 0)$ be as in Definition 4.9.*

(i) *If $\lambda_1(\mu) < |\lambda| < \lambda_2(\mu)$, then*

$$\Delta(T - \lambda\mathbf{1}) = \left(\frac{|\lambda|^2}{|\lambda|^2 + s(|\lambda|,0)^2}\,\Delta(T^*T + s(|\lambda|,0)^2\mathbf{1})\right)^{\frac{1}{2}}.$$

(ii) *If $|\lambda| \le \lambda_1(\mu)$, then $\Delta(T - \lambda\mathbf{1}) = \Delta(T)$.*

(iii) *If $|\lambda| \ge \lambda_2(\mu)$, then $\Delta(T - \lambda\mathbf{1}) = |\lambda|$.*

PROOF. The theorem is obviously true for $\lambda = 0$. Moreover, as in the proof of Lemma 4.14, it suffices to consider the case $\lambda > 0$. Note that

$$\Delta(T - \lambda\mathbf{1})^2 = \lim_{t\to 0+} \Delta\big((T-\lambda\mathbf{1})^*(T-\lambda\mathbf{1}) + t^2\mathbf{1}\big). \tag{4.14}$$

Hence, (i) follows from Lemma 4.10 and Lemma 4.14. If $0 < \lambda \le \lambda_1(\mu)$, then by Remark 4.11, $\lim_{t\to 0+} s(\lambda, t) = 0$. Hence, (ii) also follows from Lemma 4.14. Now suppose $\lambda \ge \lambda_2(\mu)$. Then $s(\lambda, t) \to \infty$ as $t \to 0+$. The right-hand side of (4.11) is equal to

$$\frac{\lambda^2 s(\lambda,t)^2}{\lambda^2 + (s(\lambda,t)-t)^2}\,\frac{\Delta(T^*T - s(\lambda,t)^2\mathbf{1})}{s(\lambda,t)^2},$$

where the first factor converges to λ^2 as $t \to 0+$, and the second factor converges to 1 (cf. (4.12)). (iii) now follows from (4.11) and (4.14).

REMARK 4.16. Note that

$$\lambda_2(\mu) = \left(\int_0^\infty u^2\, d\mu_{|T|}(u)\right)^{\frac{1}{2}} = \|T\|_2$$

and

$$\lambda_1(\mu) = \left(\int_0^\infty u^{-2}\, d\mu_{|T|}(u)\right)^{-\frac{1}{2}} = \|T^{-1}\|_2^{-1},$$

where $\|T^{-1}\|_2 := +\infty$ in case $\ker(T) \neq 0$.

THEOREM 4.17. *Let T be an R-diagonal element in $\mathcal{M}^\Delta$ with Brown measure μ_T, and suppose $\mu_{|T|}$ is not a Dirac measure.*

(a) *If* $\ker(T) = 0$, *then*

$$\operatorname{supp}(\mu_T) = \big\{\lambda \in \mathsf{C} \mid \|T^{-1}\|_2^{-1} \leq |\lambda| \leq \|T\|_2\big\}.$$

Moreover, the S-transform of $\mu_{|T|^2}$ is well-defined and strictly increasing on $(-1, 0)$ *with*

$$\mathcal{S}_{\mu_{|T|^2}}((-1,0)) = \big(\|T\|_2^{-2}, \|T^{-1}\|_2^2\big),$$

and μ_T is the unique probability measure on C *which is invariant under rotations and satisfies*

$$\mu_T\big(B(0, \mathcal{S}_{\mu_{|T|^2}}(t-1)^{-\frac{1}{2}})\big) = t, \qquad 0 < t < 1.$$

(b) *If* $\ker(T) \neq 0$, *let P denote the projection onto* $\ker(T)$. *Then*

$$\operatorname{supp}(\mu_T) = \{\lambda \in \mathsf{C} \mid |\lambda| \leq \|T\|_2\}.$$

Moreover, the S-transform of $\mu_{|T|^2}$ is well-defined and strictly increasing on $(\tau(P) - 1, 0)$ *with*

$$\mathcal{S}_{\mu_{|T|^2}}((\tau(P)-1, 0)) = (\|T\|_2^{-2}, \infty),$$

and μ_T is the unique probability measure on C *which is invariant under rotations and satisfies*

$$\mu_T\big(B(0, \mathcal{S}_{\mu_{|T|^2}}(t-1)^{-\frac{1}{2}})\big) = t, \qquad \tau(P) < t < 1.$$

PROOF. By definition, $d\mu_T(\lambda) = \frac{1}{2\pi}\nabla^2\big(\log\Delta(T - \lambda\mathbf{1})\big)d\lambda$ (in the distribution sense). Hence, μ_T can be determined from Theorem 4.15 in the same way as [9, Theorem 4.4.] is obtained from [9, (4.5)]:

Using the same notation as in [9], we define functions $f, g : (0, \infty) \to \mathsf{R}$ by

$$f(v) = \int_0^\infty \frac{1}{1 + v^2 w^2}\, d\mu_{|T|}(w),$$

and

$$g(v) = \frac{1 - f(v)}{v^2 f(v)}.$$

Moreover, for $\lambda \in (\|T^{-1}\|_2^{-2}, \|T\|_2^2)$, let $v(\lambda)$ denote the unique $v \in (0,\infty)$ such that $g(v) = \lambda^2$. Then, in our notation,

$$f(v) = \tau\big((\mathbf{1} + v^2T^*T)^{-1}\big) = v^{-1}h(v^{-1}),$$

and

$$g(v) = v^{-1}\left(\frac{1}{h(v^{-1})} - v^{-1}\right) = k(v^{-1}, 0).$$

Hence,

$$v(\lambda) = \frac{1}{s(\lambda, 0)},$$

and it follows that the formula (4.15) in [9],

$$\log \Delta(T - \lambda\mathbf{1}) = \frac{1}{2}\int_0^\infty \log(1 + v^2w^2)\, d\mu_{|T|}(w) + \frac{1}{2}\log\left(\frac{\lambda^2}{1 + v^2\lambda^2}\right),$$

$\lambda \in (\|T^{-1}\|_2^{-2}, \|T\|_2^2)$, is equivalent to the one in Theorem 4.15 (i). The rest of the proof of Theorem 4.17 is identical to the second part of the proof of [9, Theorem 4.4], since boundedness of T is not a necessary assumption in the latter.

REMARK 4.18. Let $T \in \mathcal{M}^\Delta$ be R-diagonal. Then $\operatorname{supp}(\mu_T) \subseteq \sigma(T)$, and according to Theorem 4.17,

$$\operatorname{supp}(\mu_T) = \{\lambda \in \mathsf{C} \mid \|T^{-1}\|_2^{-1} \le |\lambda| \le \|T\|_2\}.$$

Moreover, by arguments similar to the ones given in [9, proof of Proposition 4.6], one can show that

(a) if $0 < |\lambda| < \|T^{-1}\|_2^{-1}$, then $\lambda \in \sigma(T)$ iff T does not have a bounded inverse, and

(b) if $|\lambda| > \|T\|_2$, then $\lambda \in \sigma(T)$ iff T is not bounded.

5. Properties of $z = xy^{-1}$

Let $\mathcal{M} = L(\mathsf{F}_4)$ be the von Neumann algebra associated with the free group on 4 generators. According to [17] or [18], $\mathcal{M}$ is a II$_1$-factor generated by a semicircular system (s_1, s_2, s_3, s_4), i.e. the s_i's are freely independent self-adjoint elements w.r.t. the unique tracial state τ on $\mathcal{M}$, and s_i has distribution

$$d\mu_{s_i}(t) = \frac{1}{2\pi}\sqrt{4 - t^2}\, 1_{[-2,2]}(t)\, dt, \qquad 1 \le i \le 4.$$

Put

$$x = \frac{s_1 + is_2}{\sqrt{2}} \quad \text{and} \quad y = \frac{s_3 + is_4}{\sqrt{2}}.$$

Then $\mathcal{M} = W^*(x, y)$, and (x, y) is a circular system in the sense of [18]. Also, by [18], $|y|$ has the distribution

$$d\mu_{|y|}(t) = \frac{2}{\pi}\sqrt{4 - t^2}\, 1_{[0,2]}(t)\, dt.$$

In particular, $\ker(y) = 0$. In this section we will study the unbounded operator

$$z = xy^{-1}$$

as well as its powers z^n, $n = 2, 3, \ldots$. We will need the following simple observation:

LEMMA 5.1. *Let $(\mu_n)_{n=1}^\infty$ and μ be probability measures on R with densities $(f_n)_{n=1}^\infty$ and f, respectively, w.r.t. Lebesgue measure. If $f_n \overset{n\to\infty}{\longrightarrow} f$ a.e. w.r.t. Lebesgue measure, then $\mu_n \overset{n\to\infty}{\longrightarrow} \mu$ weakly.*

PROOF. Recall that $\mu_n \overset{n\to\infty}{\longrightarrow} \mu$ weakly iff for all $\phi \in C_0(\mathsf{R})$,

$$\lim_{n\to\infty} \int_{\mathsf{R}} \phi\, d\mu_n = \int_{\mathsf{R}} \phi\, d\mu. \tag{5.1}$$

Clearly, it suffices to consider $\phi \in C_0(\mathsf{R})$ with $0 \le \phi \le 1$, and for such ϕ, (5.1) follows for such ϕ by application of Fatou's Lemma to each of the sequences of integrals $\left(\int_{\mathsf{R}} \phi f_n\, dm\right)_{n=1}^\infty$ and $\left(\int_{\mathsf{R}} (1-\phi) f_n\, dm\right)_{n=1}^\infty$.

THEOREM 5.2. *Let $(\mathcal{M}, \tau)$ and $z = xy^{-1}$ be as above.*

(a) *z is an unbounded, R-diagonal operator.*

(b) *The distribution of z is given by*

$$d\mu_{|z|}(t) = \frac{2}{\pi}\frac{1}{1+t^2}\, 1_{(0,\infty)}(t)\, dt. \tag{5.2}$$

(c) *For $p \in (0, 1)$, $z, z^{-1} \in L^p(\mathcal{M}, \tau)$, and*

$$\|z\|_p^p = \|z^{-1}\|_p^p = \left[\cos\left(\frac{p\pi}{2}\right)\right]^{-1} < \infty. \tag{5.3}$$

(d) *$z, z^{-1} \in \mathcal{M}^\Delta$, and the Brown measure of z is given by*

$$d\mu_z(s) = \frac{1}{\pi(1+|s|^2)^2}\, ds, \tag{5.4}$$

where $ds = d\,\mathrm{Re}\, s\, d\,\mathrm{Im}\, s$ is Lebesgue measure on C.

PROOF. (a) Let $x = u|x|$ and $y = v|y|$ be the polar decompositions of x and y. Then, according to [18], u, $|x|$, v and $|y|$ are $*$-free elements, and u and v are Haar unitaries. In particular, x and y are R-diagonal and so is y^{-1} (cf. Proposition 3.6). Moreover, y^{-1} has polar decomposition

$$y^{-1} = v^*(v|y|^{-1}v^*) = v^*|y^*|^{-1},$$

which implies that y^{-1} is affiliated with $W^*(y)$. Hence, x and y^{-1} are $*$-free, and it follows from Proposition 3.8 that $z = xy^{-1}$ is R-diagonal with

$$\mathcal{S}_{\mu_{|z|^2}}(t) = \mathcal{S}_{\mu_{|x|^2}}(t)\,\mathcal{S}_{\mu_{|y^{-1}|^2}}(t), \qquad t \in (-1, 0).$$

The distribution of $|x|^2$ has density

$$d\mu_{|x|^2}(t) = \frac{1}{2\pi}\sqrt{\frac{4-t}{t}}\,1_{(0,4)}(t)\,dt,$$

and thus $\mathcal{S}_{\mu_{|x|^2}}$ is given by

$$\mathcal{S}_{\mu_{|x|^2}}(t) = \frac{1}{1+t}$$

for all t in a neighborhood of $(-1, 0)$ (cf. [9, example 5.2]). Since $|y^{-1}| = |y^*|^{-1} \underset{*\mathcal{D}}{\sim} |y|^{-1} \underset{*\mathcal{D}}{\sim} |x|^{-1}$, we get from Proposition 3.13 that

$$\mathcal{S}_{\mu_{|y^{-1}|^2}}(t) = \frac{1}{\mathcal{S}_{\mu_{|x|^2}}(-1-t)} = -t, \qquad t \in (-1, 0).$$

Then

$$\mathcal{S}_{\mu_{|z|^2}}(t) = -\frac{t}{1+t}, \qquad t \in (-1, 0), \tag{5.5}$$

and

$$\chi_{\mu_{|z|^2}}(t) = \frac{t}{1+t}\mathcal{S}_{\mu_{|z|^2}}(t) = -\left(\frac{t}{1+t}\right)^2, \qquad t \in (-1, 0).$$

The inverse function of $\chi_{\mu_{|z|^2}}$ is then

$$\psi_{\mu_{|z|^2}}(u) = \frac{-\sqrt{-u}}{1+\sqrt{-u}}, \qquad u \in (-\infty, 0),$$

and it follows that

$$G_{\mu_{|z|^2}}(\lambda) = \frac{1}{\lambda}\left(1 + \psi_{\mu_{|z|^2}}\left(\frac{1}{\lambda}\right)\right) = \frac{1}{\lambda - \sqrt{-\lambda}}, \qquad \lambda < 0. \tag{5.6}$$

Let $\sqrt{w}$ denote the principal value of the square root of w for $w \in \mathbf{C} \setminus (\infty, 0]$. Then both sides of (5.6) are analytic in $\mathbf{C} \setminus [0, \infty)$. Thus, (5.6) holds for all $\lambda \in \mathbf{C} \setminus [0, \infty)$, and it follows that for $t > 0$,

$$(5.7) \quad -\frac{1}{\pi} \lim_{u \to 0+} \operatorname{Im} G_{\mu_{|z|^2}}(t + iu) = -\frac{1}{\pi} \operatorname{Im}\left(\frac{1}{t + i\sqrt{t}}\right) = \frac{1}{\pi} \frac{1}{\sqrt{t}(t+1)}.$$

For $\beta \in (0, 1)$,

$$(5.8) \qquad \int_0^\infty \frac{t^{\beta-1}}{1+t}\, dt = \frac{\pi}{\sin(\beta\pi)},$$

(cf. [12, p. 592, formula 613]). The right-hand side of (5.7) therefore defines the density of a probability measure, and then, by Lemma 5.1, the probability measures

$$\frac{1}{\pi} \operatorname{Im} G_{\mu_{|z|^2}}(t + iu)\, dt, \qquad u > 0,$$

converge weakly to

$$(5.9) \qquad \frac{1}{\pi} \frac{1}{\sqrt{t}(t+1)} 1_{(0,\infty)}(t)\, dt,$$

as $u \to 0+$. Hence, by the inverse Stieltjes transform, $d\mu_{|z|^2}(t)$ is given by (5.9), and then

$$d\mu_{|z|}(t) = \frac{2}{\pi} \frac{1}{1+t^2} 1_{(0,\infty)}(t)\, dt.$$

This proves (a) and (b).

In order to prove (c), note that according to (5.8),

$$\tau(|z|^p) = \frac{2}{\pi} \int_0^\infty \frac{t^p}{1+t^2}\, dt = \frac{1}{\pi} \int_0^\infty \frac{w^{\frac{p-1}{2}}}{1+w}\, dw = \left[\sin\left(\frac{\pi(p+1)}{2}\right)\right]^{-1},$$

proving (c). Since $L^p(\mathcal{M}, \tau) \subseteq \mathcal{M}^\Delta$, $p > 0$, $z, z^{-1} \in \mathcal{M}^\Delta$. According to Theorem 4.17, μ_z is then the unique probability measure on $\mathbf{C}$ which is invariant under rotations and satisfies

$$\mu_z\big(B(0, \mathcal{S}_{\mu_{|z|^2}}(t-1)^{-\frac{1}{2}})\big) = t, \qquad 0 < t < 1.$$

Then by (5.5),

$$\mu_z\left(B\left(0, \sqrt{\frac{t}{1-t}}\right)\right) = t, \qquad 0 < t < 1,$$

that is,

$$\mu_z(B(0,r)) = \frac{r^2}{1+r^2}, \qquad r > 0.$$

Hence, $\frac{d}{dr}\mu_z(B(0,r)) = \frac{2r}{(1+r^2)^2}$, and combining this with the fact that μ_z is invariant under rotations, we find that μ_z has density w.r.t. Lebesgue measure on C given by

$$\frac{1}{2\pi r}\frac{2r}{(1+r^2)^2} = \frac{1}{\pi}\frac{1}{(1+r^2)^2}, \qquad r > 0,$$

where $r = |s|$, $s \in \mathsf{C} \setminus \{0\}$. This proves (d).

LEMMA 5.3. *Let μ be a probability measure on $[0,\infty)$ and, as in section 5, put*

$$h(s) = \int_0^\infty \frac{s}{s^2+u^2}\, d\mu(u), \qquad s \in (0,\infty).$$

Then for $0 < p < 2$,

$$\int_0^\infty u^{-p}\, d\mu(u) = \frac{2}{\pi}\sin\left(\frac{\pi p}{2}\right)\int_0^\infty s^{-p}h(s)\, ds. \tag{5.10}$$

PROOF. By Tonelli's theorem,

$$\int_0^\infty s^{-p}h(s)\, ds = \int_0^\infty \left(\int_0^\infty \frac{s^{1-p}}{s^2+u^2}\, ds\right) d\mu(u).$$

Letting $s = ut^{\frac{1}{2}}$, we find (using (5.8)) that

$$\int_0^\infty \frac{s^{1-p}}{s^2+u^2}\, ds = \frac{1}{2}u^{-p}\int_0^\infty \frac{t^{-\frac{p}{2}}}{1+t}\, dt = \frac{\pi}{2}\left[\sin\left(\frac{\pi p}{2}\right)\right]^{-1} u^{-p}.$$

This proves (5.10).

THEOREM 5.4. *Let $(\mathcal{M},\tau)$ and z be as in Theorem 5.2, and let $n \in \mathsf{N}$.*

(a) *z^n is an unbounded R-diagonal operator.*

(b)

$$\int_0^\infty \frac{s}{s^2+u^2}\, d\mu_{|z|^n}(u) = \left(s + s^{\frac{n-1}{n+1}}\right)^{-1}, \qquad s > 0. \tag{5.11}$$

(c) *For $p \in \left(0, \frac{2}{n+1}\right)$, z^n and z^{-n} both belong to $L^p(\mathcal{M},\tau)$, and*

$$\|z^n\|_p^p = \|z^{-n}\|_p^p = \frac{(n+1)\sin\left(\frac{\pi p}{2}\right)}{\sin\left(\frac{(n+1)\pi p}{2}\right)}. \tag{5.12}$$

(d) *If* $p \in \left(0, \frac{2}{n+1}\right)$ *and* $\lambda \in \mathsf{C}$, *then* $\ker(z^n - \lambda \mathbf{1}) = 0$. *Moreover,* $(z^n - \lambda\mathbf{1})^{-1} \in L^p(\mathcal{M}, \tau)$ *with*

$$\|(z^n - \lambda\mathbf{1})^{-1}\|_p \le \|z^{-n}\|_p. \tag{5.13}$$

PROOF. According to Proposition 3.9, z^n is R-diagonal. Moreover, since

$$\mathcal{S}_{\mu_{|z|^2}}(t)^n = \left(-\frac{t}{1+t}\right)^n, \qquad t \in (-1, 0),$$

$$\chi_{\mu_{|z^n|^2}}(t) = \frac{1}{1+t}\mathcal{S}_{\mu_{|z^n|^2}}(t) = -\left(-\frac{t}{1+t}\right)^{n+1}, \qquad t \in (-1, 0),$$

with inverse function

$$\psi_{\mu_{|z^n|^2}}(u) = -\frac{(-u)^{\frac{1}{n+1}}}{1 + (-u)^{\frac{1}{n+1}}}, \qquad u \in (-\infty, 0).$$

Hence, for $\lambda \in (-\infty, 0)$,

$$G_{\mu_{|z^n|^2}}(\lambda) = \frac{1}{\lambda}\left(1 + \psi_{\mu_{|z^n|^2}}\left(\frac{1}{\lambda}\right)\right) = \frac{1}{\lambda\left(1 + (-\lambda)^{-\frac{1}{n+1}}\right)}. \tag{5.14}$$

Let

$$h_n(s) = \int_0^\infty \frac{s}{s^2 + u^2}\, d\mu_{|z^n|}(u), \qquad s \in (0, \infty).$$

Then

$$h_n(s) = s\,\tau\big((s^2\mathbf{1} + |z^n|^2)^{-1}\big) = -s\, G_{\mu_{|z^n|^2}}(-s^2) \overset{(5.14)}{=} \left(s + s^{\frac{n-1}{n+1}}\right)^{-1}.$$

This proves (b).

Since $z = xy^{-1}$, where (x, y) is a circular family, it is clear that $z^{-n} \underset{*\mathcal{D}}{\sim} z^n$ for all $n \in \mathsf{N}$. Hence, $\|z^n\|_p = \|z^{-n}\|_p$ for all $p > 0$. Note that for $p > 0$,

$$\|z^{-n}\|_p^p = \tau(|z^{-n}|^p) = \tau(|(z^n)^*|^{-p}) = \tau(|z^n|^{-p}).$$

Thus, by Lemma 5.3, for $p \in (0, 2)$,

$$\|z^{-n}\|_p^p = \int_0^\infty u^{-p}\, d\mu_{|z^n|}(u) = \frac{2}{\pi}\sin\left(\frac{\pi p}{2}\right)\int_0^\infty s^{-p} h_n(s)\, ds. \tag{5.15}$$

By application of (5.11) we find that

$$\int_0^\infty s^{-p} h_n(s)\, ds = \int_0^\infty \frac{s^{-p-\frac{n-1}{n+1}}}{s^{\frac{2}{n+1}} + 1}\, ds = \frac{n+1}{2}\int_0^\infty \frac{t^{-\frac{(n+1)p}{2}}}{1+t}\, dt.$$

Then by (5.15) and (5.8), for $0 < p < \frac{2}{n+1}$,

$$
\begin{aligned}
\|z^{-n}\|_p^p &= (n+1)\sin\left(\frac{\pi p}{2}\right)\left[\sin\left(\pi\left(1-\frac{(n+1)p}{2}\right)\right)\right]^{-1} \\
&= (n+1)\sin\left(\frac{\pi p}{2}\right)\left[\sin\left(\frac{(n+1)\pi p}{2}\right)\right]^{-1},
\end{aligned} \tag{5.16}
$$

and this proves (c). Note that the right-hand side of (5.16) converges to ∞ as $p \to \frac{2}{n+1}-$. Hence, $z^{-n} \notin L^{\frac{2}{n+1}}(\mathcal{M}, \tau)$, and the same holds for z^n. In particular, z^n is not bounded, and this proves (a). In order to prove (d), let $\lambda \in \mathbf{C} \setminus \{0\}$, and put

$$h_{n,\lambda}(t) = \int_0^\infty \frac{t}{t^2+u^2}\, d\mu_{|z^n-\lambda\mathbf{1}|}(u), \qquad t > 0.$$

Then by Proposition 4.13,

$$h_{n,\lambda}(t) = h_n(s_n(|\lambda|, t)), \qquad t > 0,$$

where $s_n(|\lambda|, t)$ is given by Definition 4.9 in the case $\mu = \mu_{|z^n|}$. Note that, according to Definition 4.9,

$$s_n(|\lambda|, t) > t, \qquad t > 0.$$

Moreover, by (5.11), h_n is monotonically decreasing on $(0, \infty)$. Thus,

$$h_{n,\lambda}(t) \le h_n(t), \qquad t > 0.$$

It now follows from Lemma 5.3 that for $p \in (0, 2)$,

$$\int_0^\infty u^{-p}\, d\mu_{|z^n-\lambda\mathbf{1}|}(u) \le \int_0^\infty u^{-p}\, d\mu_{|z^n|}(u). \tag{5.17}$$

According to (c), the right-hand side of (5.17) is finite for $p \in \left(0, \frac{2}{n+1}\right)$. Hence, for such p, $\ker(z^n - \lambda\mathbf{1}) = 0$, $(z^n - \lambda\mathbf{1})^{-1} \in L^p(\mathcal{M}, \tau)$, and

$$\|(z^n - \lambda\mathbf{1})^{-1}\|_p^p \le \|z^{-n}\|_p^p.$$

REMARK 5.5. Note that Theorem 5.4 (a) and (c) generalize Theorem 5.2 (a) and (c) to all $n \in \mathbf{N}$. It is not hard to generalize Theorem 5.2 (b) and (d) as well. One finds that the distribution of $|z^n|$ is given by

$$d\mu_{|z^n|}(t) = \frac{2}{\pi}\,\frac{\sin\left(\frac{\pi}{n+1}\right)}{t\left(t^{\frac{2}{n+1}} + 2\cos\left(\frac{\pi}{n+1}\right) + t^{-\frac{2}{n+1}}\right)r}\, 1_{(0,\infty)(t)}\, dt,$$

and the Brown measure of z^n is given by

$$d\mu_{z^n}(s) = \frac{1}{n\pi} \frac{|s|^{\frac{2}{n}-2}}{\left(1+|s|^{\frac{2}{n}}\right)^2}\, d\,\mathrm{Re}\, s\, d\,\mathrm{Im}\, s.$$

We leave the details of proof to the reader.

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