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NON-STABLE K-THEORY FOR QB-RINGS

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(Dedicated to the memory of Gert K. Pedersen)

Abstract

We study the class of QB-rings that satisfy the weak cancellation condition of separativity for finitely generated projective modules. This property turns out to be crucial for proving that all (quasi-)invertible matrices over a QB-ring can be diagonalised using row and column operations. The main two consequences of this fact are: (i) The natural map $\mathrm{GL}_1(R) \to K_1(R)$ is surjective, and (ii) the only obstruction to lift invertible elements from a quotient is of K-theoretical nature. We also show that for a reasonably large class of QB-rings that includes the prime ones, separativity always holds.

Introduction

The object of this paper is the study of K-theoretical aspects of the class of QB-rings. This class was introduced by G. K. Pedersen and the authors in [8], and was subsequently studied in [9], [10], [28]. We say that a unital ring R is a QB-ring if any left unimodular row can be reduced in a special way. Specifically, R is a QB-ring if

$$Ra + Rb = R \Longrightarrow a + yb \in R_q^{-1}.$$

Here, R_q^{-1} is the set of *quasi-invertible* elements of the ring R (see below for the precise definition), which in the prime case is nothing else but the union of right and left invertible elements. The reader will have noticed that by replacing R_q^{-1} with the set R^{-1} of invertible elements we are back to the definition of stable rank one. The class of QB-rings is, however, much larger than the class of rings with stable rank one, yet they enjoy specially good structural properties. For example, the property of being a QB-ring is stable under matrix formation, passage to corners, surjective pullbacks, suitable direct and inverse limits (see [8], [10], [28]). The behaviour under extensions is considerably more complicated than that for rings with stable rank one, hence an extension

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of two QB-rings is not always a QB-ring. This reflects the fact that to work with QB-rings one needs a larger display of technology.

The notion of a QB-ring just explained comes from the geometrical concept of extremal richness for C^*-algebras, coined by L. G. Brown and G. K. Pedersen in [14] and explored further in other papers ([16], [15], [17], [23], [30], among others). We say that a C^*-algebra A is extremally rich provided its unit ball is the convex hull of its extreme points. This is in turn equivalent to the fact that the set of quasi-invertible elements is dense in the norm-topology.

The fact that rings with stable rank one are particularly well behaved with respect to non-stable K-theoretical properties motivates the quest for similar results for this bigger class of rings. For example, it would be desirable that the Whitehead group of any QB-ring is generated by the group of units of the ring itself. Results in this direction have already been obtained for C^*-algebras in [17].

One of the key aspects of rings with stable rank one is that their category of finitely generated projective modules is cancellative. That is,

$$A \oplus C \cong B \oplus C \Longrightarrow A \cong B,$$

whenever A, B and C are finitely generated projective modules.

Of course, we cannot expect that QB-rings enjoy full cancellation as simple examples show, but weaker forms of cancellation might still hold. One of such conditions is that of *separative cancellation*. This was introduced in Ring Theory in [5] borrowed from Semigroup Theory (see, e.g. [19], [21]) and has been around in the C^*-algebra scene for some time, notably in [17] (see also [29]). If, for finitely generated projective modules A, B and C over a ring R,

$$A \oplus C \cong B \oplus C \quad \text{with} \quad C \lesssim nA, \;\; C \lesssim mB \Longrightarrow A \cong B,$$

then the ring R is said to be *separative*. (Here, $P \lesssim Q$ means that P is isomorphic to a direct summand of Q.) This condition plays an important role in cancellation problems for the class of exchange rings. For example, if R is a separative exchange ring, then its stable rank can only be 1, 2 or ∞, and every (von Neumann) regular matrix can be brought to diagonal form by elementary row and column operations (see [5], [6], [7]).

Restricting to the class of separative QB-rings, we prove in Section 2 (Theorem 2.5) that any quasi-invertible matrix is also diagonalizable via elementary operations. This can be thought as optimal, since quasi-invertible elements are ubiquitous in QB-rings: they are always (von Neumann) regular, maximal in a suitable sense and algebraically dense (see [8]). A key ingredient for the proof is the previous result that, in an arbitrary QB-ring, any quasi-invertible matrix can be reduced to a special standard form. The preparatory work that culminates in this fact (together with other tools) is carried out in Section 1.

We next exploit the diagonalisation theorem in Section 3. Namely, we prove that the natural map $\mathrm{GL}_1(R, I) \to K_1(R, I)$ is surjective (for any QB-ideal I of a separative ring R). As a consequence, the natural map $\mathrm{GL}_1(R) \to K_1(R)$ is also surjective whenever R is a separative QB-ring.

In analogy with results obtained in [29] and [17], we study the condition of lifting units modulo special ideals, and we find a similar theorem to [29, Theorem 2.4], that applies to QB-ideals of separative rings and shows that QB-rings have good index theory.

In view of the good behaviour of QB-rings under the presence of separativite cancellation, it is natural to ask how widely this condition holds. We prove that all simple QB-rings are either purely infinite simple or else they have stable rank one (and in particular they are separative). In the not necessarily simple situation, we study in Section 4 the class of *central* QB-rings, in which their quasi-invertible elements are somehow parametrized by the central idempotents of the ring. We prove in Theorem 4.10 that any central QB-ring is separative. The problem in the general case (as well as for the class of exchange rings) remains open.

1. Composability

In this section we define the notion of equivalence between two quasi-invertible matrices over any (unital) ring R and study some of its properties. Our approach resembles the construction of $K_1(R)$ for a ring R and we prove a version of the Whitehead lemma (Proposition 1.4). The main result, established in Theorem 1.8, is the fact that a quasi-invertible matrix over any QB-ring has a special form modulo equivalence. This result, technical in nature, will be a key step towards the main result in Section 2.

Recall from [8] that two elements x and y in a ring R are said to be *centrally orthogonal*, in symbols $x \perp y$, provided that $xRy = yRx = 0$. We say that an element u in R is *quasi-invertible*, provided there exist elements v, w in R such that $(1 - uv) \perp (1 - wu)$. One can see ([8, Proposition 2.2]) that in this case we may choose $v = w$ and that $u = uvu$, $v = vuv$. Note that being quasi-invertible means that u is left invertible modulo an ideal I of R, and right invertible modulo another ideal J of R, in such a way that $IJ = JI = 0$. We denote the set of quasi-invertible elements by R_q^{-1}.

PROPOSITION 1.1. *If $u \in R_q^{-1}$ with quasi-inverse v, then $u^n \in R_q^{-1}$ with quasi-inverse v^n for every $n \geq 0$.*

PROOF. The expansion

$$1 - u^n v^n = \sum_{k=0}^{n-1} u^k (1 - uv) v^k$$

shows that $(1-u^n v^n) \perp (1-v^n u^n)$. It also shows that $(1-u^n v^n)(1-vu) = 0 = (1 - vu)(1 - u^n v^n)$, from which we deduce that $(u^n v^n)(vu) = (vu)(u^n v^n)$ for all n. An easy induction argument now proves that $uvu = u$ implies $u^n v^n u^n = u^n$ for all n, and thus the desired conclusion follows.

REMARK 1.2. Despite the result above, the set R_q^{-1} will never be a semigroup unless $R_q^{-1} = R^{-1}$. The problem is the product between an element u and its quasi-inverse v in R_q^{-1}. If $u \notin R^{-1}$, then either $uv \neq 1$ or $vu \neq 1$. But an idempotent $p \neq 1$ can never be quasi-invertible. To realize this, recall from [8, 2.1] that we would then have $p+I \in (R/I)_\ell^{-1}$ and $p+J \in (R/J)_r^{-1}$ for some orthogonal pair of ideals I and J. However, 1 is the only left or right invertible idempotent in any ring, which means that $1 - p = (1 - p)^2 \in IJ = 0$, contrary to our assumption.

PROPOSITION 1.3. *Let R be a unital ring, and let u, v be quasi-invertible elements in R, with quasi-inverses x and y, respectively. Then the following conditions are equivalent:*

(a) $(1 - ux) \perp (1 - yv)$ *and* $(1 - xu) \perp (1 - vy)$*;*

(b) $uv, vu \in R_q^{-1}$*;*

(c) $\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \in M_2(R)_q^{-1}$.

PROOF. (a)⇒(b). Write

$$1 - uvyx = 1 - ux + u(1 - vy)x, \quad 1 - yxuv = 1 - yv + y(1 - xu)v.$$

Since $1 - ux$ is centrally orthogonal to $1 - xu$ and $1 - yv$, and since $1 - vy$ is centrally orthogonal to $1 - yv$ and $1 - xu$, we conclude that $(1 - uvyx) \perp (1 - yxuv)$, whence $uv \in R_q^{-1}$.

For vu, we proceed similarly, writing

$$1 - vuxy = 1 - vy + v(1 - ux)y, \quad 1 - xyvu = 1 - xu + x(1 - yv)u.$$

(b)⇒(a). Assume that $uv \in R_q^{-1}$. Then there exists an element z in R such that $(1 - (uv)z) \perp (1 - z(uv))$. Since $(1 - zuv)y(1 - uvz) = 0$, we get that $y = zuvy + yuvz - zuvyuvz$, and so

$$1 - yv = (1 - zuv)(1 - yuvzv).$$

Analogously, since $(1 - zuv)x(1 - uvz) = 0$, we get $x = zuvx + xuvz - zuvxuvz$, whence

$$1 - ux = (1 - uzuvx)(1 - uvz).$$

Thus we conclude that $(1 - ux) \perp (1 - yv)$. Using that $vu \in R_q^{-1}$ in a similar way, we get that $(1 - xu) \perp (1 - vy)$.

(a)$\Rightarrow$(c) is a trivial matrix calculation.

(c)$\Rightarrow$(a). Assume that $\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \in M_2(R)_q^{-1}$, and choose a quasi-inverse $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. A straightforward (albeit tedious) matrix calculation shows that $u = uau$, $v = vdv$ and also that

$$\begin{aligned} &(1 - ua) \perp (1 - au), \quad (1 - vd) \perp (1 - dv), \\ &(1 - ua) \perp (1 - dv), \quad (1 - au) \perp (1 - vd). \end{aligned}$$

Observe that $1 - ux = (1 - ux)(1 - ua)$ and that $1 - yv = (1 - dv)(1 - yv)$. It follows that $(1 - ux) \perp (1 - yv)$, and similarly $(1 - xu) \perp (1 - vy)$.

As in [15, 2.6], if u and v in R_q^{-1} satisfy one of the (equivalent) conditions of Proposition 1.3, we say that u and v are *composable*.

We also say that two elements u and v in $M_n(R)_q^{-1}$ are *equivalent* provided there exist matrices α and β in $E_n(R)$ such that $u = \alpha v \beta$. Thus, in analogy with [15, Proposition 2.7] (see also, e.g. [24, Lemma 9.7] or [32, Proposition 2.1.4]), we have:

PROPOSITION 1.4. *Let R be a unital ring, and let u and v be composable elements in R_q^{-1}. Then the matrices*

$$\begin{pmatrix} uv & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} vu & 0 \\ 0 & 1 \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$$

are all equivalent.

PROOF. Choose quasi-inverses x and y for u and v respectively. By Proposition 1.3 we have that $(1 - ux) \perp (1 - yv)$. Let α and β be the elements in $E_2(R)$ given by:

$$\alpha = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -x & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -u & 1 \end{pmatrix},$$

$$\beta = \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} \begin{pmatrix} 1 & (1 - xu)v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -y & 1 \end{pmatrix}.$$

Now check that

$$\alpha\begin{pmatrix}1 & 0\\ 0 & uv\end{pmatrix}\beta=\begin{pmatrix}u & 0\\ 1-xu & xuv\end{pmatrix}\beta=\begin{pmatrix}u & 0\\ 0 & v\end{pmatrix}.$$

The proof works similarly for $\begin{pmatrix}vu & 0\\ 0 & 1\end{pmatrix}$.

Corollary 1.5 (cf. [15, 2.8]). *Let R be a unital ring, and let u be a quasi-invertible element and v an invertible element. Then $\begin{pmatrix}u & 0\\ 0 & 1\end{pmatrix}$ and $\begin{pmatrix}vuv^{-1} & 0\\ 0 & 1\end{pmatrix}$ are equivalent.*

Proof. Obviously, v and uv^{-1} are composable.

The reader will have noticed that quasi-inverses of quasi-invertible elements are not unique, but their precise form was given in [8, Theorem 2.3]. For technical reasons (namely, the proof of Theorem 1.8 below), we shall need a slight generalization of that result, that extends its validity to the context of the so-called skew corners. We recall the main definitions and give some details of the proof.

Given idempotents p and q in a unital ring R such that $pRq\neq 0$, we say that an element $u\in pRq$ is *quasi-invertible* (in symbols $u\in(pRq)_q^{-1}$) provided that there exists an element v in qRp such that $u=uvu$ and $(p-uv)\perp(q-vu)$ (we may also choose v so that $v=vuv$). (See [8, §5].)

Proposition 1.6. *Let R be a unital ring, and let p and q be two idempotents in R such that $pRq\neq 0$. If $u\in(pRq)_q^{-1}$ with a quasi-inverse v in qRp, then each element of the form*

$$v'=v+a(p-uv)+(q-vu)b, \tag{1}$$

with a in qR and b in Rp, is a quasi-inverse for u and we have

$$(p-uv')\perp(q-v'u),\quad (p-uv')\perp(q-vu),\quad (p-uv)\perp(q-v'u).$$

Conversely, if v' in qRp is a partial inverse for u, then v' has the form in (1), *with $a=b=v'$.*

Proof. Since $uv'=uv+ua(p-uv)$, we get $uv'u=uvu=u$. Now $(p-u(ap))(p-uv)=p-uv-uap+uauv=p-uv'$, and similarly $q-v'u=(q-vu)(q-(qb)u)$. This shows that $(p-uv')\perp(q-v'u)$, and also that v' is a quasi-inverse for u.

Observe that $(q - v'u)(q - vu) = q - vu$, whence $(p - uv') \perp (q - vu)$. The relation $(p - uv) \perp (q - v'u)$ follows similarly.

Conversely, assume that $v' \in qRp$ and that it is a partial inverse for u. Then $0 = (q - vu)v'(p - uv) = v' - vuv' - v'uv + v$. Thus

$$v' = 2v' - vuv' - v'uv + v = v + (q - vu)v' + v'(p - uv)\,,$$

so that $v' \in (qRp)_q^{-1}$ with $(p - uv') \perp (q - v'u)$, by the first part of the proof.

LEMMA 1.7. *Let R be a unital ring. Let a be a quasi-invertible element in R, and choose a quasi-inverse b for a. Let u and v be invertible elements in R and take any x in R.*

(a) *Let $b_1 = bu + (1 - ba)x$. Then $a_1 := u^{-1}a \in R_q^{-1}$ and has b_1 as quasi-inverse.*

(b) *Let $b_1 = ubv$. Then $a_1 := v^{-1}au^{-1} \in R_q^{-1}$ and has b_1 as a quasi-inverse.*

PROOF. (a) Let $y = b_1u^{-1} = b + (1 - ba)xu^{-1}$. By [8, Theorem 2.3], y is a quasi-inverse for a. Now, we have that

$$a_1b_1a_1 = (u^{-1}a)yu(u^{-1}a) = u^{-1}a = a_1,$$
$$b_1a_1b_1 = (yu)u^{-1}a(yu) = yu = b_1,$$

and also

$$1 - a_1b_1 = u^{-1}(1 - ay)u, \quad 1 - b_1a_1 = 1 - ya,$$

whence the conclusion follows.

(b) Straightforward computations show that in this case also $a_1 = a_1b_1a_1$ and that $b_1 = b_1a_1b_1$, and moreover $1 - a_1b_1 = v^{-1}(1 - ab)v$ and $1 - b_1a_1 = u(1 - ba)u^{-1}$.

THEOREM 1.8. *Let R be a unital QB-ring. Then, every element in $M_2(R)_q^{-1}$ is equivalent to a matrix of the form*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $a, d \in R_q^{-1}$, and if s, t are quasi-inverses for a and d respectively, then

$$b \in ((1 - as)R(1 - td))_q^{-1}, \quad \textit{and} \quad c \in ((1 - dt)R(1 - sa))_q^{-1}$$

(and $b = 0$ if $(1 - as)R(1 - td) = 0$, and $c = 0$ if $(1 - dt)R(1 - sa) = 0$).

PROOF. Let α be a quasi-invertible element in $M_2(R)$, and choose a quasi-inverse β, so that $\alpha = \alpha\beta\alpha$, $\beta = \beta\alpha\beta$, and $(1_2 - \alpha\beta) \perp (1_2 - \beta\alpha)$. The process carried out in [8, Theorem 6.4] to show that $M_2(R)$ is also a QB-ring can be performed in this setting to the trivial equation:

$$\beta\alpha + (1_2 - \beta\alpha) = 1_2.$$

The inspection of this process reveals that (in a finite number of steps n) we get matrices α_n and β_n, so that

$$\beta_n = \begin{pmatrix} u & w_{12} \\ w_{21} & v \end{pmatrix},$$

where $u, v \in R_q^{-1}$, and if x, y are quasi-inverses for u and v respectively, then

$$w_{12} \in ((1-ux)R(1-yv))_q^{-1} \cup \{0\}, \quad \text{and} \quad w_{21} \in ((1-vy)R(1-xu))_q^{-1} \cup \{0\}.$$

We also have that $\alpha_0 = \alpha$ and $\beta_0 = \beta$, and each pair α_i, β_i is obtained from the previous one according to the operations (a) or (b) in Lemma 1.7, using as invertible elements elementary matrices (and replacing a by α_i, and b by β_i). In particular, α_n and β_n are quasi-inverses for one another, and α_n is obtained from α by elementary changes. Thus, without loss of generality, we may assume that $\alpha = \alpha_n$ and $\beta = \beta_n$. Hence, writing $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have the equation:

$$\begin{pmatrix} u & w_{12} \\ w_{21} & v \end{pmatrix} = \begin{pmatrix} u & w_{12} \\ w_{21} & v \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u & w_{12} \\ w_{21} & v \end{pmatrix}.$$

The $(1, 1)$ equation gives:

$$u = uau + w_{12}cu + ubw_{21} + w_{12}dw_{21},$$

where the last summand is actually zero, because $(1 - ux) \perp (1 - xu)$. Multiplying this equation on the left by $1 - ux$ we get $w_{12}cu = 0$, and multiplying afterwards on the right by $1 - xu$ we get $ubw_{21} = 0$. Hence $u = uau$. By [8, Theorem 2.3] (or also Proposition 1.6), $a \in R_q^{-1}$. In fact, $a = x + (1-xu)a + a(1-ux)$ and $(1-au) \perp (1-ua)$, whence also $a = aua$. Similar computations using the $(2, 2)$ equation yield $d = dvd$, $v = vdv$ and $(1 - dv) \perp (1 - vd)$, so also $d \in R_q^{-1}$.

Now the $(1, 2)$ equation gives:

$$w_{12} = uaw_{12} + w_{12}cw_{12} + ubv + w_{12}dv\,.$$

After left multiplication by $1 - ux$ and right multiplication by $1 - yv$ we get $w_{12} = w_{12}cw_{12}$. If, instead, we multiply on the left by ux and on the right by yv, we get $ubv = 0$. Similarly, working with the (2, 1) equation, we get $w_{21} = w_{21}bw_{21}$ and $vcu = 0$.

In principle there is no reason to believe that, e.g. $(1 - au)b(1 - vd) = b$, hence we need to perform some more changes on α, as follows:

$$\begin{pmatrix} 1 & 0 \\ -cu & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & -ub \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & (1-au)b \\ c(1-ua) & d - cub \end{pmatrix}.$$

Set $d_1 := d - cub$. Since $cubv = 0$, we have that $cub(1 - vd) = cub$, hence by [8, Theorem 2.3] (or also Proposition 1.6) d_1 and v are quasi-inverses for one another. Thus $d_1vd_1 = d_1$. On the other hand, and using also that $vcu = 0$, we see that $d_1vd_1 = d$, hence $d = d_1$. Finally, we compute:

$$\begin{pmatrix} 1 & -(1-au)b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & (1-au)b \\ c(1-ua) & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -vc(1-ua) & 1 \end{pmatrix}$$
$$= \begin{pmatrix} a & (1-au)b(1-vd) \\ (1-dv)c(1-ua) & d \end{pmatrix}.$$

It is clear that we only have to consider the cases where the skew corners $(1-au)R(1-vd)$ or $(1-dv)R(1-ua)$ are different from zero. Observe also that $(1 - au)R(1 - vd) = (1 - xu)R(1 - vy)$ and that $(1 - dv)R(1 - ua) = (1 - yv)R(1 - ux)$. Hence, Proposition 1.6 implies that $(1 - au)b(1 - vd) \in ((1 - au)R(1 - vd))_q^{-1}$. Similarly, we get that $(1 - dv)c(1 - ua) \in ((1 - dv)R(1 - ua))_q^{-1}$.

For a unital ring R, we denote the monoid of isomorphism classes of finitely generated right projective R-modules by $V(R)$ (where the addition is given by the direct sum of representatives).

Alternatively, $V(R)$ can be described by using an idempotent picture that will prove more suitable to us in some cases. Given idempotents p and q in R, we say that p and q are *(Murray-von Neumann) equivalent*, and write $p \sim q$, provided there exist elements x in pRq and y in qRp such that $p = xy$ and $q = yx$. Let $M_\infty(R) = \varinjlim M_n(R)$ under the mappings $x \mapsto \operatorname{diag}(x, 0)$. Denote by $[p]$ the equivalence class of an idempotent in $M_\infty(R)$. Then $V(R)$ can also be viewed as the set of all equivalence classes of idempotents from $M_\infty(R)$, with addition defined by $[p] + [q] = [p \oplus q]$ ($p \oplus q$ stands for $\operatorname{diag}(p, q)$). Write $p \lesssim q$ if there is an idempotent p' such that $p \sim p' \leq q$. The passage from one picture of $V(R)$ to the other consists of identifying every finitely projective right module with an idempotent in some matrix ring over R in the standard way. Whilst $\sim$ translates into isomorphism, the symbol $\lesssim$ translates into the statement “is isomorphic to a direct summand of”.

If R is a unital ring and u is a quasi-invertible element in R, with a quasi-inverse v, we set $p = 1 - uv$ and $q = 1 - vu$. We shall refer to the idempotents p and q as the *defect idempotents* associated with the quasi-invertible element u. It was observed in [10, p. 77] that, even though quasi-inverses are not unique, the defect idempotents corresponding to different quasi-inverses are equivalent (in fact, they are conjugate by a unit of the ring). We therefore denote these classes by λ_u and ρ_u.

COROLLARY 1.9. *Let R be a unital QB-ring, and let α be a quasi-invertible element in $M_2(R)$. Then the classes λ_α and ρ_α admit a diagonal representative.*

PROOF. By Theorem 1.8, there exist matrices β and γ in $E_2(R)$ such that $\beta\alpha\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $a, d \in R_q^{-1}$, and if x, y are quasi-inverses for a and d respectively, then

$$b \in ((1-ax)R(1-yd))_q^{-1}, \quad \text{and} \quad c \in ((1-dy)R(1-xa))_q^{-1}$$

(and $b = 0$ if $(1-ax)R(1-yd) = 0$, and $c = 0$ if $(1-dy)R(1-xa) = 0$). Let $\alpha' = \begin{pmatrix} x & s \\ t & y \end{pmatrix}$, where s is a quasi-inverse for b (or $s = 0$), and t is a quasi-inverse for c (or $t = 0$). It is straightforward to check that $\beta\alpha\gamma\alpha'$ is a diagonal idempotent. Since $\beta(1_2 - \alpha(\gamma\alpha'\beta))\beta^{-1} = 1_2 - \beta\alpha\gamma\alpha'$, we see that $\lambda_\alpha = [1_2 - \beta\alpha\gamma\alpha']$. In a similar way, $1_2 - \alpha'\beta\alpha\gamma = \gamma^{-1}(1_2 - (\gamma\alpha'\beta)\alpha)\gamma$, whence $\rho_\alpha = [1_2 - \alpha'\beta\alpha\gamma]$, and the result follows.

2. Diagonalization of matrices and cancellation conditions

Our goal in this section is to show that all quasi-invertible matrices over a separative QB-ring can be diagonalized by performing elementary row and column operations. In particular, we see that a large subset of the set of von Neumann regular elements can be understood. This parallels the corresponding result established in [7] (see also [6]), where it is shown that *all* von Neumann regular matrices over a separative exchange ring can be diagonalised with elementary operations. It also recovers the corresponding result for extremally rich C^*-algebras ([17]), although our methods are different. The main consequences of our theorem will be derived in the next section, where some interesting facts concerning the K-Theory of QB-rings will be established.

The ingredient we require, as already mentioned, is that our rings R are separative. It is convenient for our purposes in the current secion to use the idempotent picture of the monoid $V(R)$ described above, and thus R is separative if, for idempotents p, q and r in $M_\infty(R)$, we have

$$p \oplus r \sim q \oplus r \quad \text{and} \quad r \lesssim n \cdot p, m \cdot q \qquad \Longrightarrow p \sim q.$$

All known examples of QB-rings satisfy this weakened cancellation condition. In view of the results in this and the next section, the following question imposes itself:

QUESTION 2.1. Are all QB-rings separative?

We shall present in the next sections some instances where the answer to this question turns out to be positive.

In the case of operator algebras, L. G. Brown and G. K. Pedersen have proved the remarkable result that all extremally rich C^*-algebras with real rank zero are separative, see [17]. Since the C^*-algebras that are exchange rings are precisely the ones with real rank zero ([5, Theorem 7.2]), it would be interesting to know whether or not every *exchange* QB-ring is separative. This is connected to the Fundamental Separativity Question formulated for exchange rings (see [5]).

In order to achieve our main result we assemble three lemmas, as follows. Let $n \in \mathbb{N}$. Recall that $E_n(R)$ is used to denote the subgroup of $\mathrm{GL}_n(R)$ generated by the elementary matrices of size n. If I is an ideal of R, denote by $E_n(R, I)$ the normal subgroup of $E_n(R)$ generated by the elements in $E_n(I)$. Define $\mathrm{GL}_n(R, I)$ as the kernel of the map $\mathrm{GL}_n(R) \to \mathrm{GL}_n(R/I)$. In other words, the elements in $\mathrm{GL}_n(R, I)$ are those matrices α in $\mathrm{GL}_n(R)$ such that $\pi_I(\alpha) = 1_n$.

Let $\mathrm{GL}(R, I) = \varinjlim \mathrm{GL}_n(R, I)$ through the maps $x \mapsto \mathrm{diag}(x, 1)$ and let $E(R, I) = \varinjlim E_n(R, I)$. Then $E(R, I)$ is a normal subgroup of $\mathrm{GL}(R, I)$ (see, e.g. [32]).

LEMMA 2.2. *Let R be a ring and let I be a two-sided ideal of R. Let $a \in R$ and assume that $a - 1 \in I$ and that a is von Neumann regular (i.e. $a = axa$ and $x = xax$ for some x in R). Assume there exists an idempotent e in I and a unit u in R such that $u(1 - e)u^{-1} = ax$. Then there is a matrix α in $E_2(R, I)$ such that $(a, ue)\alpha = (1, 0)$.*

PROOF. Let $\alpha_1 = \begin{pmatrix} 1 & xu \\ 0 & 1 \end{pmatrix}$. Then $(a, ue)\alpha_1 = (a, u)$. Next, take $\alpha_2 = \begin{pmatrix} 1 & 0 \\ u^{-1}(1-a) & 1 \end{pmatrix}$. Then $(a, u)\alpha_2 = (1, u)$. Finally, let $\alpha_3 = \begin{pmatrix} 1 & -u \\ 0 & 1 \end{pmatrix}$, so that $(1, u)\alpha_3 = (1, 0)$.

Let $\alpha = \alpha_1\alpha_2\alpha_3$. Note that $\alpha = \alpha_1\alpha_3\alpha_3^{-1}\alpha_2\alpha_3$. Since $1 - a \in I$, we have $\alpha_2 \in E_2(I)$, so that $\alpha_3^{-1}\alpha_2\alpha_3 \in E(R, I)$. On the other hand, $\alpha_1\alpha_3 = \begin{pmatrix} 1 & (x-1)u \\ 0 & 1 \end{pmatrix} \in E_2(I)$ (since also $x - 1 \in I$).

The following observation is easy and has been made in [17]. Recall that an idempotent p in a ring R is called *full* provided that $RpR = R$ (hence the module pR is a generator in the category of finitely generated projective modules).

LEMMA 2.3. *Let a be a quasi-invertible element in a ring R, and let x be a quasi-inverse for a. Then the idempotent ax is full in R.*

PROOF. Since $(1-ax)(1-xa)=0$, we have that $1-ax=(1-ax)xa$. Therefore

$$ax+[(1-ax)x]axa = ax+(1-ax)xa = ax+1-ax = 1.$$

The next lemma requires the notion of a non-unital QB-ring, as defined in [8, §4]. This will allow us to obtain further results in relative K-Theory (see Section 3). We recall the definition (that will be used in the next section) and an equivalent formulation that is more convenient for our proof below.

Let I be any ring (unital or not). We shall denote $\widetilde{I} = I \oplus \mathsf{Z}$, which is a canonical unitization of I with componentwise addition and multiplication given by the rule $(a,n)(b,m) = (ab+an+bm, nm)$. This is a unital ring that contains I as a two-sided ideal.

We say that I is a *QB-ring* provided that whenever $(1-x)(1-a)+b=1$ (in $\widetilde{I}$) for x, a, b in I, then there is an element y in I such that $1-x+by \in (\widetilde{I})_q^{-1}$. If I sits as a two-sided ideal of a unital ring R, then it was proved in [8, Theorem 4.9] that I is a QB-ring if and only if, whenever $(1-x)a+b=1$, for x in I, and a, b in R, there is an element y in R such that $1-x+by \in R_q^{-1}$. We shall say in this situation that I is a *QB-ideal* of the ring R.

LEMMA 2.4. *Let R be a separative ring, and let I be a QB-ideal of R. Assume that J is an ideal of R such that $JI=0$. Let $\alpha \in M_2(R)$ be such that*

$$\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $a-1$, b, $c \in I$, and $\pi_J(\alpha) \in M_2(R/J)_r^{-1}$. Then there exist matrices β, γ in $E_2(R,I)$ such that $\beta\alpha\gamma = \begin{pmatrix} 1 & 0 \\ 0 & d_1 \end{pmatrix}$, with $\pi_J(d_1) \in (R/J)_r^{-1}$.

PROOF. Denote, for the proof, $\pi_J(x) = \overline{x}$. We have an equation

$$\begin{pmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{pmatrix} \begin{pmatrix} \overline{a'} & \overline{b'} \\ \overline{c'} & \overline{d'} \end{pmatrix} = \begin{pmatrix} \overline{1} & 0 \\ 0 & \overline{1} \end{pmatrix} \tag{2}$$

from which we obtain $aa'+bc'+j=1$ for some j in J. Write $a=1-a_1$, where $a_1 \in I$, and $a'=1-a_1'$, so we have $(1-a_1)(1-a_1')+(bc'+j)=1$. By [8, Lemma 4.6], we can rearrange this equality so that $(1-a_1)(1-x_1)+$

$(bc' + j)x_2 = 1$, for elements x_1 and x_2 in I, and since $JI = 0$, we get $(1 - a_1)(1 - x_1) + bc'x_2 = 1$.

Now we use that I is a QB-ideal of the ring R, hence there exists z in R satisfying $a + bc'x_2z = 1 - a_1 + bc'x_2z \in R_q^{-1}$. We compute that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ c'x_2z & 1 \end{pmatrix} = \begin{pmatrix} a + bc'x_2z & b \\ c + dc'x_2z & d \end{pmatrix},$$

where $c'x_2z \in I$. Hence, without loss of generality, we can assume that $a \in R_q^{-1}$ and that equation (2) still holds.

Next, let x be a quasi-inverse for a, so that $a = axa$, $x = xax$ and $(1 - xa) \perp (1 - ax)$. Observe that $1 - x \in I$, and thus $1 - ax, 1 - xa \in I$. Compute:

$$\begin{pmatrix} 1 & 0 \\ -cx & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & -xb \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & (1 - ax)b \\ c(1 - xa) & d_1 \end{pmatrix}.$$

Hence, without loss of generality, we can also assume that $axb = cxa = 0$, and that equation (2) is still valid.

Look at the equality $aa' + bc' + j = 1$ for some (possibly different) j in J. Multiply this equation left and right by $1 - ax$, an element of I. Since $JI = 0$, $(1 - ax)a = 0$ and $axb = 0$, we obtain $bc'(1 - ax) = 1 - ax$. Let $e = c'(1 - ax)b = c'b$. Then e is an idempotent in I and $e \sim 1 - ax$. Note that

$$(1 - e)c'(1 - ax) = (1 - c'b)c'(1 - ax) = c'(1 - ax) - c'(1 - ax) = 0,$$

and also

$$b(1 - e) = b(1 - c'b) = (1 - bc'(1 - ax))b = axb = 0.$$

We now claim that $1 - e$ is full in R.

To see this, compute

$$d(1 - e)d' = dd' - dc'bd'.$$

We know that $ca' + dc' = k \in J$. Then

$$\begin{aligned} dc'bd' &= (k - ca')bd' = (k - ca')(1 - ax)bd' = -ca'(1 - ax)bd' \\ &= -c(1 - xa)a'(1 - ax)bd' = 0, \end{aligned}$$

because $JI = 0$, $1 - ax \in I$, and $(1 - ax) \perp (1 - xa)$. Therefore $d(1 - e)d' = dd'$.

On the other hand,

$$c(1-e)(1-xa)b' = c(1-c'(1-ax)b)(1-xa)b' = cb',$$

since $axb = 0$ and $(1-xa) \perp (1-ax)$. Hence $d(1-e)d'+c(1-e)(1-xa)b' = cb' + dd' \in 1 + J$. Since $e \in I$ and $JI = 0$, we obtain

$$d(1-e)d'e + c(1-e)(1-xa)b'e = e.$$

This implies that $e \in R(1-e)R$. Since we also have $1-e \in R(1-e)R$, we get $R = R(1-e)R$, which establishes the claim.

In $V(R)$ we have that

$$[ax] + [1-ax] = [1] = [1-e] + [e] = [1-e] + [1-ax],$$

where $1-e$ is full and ax is also full (by Lemma 2.3). Using the separativity of R, we conclude that $ax \sim 1-e$. Write $ax = st$ and $1-e = ts$, where $s \in axR(1-e)$ and $t \in (1-e)Rax$. We had previously that $e = c'(1-ax)b$ and $1-ax = bc'(1-ax)$. We conclude that the element $u = s+b$ is invertible in R, with inverse $u^{-1} = t + c'(1-ax)$, and $u(1-e)u^{-1} = ax$. Observe that, by our computations, $ue = b$.

Invoking Lemma 2.2, we find a matrix α in $E_2(R, I)$ such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \alpha = \begin{pmatrix} 1 & 0 \\ c_1 & d_1 \end{pmatrix},$$

where, by our computations, $c_1 \in I$. The last routine calculation is done multiplying on the left by $\begin{pmatrix} 1 & 0 \\ -c_1 & 1 \end{pmatrix} \in E_2(I)$.

Let M be an abelian monoid. We equip M with its natural (algebraic) pre-order, given by $x \leq y$ if $x+z = y$ for some z in M. We say that M is *separative* if

$$a + c = b + c \text{ with } c \leq na, c \leq mb \Longrightarrow a = b.$$

The example we have in mind is that of $V(R)$ for a ring R. Clearly, $V(R)$ is a separative monoid precisely when R is a separative ring. The need for the more general definition will be clear from the following definition and in the proof of the theorem below.

If I is a submonoid of an abelian monoid M, then I is said to be an *order-ideal* (or an *o-ideal*) of M if $x \leq y$ with y in I forces $x \in I$. One can define a congruence on M by declaring that $x \sim y$ if there are elements z and t in I such that $x + z = y + t$. We denote $M/I = M/{\sim}$, and this becomes an abelian monoid under $\pi_I(x) + \pi_I(y) = \pi_I(x + y)$, where $\pi_I(x)$ denotes the congruence class of x.

In the particular case that I is an ideal of a ring R, it is easily checked that $V(I)$ is an order-ideal of $V(R)$. There is a natural map $V(R)/V(I) \to V(R/I)$, which is surjective if idempotent matrices lift modulo I. It is easy to verify that if R is a separative ring, then $V(R)/V(I)$ is a separative monoid, although it is not necessarily true that R/I is also separative (see [5]). We shall denote by $\pi_J([p])$ the class in $V(R)/V(J)$ of an element $[p]$ in $V(R)$ (rather than using $\pi_{V(J)}$).

We are now ready for the harvest.

THEOREM 2.5. *Let R be a separative QB-ring, and let $\alpha \in M_2(R)_q^{-1}$. Then there exist matrices β, γ in $E_2(R)$ such that*

$$\beta\alpha\gamma = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix},$$

with $d \in R_q^{-1}$.

PROOF. Without loss of generality (using Theorem 1.8), we may assume that

$$\alpha = \begin{pmatrix} a & w_{12} \\ w_{21} & b \end{pmatrix},$$

where $a, b \in R_q^{-1}$ with (respective) quasi-inverses x and y such that

$$w_{12} \in ((1-ax)R(1-yb))_q^{-1} \cup \{0\} \quad \text{and} \quad w_{21} \in ((1-by)R(1-xa))_q^{-1} \cup \{0\}.$$

If $w_{ij} = 0$ for all i, j, then by combining Propositions 1.3 and 1.4, we have that $ab \in R_q^{-1}$ and α is equivalent to $\begin{pmatrix} 1 & 0 \\ 0 & ab \end{pmatrix}$.

Thus, we may assume that some of the w_{ij} are not zero. It is enough to consider the case that $w_{ij} \neq 0$ for all i and j. In this case, pick quasi-inverses w'_{ij} for w_{ij} and check that

$$\begin{pmatrix} a & w_{12} \\ w_{21} & b \end{pmatrix} \begin{pmatrix} x & w'_{21} \\ w'_{12} & y \end{pmatrix} = \begin{pmatrix} ax + w_{12}w'_{12} & 0 \\ 0 & by + w_{21}w'_{21} \end{pmatrix}.$$

Let $J = R(1 - ax - w_{12}w'_{12})R + R(1 - by - w_{21}w'_{21})R$. Denoting by $\overline{v}$ the classes modulo J, we have the following equation in R/J:

$$\begin{pmatrix} \overline{a} & \overline{w_{12}} \\ \overline{w_{21}} & \overline{b} \end{pmatrix} \begin{pmatrix} \overline{x} & \overline{w'_{21}} \\ \overline{w'_{12}} & \overline{y} \end{pmatrix} = \begin{pmatrix} \overline{1} & 0 \\ 0 & \overline{1} \end{pmatrix}.$$

We start as in the previous lemma, so that we multiply the equation

$$\overline{ax} + \overline{w_{12}}\,\overline{w'_{12}} = 1$$

on the left by $\overline{1-ax}$, and we get $\overline{w_{12}}\,\overline{w'_{12}} = \overline{1-ax}$. Set $e = \overline{w'_{12}}\,\overline{w_{12}}$, which is an idempotent, equivalent to $\overline{1-ax}$. Our goal is to see that also $1-e$ is equivalent to $\overline{ax}$, so that we will be in position to apply Lemma 2.2.

Note the following relations:

$$b(1 - w'_{12}w_{12})y = by, \quad w_{21}(1 - w'_{12}w_{12})w'_{21} = w_{21}w'_{21}.$$

(The second one holds since, for example, $(1-ax) \perp (1-xa)$.) Therefore, the idempotent $by + w_{21}w'_{21}$ can be rewritten as $\epsilon_1\epsilon_2\epsilon_3$, where

$$\epsilon_1 = \begin{pmatrix} b & w_{21} \\ 0 & 0 \end{pmatrix}, \quad \epsilon_2 = \operatorname{diag}(1 - w'_{12}w_{12}, 1 - w'_{12}w_{12}),$$
$$\text{and} \quad \epsilon_3 = \begin{pmatrix} y & 0 \\ w'_{21} & 0 \end{pmatrix}.$$

If we let $f = \epsilon_2\epsilon_3\epsilon_1\epsilon_2$, then we have that $[by + w_{21}w'_{21}] = [f] \leq 2[1 - w'_{12}w_{12}]$ in $V(R)$.

Recall that $\pi_J([p])$ denotes the class in $V(R)/V(J)$ of an element $[p]$ from $V(R)$. Since $1 - by - w_{21}w'_{21} \in J$, we get in $V(R)/V(J)$ that

$$\begin{aligned}\pi_J([1-ax]) \leq \pi_J([1]) &= \pi_J([by + w_{21}w'_{21}]) \\ &= \pi_J([f]) \leq 2\pi_J([1 - w'_{12}w_{12}]).\end{aligned}$$

On the other hand, since ax is full in R (by Lemma 2.3 and also its proof) we have that $\pi_J([1]) \leq 2\pi_J([ax])$ in $V(R)/V(J)$, hence also $\pi_J([1-ax]) \leq 2\pi_J([ax])$.

Notice also that $\pi_J([w'_{12}w_{12}]) = \pi_J([1-ax])$ in $V(R)/V(J)$, because $1 - ax - w_{12}w'_{12} \in J$ and $[w'_{12}w_{12}] + [1 - ax - w_{12}w'_{12}] = [1-ax]$ in $V(R)$.

Now, as in the comments preceding the proof, $V(R)/V(J)$ is a separative monoid. Therefore, the fact that

$$\begin{aligned}\pi_J([1-ax]) + \pi_J([ax]) &= \pi_J([w'_{12}w_{12}]) + \pi_J([1 - w'_{12}w_{12}]) \\ &= \pi_J([1-ax]) + \pi_J([1 - w'_{12}w_{12}]),\end{aligned}$$

coupled with the inequalities we just observed above implies that $\pi_J([ax]) = \pi_J([1 - w'_{12}w_{12}])$. This carries over to $V(R/J)$, hence we get that $\overline{ax}$ is equivalent to $1-e$, as claimed.

Write $\overline{ax} = st$ and $ts = 1-e$ for appropriate s and t. Then the element $u = s + \overline{w_{12}}$ is a unit in R/J such that $u(1-e)u^{-1} = \overline{ax}$ (and one can check that $ue = \overline{w_{12}}$). We can therefore apply Lemma 2.2 (with the ideal being R/J) to get $\beta' \in E_2(R/J)$ such that $\overline{\alpha}\beta' = \begin{pmatrix} 1 & 0 \\ c_1 & d_1 \end{pmatrix}$. We may obviously assume that $\beta' = \overline{\beta}$, for some $\beta \in E_2(R)$. Necessarily also, d_1 is right invertible in R/J. Thus, after right multiplication by a suitable elementary matrix, we can assume that $c_1 = 0$. Upstairs, this translates into the following

$$\alpha\beta = \begin{pmatrix} 1 + x_{11} & x_{21} \\ x_{12} & d \end{pmatrix},$$

where $x_{ij} \in J$.

Now, let $I = R(1 - xa - w'_{21}w_{21})R + R(1 - yb - w'_{12}w_{12})R$. It follows from our relations that $IJ = JI = 0$. Since also

$$\begin{pmatrix} x & w'_{21} \\ w'_{12} & y \end{pmatrix}\begin{pmatrix} a & w_{12} \\ w_{21} & b \end{pmatrix} = \begin{pmatrix} xa + w'_{21}w_{21} & 0 \\ 0 & yb + w'_{12}w_{12} \end{pmatrix},$$

we know that $\pi_I(\alpha\beta) \in M_2(R/I)_l^{-1}$. Thus the left version of Lemma 2.4 applies, so that we find γ and $\epsilon \in E_2(R)$ such that $\gamma\alpha\beta\epsilon = \begin{pmatrix} 1 & 0 \\ 0 & d' \end{pmatrix}$, and necessarily $d' \in R_q^{-1}$.

In the case of exchange rings, the same type of diagonalisation result (for all von Neumann regular matrices) turns out to characterize separativity (see [6, Theorem 3.4]). Given the symmetry between these two classes of rings, it is thus natural to expect that a similar result holds for QB-rings. In the remaining part of this section we explore this possibility and establish a partial result (see Theorem 2.13).

Given any (unital) ring R, denote by $D(R)$ the two-sided ideal generated by all defect idempotents of R, that is, the idempotents of the form $1 - ux$ where $u \in R_q^{-1}$ and x is a quasi-inverse for u. We shall refer to $D(R)$ as the *defect ideal.*

LEMMA 2.6 (cf. [16, Proposition 3.2]). *Let R be a unital QB-ring. Then $R/D(R)$ has stable rank one and $D(R)$ is the smallest ideal of R such that this holds.*

PROOF. Using [8, Proposition 3.9], we only need to verify that $(R/D(R))_q^{-1} = (R/D(R))^{-1}$. Since R is a QB-ring, we have that $(R/D(R))_q^{-1} = (R_q^{-1} + D(R))/D(R)$. We therefore take u in R_q^{-1}, and pick any quasi-inverse x for u. Then $1 - ux, 1 - xu \in D(R)$, and hence the class $u + D(R)$ of u modulo $D(R)$ is invertible (with inverse $x + D(R)$). This shows that $(R/D(R))_q^{-1} \subseteq (R/D(R))^{-1}$, and since the converse inclusion is obvious, the first part of our assertion holds.

Next, suppose that I is an ideal of R such that R/I has stable rank one. Since R/I is also a QB-ring, this implies that $(R/I)_q^{-1} = (R/I)^{-1}$, and by [8, Proposition 7.1], this set equals $(R_q^{-1} + I)/I$. Let $u \in R_q^{-1}$ and take a quasi-inverse x. Then $u = uxu$ and since the class of u is invertible in R/I, we have that $1 - ux, 1 - xu \in I$. Therefore all defect idempotents belong to I, from which we infer that $D(R) \subseteq I$, as desired.

Corollary 2.7. *Let R be a unital QB-ring.*

(i) *If $n \geq 1$, then $D(M_n(R)) = M_n(D(R))$.*

(ii) *If e is an idempotent, $D(eRe) \subseteq eD(R)e$, and equality holds in the case that e is full in R.*

Proof. (i) Since $M_n(R)/M_n(D(R)) \cong M_n(R/D(R))$ and $R/D(R)$ has stable rank one, we have that $M_n(R)/M_n(D(R))$ has stable rank one too. If I is an ideal of R such that $M_n(R)/M_n(I) \cong M_n(R/I)$ has stable rank one, then R/I, being a corner of the matrix ring, will also have stable rank one, hence $D(R) \subseteq I$ and $M_n(D(R)) \subseteq M_n(I)$.

(ii) Since $eRe/eD(R)e = (e + D(R))R/D(R)(e + D(R))$, we see that $eRe/eD(R)e$ has stable rank one, and so $D(eRe) \subseteq eD(R)e$. Assume now that e is full. If I is an ideal of eRe, we have that $I = eI'e$ for some ideal I' of R (see, e.g. [22, Theorem 21.11]). If $eRe/I = (e+I')R/I'(e+I')$ has stable rank one, then using again that e is full we find that R/I' will also have stable rank one, hence $D(R) \subseteq I'$ and so $eD(R)e \subseteq eI'e = I$.

Lemma 2.8. *Let R be a unital ring and let e be a full idempotent of R. Then, the unital map $\iota \colon eRe \to R$ given by $\iota(x) = x + 1 - e$ induces an isomorphism $\iota_* \colon K_1(eRe) \to K_1(R)$.*

The lemma below is basically [17, Proposition 2.2]. We include a proof because the argument can be stretchted to give some more information that will be used later on. Recall that, if R is a unital ring, then $K_0(R)$ is the Grothendieck group of the monoid $V(R)$.

Lemma 2.9. *Let R be a unital QB-ring, and let p and q be idempotents in R such that $[p] = [q]$ in $K_0(R)$. Then, either $p \perp q$ or else pRq is a QB-corner, and for every x in $(pRq)_q^{-1}$ with quasi-inverse y in $(qRp)_q^{-1}$, we have that $p - xy$, $q - yx \in D(R)$.*

Proof. Since $[p] = [q]$, we have $[1 - p] = [1 - q]$ in $K_0(R)$. Therefore, for a suitable m, we have $(1 - p) \oplus m \cdot 1 \sim (1 - q) \oplus m \cdot 1$ in $M_{m+1}(R)$, that is, $1_{m+1} - \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \sim 1_{m+1} - \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}$, where 1_{m+1} denotes the unit of $M_{m+1}(R)$.

Now, by [8, Corollary 5.8], either $\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \perp \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}$ (and that implies $p \perp q$), or else

$$\begin{pmatrix} pRq & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} M_{m+1}(R) \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}$$

is a QB-corner, that is, pRq is a QB-corner.

In this last situation, write $1_{m+1} - \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} = uv$ and $1_{m+1} - \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix} = vu$ for some u, v in $M_{m+1}(R)$.

Now let $x \in (pRq)_q^{-1}$ with quasi-inverse $y \in (qRp)_q^{-1}$. Then, by the proof of [8, Theorem 5.5], $u + x \in M_{m+1}(R)_q^{-1}$ with quasi-inverse $v + y$, and

$$p - xy \sim 1_{m+1} - (u + x)(v + y) \in D(M_{m+1}(R)) = M_{m+1}(D(R)),$$

by Corollary 2.7. Therefore $p - xy \in D(R)$ and similarly $q - yx \in D(R)$.

LEMMA 2.10. *Let R be a (unital) ring. If α in $M_2(R)_q^{-1}$ is equivalent to a matrix of the form $\begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}$, where $d \in R_q^{-1}$, then the corresponding defect idempotent for α is equivalent to a defect idempotent in R.*

PROOF. By assumption there are matrices β and γ in $E_2(R)$ such that $\beta\alpha\gamma = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}$. Let α' be any quasi-inverse for α. Then, as in Lemma 1.7, we have that $\gamma^{-1}\alpha'\beta^{-1}$ is a quasi-inverse for $\beta\alpha\gamma$, and $1_2 - \alpha\alpha'$ is equivalent (in fact, conjugate) to the defect idempotent corresponding to $\beta\alpha\gamma$ (and $\gamma^{-1}\alpha'\beta^{-1}$).

Let d' be a quasi-inverse for d. Then $\begin{pmatrix} 1 & 0 \\ 0 & d' \end{pmatrix}$ is also a quasi-inverse for $\beta\alpha\gamma$, with defect idempotent equivalent to $1 - dd'$ (a defect idempotent in R). By the considerations preceding Corollary 1.9, this is also equivalent to the defect idempotent corresponding to $\beta\alpha\gamma$ (and $\gamma^{-1}\alpha\beta^{-1}$).

By using Propositions 1.1 and 1.4, the argument in the lemma below can be almost entirely excised from the proof of [17, Theorem 2.6], and hence we omit the details.

LEMMA 2.11. *Let R be a (unital) ring such that, for any idempotent p in R, every matrix in $M_2(pRp)_q^{-1}$ is equivalent to a matrix of the form $\begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}$, where $d \in (pRp)_q^{-1}$. If u, $v \in R_q^{-1}$ with respective quasi-inverses x, y, then* *there exists an idempotent e in R such that*

$$\begin{pmatrix} 1-ux & 0 \\ 0 & 1-vy \end{pmatrix} \sim e.$$

REMARK 2.12. By an argument based on induction, it can be shown that the conclusion of the lemma can be extended to finite families of defect idempotents. That is, if $u_1, \dots, u_n$ are quasi-invertible elements with respective quasi-inverses $v_1, \dots, v_n$, then there is an idempotent e in R such that $e \sim (1-u_1v_1) \oplus \cdots \oplus (1-u_nv_n)$.

We say that a ring R is *weakly cancellative* provided that

$$p \oplus r \sim q \oplus r \text{ for idempotents } p, q, r \text{ in } R \quad \text{such that } r \lesssim n \cdot p,\ n \cdot q \Longrightarrow p \sim q.$$

This is, by its very definition, a weaker version of separativity. It can be proved to be equivalent to the following condition (that was introduced by L. G. Brown and G. K. Pedersen for C^*-algebras, and also termed weak cancellation, see [12] and [17]): Given idempotents p and q in R such that $I = RpR = RqR$ and $[p] = [q]$ in $K_0(I)$, then $p \sim q$.

For the proof of the following result, we shall denote by $\delta\colon K_1(R/I) \to K_0(I)$ the connecting map in algebraic K-Theory, and recall this fits into a (long) exact sequence

$$K_1(R) \to K_1(R/I) \to K_0(I) \to K_0(R) \to K_0(R/I)$$

(see, e.g. [32]).

THEOREM 2.13 (cf. [17, Theorem 2.6]). *Let R be a QB-ring such that, for any idempotent e in R, every matrix in $M_2(eRe)_q^{-1}$ is equivalent to a matrix of the form $\begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}$ where $d \in (eRe)_q^{-1}$. Then R is weakly cancellative.*

PROOF. Assume that $p \oplus r \sim q \oplus r$ for idempotents p, q, r in R and that $r \lesssim n \cdot p, n \cdot q$.

Put $S = M_{n+1}(R)$. By a standard argument, we can reduce to the case where we have the previous equivalence in S with $p = e_{11}$ (the usual matrix unit) and $q \in S$, both full idempotents. Since $[p] = [q]$ in $K_0(S)$, we have, by Lemma 2.9 and the fact that p and q are full, that pSq is a QB-corner. Pick any x in $(pSq)_q^{-1}$ with quasi-inverse y in $(qSp)_q^{-1}$. Then we know (also by Lemma 2.9) that the idempotents $p - xy$ and $q - yx$ belong to $D(S)$. A proof akin to the one of Lemma 2.3 shows that the idempotent $e = xy$ is also full in S.

Obviously $[p - xy] = [q - yx]$ in $K_0(S)$, hence

$$[p - xy] - [q - yx] \in \ker(K_0(D(S)) \to K_0(S)).$$

By exactness we find an element s' in $K_1(S/D(S))$ such that $\delta(s') = [p - xy] - [q - yx]$.

By Lemma 2.6, the ring $S/D(S)$ has stable rank one, hence s' admits a representative $s_0 + D(S)$ with s_0 in S, which is an invertible element in $S/D(S)$. Next, since e is full in S, we have by condition (ii) in Corollary 2.7 that $D(eSe) = eD(S)e$. It follows from this that $eSe/D(eSe) = (e + D(S))S/D(S)(e + D(S))$, which is also a ring with stable rank one. Observe that $e + D(S)$ is a full idempotent in $S/D(S)$, hence we have by Lemma 2.8 an isomorphism $K_1(eSe/eD(S)e) \cong K_1(S/D(S))$ induced by the natural map $eSe/eD(S)e \to S/D(S)$ that adds $p - e$.

Therefore, there is an invertible element $s + eD(S)e$ in $eSe/eD(S)e$ such that

$$[(s + p - e) + D(S)] = [s_0 + D(S)] \qquad \text{in} \quad K_1(S/D(S)).$$

Since eSe is also a QB-ring, we may of course assume that s is quasi-invertible in eSe and choose a quasi-inverse t (whose class modulo $eD(S)e$ will be the inverse of s). Then $p - (p - e + s)(p - e + t) = e - st$ and $p - (p - e + t)(p - e + s) = e - ts$ and both idempotents belong to $D(S)$. Note that, since ts (and st) is full in eSe we have that ts is full in S.

Since $p - e + s$ is a von Neumann regular element with partial inverse $p - e + t$, we may invoke, e.g. [26, Proposition 1.3] (which assumes, but does not use, regularity for all elements) and conclude that

$$\delta(s') = [e - ts] - [e - st] = [p - xy] - [q - yx].$$

Put

$$f = (p - xy) \oplus (e - st), \quad g = (q - yx) \oplus y(e - ts)x \quad \text{and} \quad h = st.$$

Then $f \oplus h = p$, $g \oplus h \sim q$ and also $[f] = [g]$ in $K_0(D(S))$.

Let $T = hSh$. Then $SD(T)S = SD(hSh)S = ShSD(S)ShS = D(S)$, by Corollary 2.7. Therefore $D(S)$ is generated, as an ideal, by the defect idempotents in T. Moreover, by [2, Theorem 7.4] we have $K_0(D(S)) = G(V(D(S)))$.

Then, since $[f] = [g]$ in $K_0(D(S))$ we have quasi-invertible elements $u_1, \ldots, u_m$ in T with respective quasi-inverses $v_1, \ldots, v_m$ such that

$$f \oplus (h - u_1v_1) \oplus \cdots \oplus (h - u_mv_m) \sim g \oplus (h - u_1v_1) \oplus \cdots \oplus (h - u_mv_m)$$

in $M_\infty(D(S))$.

Since $h \in eSe$ and $e \in pSp$, we see that in fact $h = h'p = h'e_{11}$ and h' is an idempotent in R. Thus $T = hSh \cong h'Rh'$. We then combine Lemma 2.11 with Remark 2.12 in order to find an idempotent h_0 in T such that $h_0 \sim (h - u_1v_1) \oplus \cdots \oplus (h - u_mv_m)$.

Finally, $p = f \oplus h = f \oplus h_0 \oplus (h - h_0) \sim g \oplus h_0 \oplus (h - h_0) \sim g \oplus h \sim q$, as desired.

QUESTION 2.14. Can the conclusion of the theorem be improved to show that R is separative, rather than just weakly cancellative? In other words, is the condition of weak cancellation stable for QB-rings? (It is known that this is the case for C^*-algebras, see [17, Theorem 3.9].)

In the positive direction we offer the following result.

COROLLARY 2.15. *Let R be a QB-ring such that, for any idempotent e in R, every matrix in $M_2(eRe)_q^{-1}$ is equivalent to a matrix of the form* $\begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}$ *where $d \in (eRe)_q^{-1}$.*

Then, for any idempotent f in $M_n(R)$, the ideal $D(fM_n(R)f)$ is separative. In particular $D(R)$ is separative.

PROOF. Our hypothesis guarantees, after using Lemma 2.11, that any idempotent in $M_n(D(R))$ is equivalent to an idempotent in $D(R)$. Thus, by Theorem 2.13, we conclude that $D(R)$ is separative.

Since $D(M_n(R)) = M_n(D(R))$ we have that $D(M_n(R))$ is also separative. Thus, in order to finish the proof, we only need to show that $D(fRf)$ is separative for any idempotent f in R.

Suppose that $p \oplus r \sim q \oplus r$ for idempotents p, q and r in $M_n(D(fRf))$ and that also $r \lesssim n \cdot p, n \cdot q$. Since $D(fRf) \subseteq fD(R)f \subseteq D(R)$ and using the first part of the proof, we have that $p \sim q$ in $M_n(D(R))$. But then $p = xy$ and $q = yx$, where $x = \mathrm{diag}(p, \ldots, p)x\,\mathrm{diag}(q, \ldots, q)$ and $y = \mathrm{diag}(q, \ldots, q)y\,\mathrm{diag}(p, \ldots, p)$. Therefore

$$x \in M_n(D(fRf))M_n(D(R))M_n(D(fRf)),$$

and

$$\begin{aligned} D(fRf)D(R)D(fRf) &= D(fRf)fD(R)fD(fRf) \\ &\subseteq D(fRf)^2 = D(fRf), \end{aligned}$$

since $D(fRf)$ is generated by idempotents. Thus $x \in M_n(D(fRf))$ and similarly for y.

3. Non-stable K-Theory

The main goal of this section is to draw consequences of the results we have achieved before on diagonalisation of matrices. Basically, what we obtain are results of K_1-surjectivity type and index theorems. These are known in the case of rings with stable rank one (see, e.g. [25]), separative exchange rings (see [7], [29]) and also extremally rich C^*-algebras ([17]).

We know already that these results are relevant for QB-rings as their stable rank is usually different from one. In fact, from the examples we know of (see [8], [10], [28]) the following question is quite pertinent:

QUESTION 3.1. Is the stable rank of a (separative) QB-ring always one, two or infinity?

We prove below the answer to this question is positive in case the ring is simple. It also shows that all simple QB-rings are separative, although this will follow from Theorem 4.10.

Our starting point is the observation that, within the class of QB-rings, stable rank one can be easily identified in the monoid $V(R)$. (See also [5] and [11].)

LEMMA 3.2. *Let R a QB-ring. Then R has stable rank one if and only if $V(R)$ is a cancellative monoid.*

PROOF. Since it is well-known that stable rank one implies cancellation in general, we need only to check the converse.

Assume that $V(R)$ is cancellative. Let $u \in R_q^{-1}$, and let x be any quasi-inverse, so that $(1-ux) \perp (1-xu)$. Since $V(R)$ has cancellation and since $ux \sim xu$, it follows that $1-ux \sim 1-xu$. This implies that $ux = xu = 1$, because $1-ux$ and $1-xu$ are centrally orthogonal. This shows that $R^{-1} = R_q^{-1}$, and the result now follows using [8, Proposition 3.9].

We need the following technical lemma.

LEMMA 3.3. *Let R be a unital QB-ring, and let I be a left ideal of R. Then I is a QB-ring (as a non-unital ring).*

PROOF. We will use the definition of a non-unital QB-ring (see [8, Definition 4.4]). Assume that we have an equation

$$xa - x - a + b = 0,$$

for some elements x, a and b in I. In R, this reads as

$$(1-x)(1-a) + b = 1.$$

Thus $a = a(1-x)(1-a) + ab$, and so

$$1 = 1 + 0 = 1 - a + a(1-x)(1-a) + ab = (1 + a(1-x))(1-a) + ab,$$

and $a(1-x) = a - ax \in I$. Thus, changing notation we may assume that

$$1 = (1-x)(1-a) + ab,$$

where x, a and $b \in I$. Since R is a QB-ring by hypothesis, there exists an element z in R such that $1-a+(za)b \in R_q^{-1}$, and $za \in I$. Set $t = a - zab$, and choose any quasi-inverse for $1-t$, written on the form $1-s$ for some s in R. The equation $1-t = (1-t)(1-s)(1-t)$ shows that $s = -t+t^2+ts+st-tst$, and therefore $s - ts \in I$.

The defect idempotents for the pair $1-t$ and $1-s$ are $t+s-ts$ and $t+s-st$. By [8, Theorem 2.3], any element of the form

$$1 - s + u(t+s-ts) + (t+s-st)v,$$

for u and v in R, is a quasi-inverse for $1-t$. Taking $u = v = 1$, we get that $1 + 2t - st + s - ts$ is a quasi-inverse for $1-t$, and $2t - st + s - ts \in I$. Therefore, without loss of generality, we may assume that $s \in I$. Thus $t+s-ts$ and $t+s-st$ are centrally orthogonal in R, and so they are also in $\widetilde{I}$. This implies that I is a QB-ring.

The following non-unital version of [8, Proposition 3.9] will be also needed. Its proof is very similar to the unital version, and therefore we omit it here.

PROPOSITION 3.4. *Let R be any QB-ring (unital or not). Denote by $\widetilde{R}$ the standard unitization of R. Then R has stable rank one if and only if $(\widetilde{R})^{-1} \cap (1-R) = (\widetilde{R})_q^{-1} \cap (1-R)$*

Recall that a (unital) simple ring R is *purely infinite* if R is not a division ring and for any non-zero element x in R, there exist elements a, and b in R such that $axb = 1$ (see [4], [8]). It follows from, e.g. [4, Proposition 2.1], that every purely infinite simple ring is separative.

THEOREM 3.5. *Let R be a simple unital ring. Then R is a QB-ring if and only if either* $\mathrm{sr}(R) = 1$ *or else R is a purely infinite simple ring.*

PROOF. Assume that R is a simple QB-ring. Suppose first that there is a finite idempotent e in R. Then $(eRe)_q^{-1} = (eRe)^{-1}$, and by [8, Proposition 3.9] eRe has stable rank one. Since R is simple, eRe and R are Morita equivalent and therefore R has also stable rank one.

So, we may assume that all non-zero idempotents in R are infinite. In order to prove that R is purely infinite simple, we have to show that every principal (non-zero) left ideal of R contains a non-zero idempotent. Pick a non-zero element x in R and consider the left ideal Rx.

Since $\mathrm{sr}(R) > 1$, we first show that $\mathrm{sr}(Rx) > 1$. There exist a natural number n and elements $s_1, \dots, s_n, t_1, \dots, t_n$ in R such that $\sum_{i=1}^{n} s_i x t_i = 1$. Let $S = M_n(R)$ and $y = x1_n$, where 1_n is the unit element in S. If $s = \sum s_i e_{1i}$ and $t = \sum_j t_j e_{j1}$, where e_{ij} stands for the usual set of matrix units in S, we then have the equation $syt = e_{11}$. Now $e := tsy$ is an idempotent in $Sy = M_n(Rx)$, which is infinite in S, because it is equivalent to 1. But now it is trivial to verify that e is also infinite when viewed as an idempotent in Sy. Therefore $Sy = M_n(Rx)$ cannot have stable rank one and thus Rx does not have stable rank one either.

By Lemma 3.3, Rx is a QB-ring. Then Proposition 3.4 and the argument in the previous paragraph imply that there exists a quasi-adversible element a in Rx which is not adversible. This means that there exists b in Rx such that the idempotents $a + b - ab$ and $a + b - ba$ are centrally orthogonal in $\widetilde{Rx}$ and at least one of them is non-zero. In any case this produces a non-zero idempotent in Rx, as desired.

The converse implication is provided by [8, Definition 3.4] and [8, Proposition 3.10].

Let I be a two-sided ideal of a unital ring R. The relative K_1 group is then defined as $K_1(R, I) = \mathrm{GL}(R, I)/E(R, I)$. (Of course, $K_1(R) = K_1(R, R)$.)

THEOREM 3.6. *Let R be a separative ring and let I be a QB-ideal of R. Then the natural map $\mathrm{GL}_1(R, I) \to K_1(R, I)$ is surjective.*

PROOF. By induction, it is enough to show that given an element $\alpha \in \mathrm{GL}_2(R, I)$, there is an invertible element $d \in \mathrm{GL}_1(R, I)$ such that $[\alpha] = [d]$. Applying Lemma 2.4 to α (and taking $J = 0$), we find elements $\beta, \gamma \in E_2(R, I)$ such that $\beta\alpha\gamma = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}$, and necessarily $d \in \mathrm{GL}_1(R, I)$. It follows that $[\alpha] = [\beta\alpha\gamma] = [d]$ in $K_1(R, I)$, as desired.

COROLLARY 3.7. *If R is a separative QB-ring, then the natural map $\mathrm{GL}_1(R) \to K_1(R)$ is surjective.*

In view of this result, it is natural to ask for the kernel of this map, which would provide a more complete understanding of $K_1(R)$. Results in this direction exist for rings with stable rank one (see, e.g. [25], and [33]).

THEOREM 3.8. *Let R be a separative QB-ring. Then, for any ideal I of R, the natural map $\mathrm{GL}_1(R/I) \to K_1(R/I)$ is surjective.*

PROOF. Let $x \in K_1(R/I)$. Then $x = [\alpha]$ for some α in $\mathrm{GL}_n(R/I)$. By using induction, we may assume that $n = 2$. Since R is a QB-ring, there exists a quasi-invertible matrix β in $M_2(R)$ such that $\pi_I(\beta) = \alpha$. Now, Theorem 2.5 provides us with matrices γ and δ in $E_2(R)$ such that $\gamma\beta\delta = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}$, and $d \in R_q^{-1}$. Notice that $\pi_I(\gamma)\alpha\pi_I(\delta) = \begin{pmatrix} 1 & 0 \\ 0 & \pi_I(d) \end{pmatrix}$, so that $\pi_I(d) \in \mathrm{GL}_1(R/I)$. Since also $x = [\pi_I(\gamma)\alpha\pi_I(\delta)]$, the result follows.

Note that the previous theorem would follow immediately from Corollary 3.7 in case we knew that R/I was separative for any separative QB-ring R. This is true for C^*-algebras, as proved in [17, Proposition 3.4], but the corresponding result in the algebraic context remains open.

We now turn to establish a lifting theorem for QB-rings that is closely related to [29, Theorem 2.4] and [17, Theorem 5.4]. Let R be any ring and let I be a two-sided ideal of R. Define the set of *Fredholm elements* of R relative to I as $F(R, I) = \pi^{-1}(\mathrm{GL}_1(R/I))$.

We of course have $\mathrm{GL}_1(R) + I \subseteq F(I, R)$, with equality if $\mathrm{sr}(R) = 1$. We define the *index map* as the semigroup homomorphism

$$\text{index}\colon F(I, R) \to K_0(I),$$

by $\mathrm{index}(x) = \delta([\pi(x)])$, where $\delta\colon K_1(R/I) \to K_0(I)$ is the connecting map in K-Theory.

THEOREM 3.9. *Let R be a separative ring, and let I be a QB-ideal. Let x be a Fredholm element relative to I. Then there exists y in* $\mathrm{GL}_1(R)$ *such that $x - y \in I$ if and only if* $\mathrm{index}(x) = 0$.

PROOF. It is enough to prove the "if" part. Assume that $\mathrm{index}(x) = 0$. Then, by exactness of the sequence

$$K_1(R) \to K_1(R/I) \to K_0(I),$$

we find y_1 in $\mathrm{GL}_k(R)$ (for some k) such that $[\pi(y_1)] = [\pi(x)]$. Therefore, if $m = 2^n$ is large enough, there exists a z in $E_m(R)$ such that $\pi(z)\pi(y_1) = \pi(x) \oplus 1_{m-1}$. Set $w_1 = zy_1$, and apply Lemma 2.4 (replacing I and R by $M_{m/2}(I)$ and $M_{m/2}(R)$, and taking $J = 0$). Thus we get an element w_2 in $\mathrm{GL}_{m/2}(R)$ and α, β in $E_m(R)$ such that $\pi(\alpha) = \pi(\beta) = 1_m$ and $\alpha w_1 \beta = w_2 \oplus 1_{m/2}$, and $\pi(w_2) = \pi(x) \oplus 1_{m/2-1}$. Therefore, in $K_1(R)$, $[w_1] = [\alpha w_1 \beta] = [w_2]$. A recursive procedure shows that we get $w_1, \ldots, w_n$ with $w_n \in \mathrm{GL}_1(R)$ satisfying $\pi(w_n) = \pi(x)$ and $[w_1] = [w_2] = \ldots = [w_n]$.

4. Central QB-rings

The main purpose of this section is to study a class of QB-rings, which we term *central*, whose quasi-invertible elements are close to being the union of left or right invertible elements. Our main result states that such rings are always separative (Theorem 4.10), and also any (semi-prime) QB-ring is 'close', in a certain sense, to a central QB-ring (see Propositions 4.1 and 4.2).

Throughout the section, we shall denote by $\mathrm{Id}_Z(R)$ the set of all central idempotents in a ring R. We say that a unital ring R is a *central QB-ring* provided that R is a QB-ring and

(3)
$$R_q^{-1} = \{u + v \in R \mid u \in (eR)_r^{-1}, v \in ((1-e)R)_l^{-1} \text{ for some } e \text{ in } \mathrm{Id}_Z(R)\}.$$

This is connected with the notion of related comparability introduced earlier by Chen. Following [18], we say that a unital ring R satisfies *related comparability* if whenever we have two idempotents p and q in R such that $1-p \sim 1-q$, then there is a central idempotent e in R such that epR is isomorphic to a direct summand in eqR and $(1-e)qR$ is isomorphic to a direct summand in $(1-e)pR$. In general, a central QB-ring satisfies related comparability by [8, Corollary 5.11] and [18, Theorem 2] (see also [8, Proposition 2.9]), but $\mathbb{Z}$ is an example where the converse does not hold. However, for exchange rings, both notions coalesce. To make the connection explicit, observe that if R is an exchange ring satisfying related comparability, then R is a QB-ring, by the arguments in [8, 8.8]. Since quasi-invertible elements are von Neumann regular, we may apply [18, Theorem 2] to conclude that R_q^{-1} has the form in (3).

Central QB-rings enjoy specially good structural properties, due to the particular form of their quasi-invertible elements. Before establishing them, we note that this class of rings appears quite frequently. Indeed, to any QB-ring we can associate a central QB-ring in a fairly "standard" way, as follows.

Let us recall first the definition of the symmetric ring of quotients of a semi-prime ring R. An *essentially defined double centralizer* consists of a triple (f, g, I) where I is an essential ideal of R and $f, g\colon I \to R$ satisfy that f is a left R-module homomorphism, g is a right R-module homomorphism, and $f(x)y = xg(y)$ for all x, y in I. Two sets of data (f, g, I) and (f', g', J) are said to be *equivalent* if f and f' agree on $I \cap J$. The set of all equivalence classes $[(f, g, I)]$ is denoted by $Q_s(R)$, which turns out to be (under natural operations) a ring with identity $[(\mathrm{id}, \mathrm{id}, R)]$. This ring was first introduced by Kharchenko and studied by other authors (see, e.g. [27], [3] and the references therein). The *extended centroid* of a semi-prime ring R is by definition the center of $Q_s(R)$, which is normally denoted by $C(R)$ or C if the context is clear. The *central closure* of R is the subring RC of $Q_s(R)$. We say that R is *centrally closed* provided $R = RC$.

Recall from [8, 6.7] that a subring S of a ring R is *primely embedded* in case $p \perp q$ in R whenever $p \perp q$ in S, for idempotents p and q in S. In particular, if $S \subseteq R$ is a prime embedding, we have $S_q^{-1} \subseteq R_q^{-1}$.

PROPOSITION 4.1. *Let R be a unital semi-prime ring, and let S be the subring of the central closure of R generated by R and the central idempotents of $Q_s(R)$. If R is a QB-ring, then S is a central QB-ring.*

PROOF. We first check that S is a QB-ring. Observe that S is the direct limit of the collection of subrings $\{\langle R, e_1, \ldots, e_n\rangle\}$, where $e_1, \ldots, e_n$ are central idempotents of $Q_s(R)$. Also, the inclusions $\langle R, e_1, \ldots, e_n\rangle \subseteq \langle R, e_1, \ldots, e_{n+1}\rangle$ are prime embeddings. Hence, it suffices to show (by [8, Proposition 6.8]) that if e is any idempotent in S, then $T := \langle R, e\rangle$ (that is, the subring of S generated by R and e) is a semi-prime QB-ring. Note that

$$T \cong eR \times (1-e)R \cong R/((1-e)R \cap R) \times R/(eR \cap R).$$

Since R is a QB-ring and a factor of a QB-ring is also a QB-ring (see [8, Corollary 3.8]), we get that T is a QB-ring. It is obvious that T is a semi-prime ring.

Now, since $Q_s(R)$ is centrally closed (see, e.g. [3, Proposition 2.1.5]), S satisfies the following property: given orthogonal ideals I and J of S, there is a central idempotent e in S such that $eI = I$ and $(1-e)J = J$. It follows that S is a central QB-ring.

Let A be a unital C^*-algebra, and let $M_{loc}(A)$ be the local multiplier algebra of A, that is, the inductive limit of the multiplier algebras $M(I)$, where I ranges over the closed essential ideals of A (see [3] for details). Let ${}^cA$ be the bounded central closure of A, which is the C^*-subalgebra of $M_{loc}(A)$ generated by A and the centre $Z(M_{loc}(A))$ of $M_{loc}(A)$. Since $Z(M_{loc}(A))$ is an AW^*-algebra (see [3, Proposition 3.1.5]) it is generated by its projections (in fact, it satisfies the much stronger condition of real rank zero, see [13]). Therefore ${}^cA$ is the C^*-subalgebra of $M_{loc}(A)$ generated by A and the central projections in $M_{loc}(A)$. The next proposition, together with Theorem 4.10 below, provide extensions of the results established by Brown and Pedersen concerning the so-called isometrically rich C^*-algebras (see [16]).

PROPOSITION 4.2. *Let A be an extremally rich C^*-algebra. Then ${}^cA$ is a central extremally rich C^*-algebra.*

PROOF. First note that every projection in $Z(M_{loc}(A))$ is contained in $Q_s(A)$, by [3, Lemma 3.1.2]. Given a projection e in $Z(M_{loc}(A))$ and a C^*-

algebra B such that $A \subset B \subset \underrightarrow{\text{alglim}}\, M(I)$ we have that the algebra $\langle B, e\rangle$ generated by B and e inside $\underrightarrow{\text{alglim}}\, M(I)$ is in fact a C^*-algebra, because

$$\langle B, e\rangle \cong B/((1-e)B \cap B) \times B/(eB \cap B).$$

By [14, Theorem 3.5], $\langle B, e\rangle$ is extremally rich provided that B is. It follows that for every finite number of central idempotents $e_1, \dots, e_n$ in $M_{loc}(A)$, the C^*-subalgebra of $M_{loc}(A)$ generated by A and $\{e_1, \dots, e_n\}$ is extremally rich. Now, from the comments above, ${}^cA$ is the direct limit of the collection of C^*-subalgebras $\{\langle A, e_1, \dots, e_n\rangle\}$, where $\{e_1, \dots, e_n\}$ ranges over the central projections of $M_{loc}(A)$. Since this is a family of extremally rich C^*-algebras, we get that ${}^cA$ is also extremally rich provided that we can prove that $\langle A, e_1, \dots, e_n\rangle \subset {}^cA$ is extreme point-preserving (see [14, Proposition 5.2]). For this just note that if $a, b \in M_{loc}(A)$, the relation $aAb = 0$ forces $aM_{loc}(A)b = 0$ ([3, Proposition 2.3.3]).

If p and q are idempotents in R, then the skew corner pRq is a *central QB-corner* if pRq is a QB-corner and $(pRq)_q^{-1}$ consists of the elements of the form $eu + (1-e)v$, where $e \in \mathrm{Id}_Z(R)$, $u, v \in pRq$, and there exist elements u', v' in qRp such that $euu' = ep$ and $(1-e)v'v = (1-e)q$.

LEMMA 4.3. *Let R be a central QB-ring. Then, given two idempotents p and q in R such that $1-p \sim 1-q$, we either have $p \perp q$ or else pRq is a central QB-corner.*

PROOF. By [8, Corollary 5.8], we either have $p \perp q$ or else pRq is a QB-corner. In the second case, write $1-p = xy$ and $1-q = yx$, and let w be an element in $(pRq)_q^{-1}$. By [8, Theorem 5.5], $x + w \in R_q^{-1}$. By hypothesis there exist a central idempotent e in R and elements a, b, a' and b' in R such that

$$x + w = ea + (1-e)b,$$

with $eaa' = e$ and $(1-e)b'b = 1-e$.

Now, after left multiplication by p we get $w = epa + (1-e)pb$, whence

$$ea + (1-e)b = x + w = x + epa + (1-e)pb$$

and thus $(1-e)b = (1-e)x + (1-e)pb$, so that

$$(1-e)bq = (1-e)pbq.$$

Similarly (and after right multiplication by q), we get

$$epa = epaq\ .$$

Therefore, since $eaa' = e$ and $(1-e)b'b = 1-e$, we easily check that $e(paq)(qa'p) = ep$, and that $(1-e)(qb'p)(pbq) = (1-e)q$. Finally,

$$w = p(x+w)q = e(paq) + (1-e)(pbq),$$

as desired.

COROLLARY 4.4. *If R is a central QB-ring and p is an idempotent in R, then pRp is also a central QB-ring.*

LEMMA 4.5. *Let R be a central QB-ring, and let $u \in R_l^{-1}$, $v \in R_r^{-1}$. Assume that $w \in ((1-ux)R(1-yv))_q^{-1}$, where $xu = vy = 1$. Then there exist a central idempotent g in R and matrices β, γ in $M_2(R)$ such that*

$$\begin{pmatrix} u & w \\ 0 & v \end{pmatrix} = \operatorname{diag}(g,g)\beta + \operatorname{diag}(1-g,1-g)\gamma,$$

and $\operatorname{diag}(g,g)\beta \in M_2(gR)_r^{-1}$, $\operatorname{diag}(1-g,1-g)\gamma \in M_2((1-g)R)_l^{-1}$.

PROOF. Observe that $1-(1-ux) = ux \sim xu = 1$ and that $1-(1-yv) = yv \sim vy = 1$. Therefore, and since $(1-ux)R(1-yv) \neq 0$, we may apply Lemma 4.3 to conclude that $(1-ux)R(1-yv)$ is a central QB-corner.

Hence, given w in $((1-ux)R(1-yv))_q^{-1}$, there exist a central idempotent g in R and elements $w_1, w_2 \in (1-ux)R(1-yv)$, and $w_1', w_2' \in (1-yv)R(1-ux)$ such that $w = gw_1 + (1-g)w_2$, and moreover $gw_1w_1' = g(1-ux)$ and $(1-g)w_2'w_2 = (1-g)(1-yv)$. Let

$$\beta = \begin{pmatrix} gu & gw_1 \\ 0 & gv \end{pmatrix}, \quad \gamma = \begin{pmatrix} (1-g)u & (1-g)w_2 \\ 0 & (1-g)v \end{pmatrix}.$$

Since $g = gux + gw_1w_1'$, $w_1y = 0$, $vw_1' = 0$ and $vy = 1$ we see that

$$\begin{pmatrix} gu & gw_1 \\ 0 & gv \end{pmatrix}\begin{pmatrix} gx & 0 \\ gw_1' & gy \end{pmatrix} = \operatorname{diag}(g,g).$$

Analogously,

$$\begin{pmatrix} (1-g)x & 0 \\ (1-g)w_2' & (1-g)y \end{pmatrix}\begin{pmatrix} (1-g)u & (1-g)w_2 \\ 0 & (1-g)v \end{pmatrix} = \operatorname{diag}(1-g,1-g).$$

THEOREM 4.6. *Let R be a central QB-ring. Then $M_n(R)$ is a central QB-ring for all n.*

PROOF. By Corollary 4.4, it suffices to show that $M_2(R)$ is a central QB-ring. Using [8, Theorem 6.4], we get that $M_2(R)$ is a QB-ring.

Let $\alpha \in M_2(R)_q^{-1}$. Since $M_2(R)$ is a QB-ring, and using Theorem 1.8, we may perform elementary row and column operations on α so that with no loss of generality we may assume that:

$$\alpha = \begin{pmatrix} u & w_{12} \\ w_{21} & v \end{pmatrix},$$

where $u, v \in R_q^{-1}$, and if x, y are (any) quasi-inverses for u and v respectively, then

$$w_{12} \in ((1-ux)R(1-yv))_q^{-1} \cup \{0\}, \quad w_{21} \in ((1-vy)R(1-xu))_q^{-1} \cup \{0\}.$$

Since R is a central QB-ring, there exist central idempotents e and f in R, and elements $a, b, c, d, a', b', c', d'$ in R such that

$$\begin{aligned} u &= ea + (1-e)b, \quad eaa' = e, \quad (1-e)b'b = 1-e, \\ v &= fc + (1-f)d, \quad fcc' = f, \quad (1-f)d'd = 1-f. \end{aligned}$$

An easy computation shows that if we define $x := ea' + (1-e)b'$ and $y := fc' + (1-f)d'$, then x and y are quasi-inverses for u and v respectively. Indeed,

$$\begin{aligned} 1-ux &= (1-e)(1-bb'), \quad 1-xu = e(1-a'a), \\ 1-vy &= (1-f)(1-dd'), \quad 1-yv = f(1-c'c). \end{aligned}$$

Now let

$$\alpha_r' = \begin{pmatrix} efa & 0 \\ 0 & efc \end{pmatrix}, \qquad \alpha_{12} = \begin{pmatrix} (1-e)fb & w_{12} \\ 0 & (1-e)fc \end{pmatrix},$$

$$\alpha_{21} = \begin{pmatrix} e(1-f)a & 0 \\ w_{21} & e(1-f)d \end{pmatrix},$$

$$\alpha_l' = \begin{pmatrix} (1-e)(1-f)b & 0 \\ 0 & (1-e)(1-f)d \end{pmatrix},$$

and notice that $\alpha = \alpha_r' + \alpha_{12} + \alpha_{21} + \alpha_l'$. (In fact, this is the decomposition of α along the central diagonal idempotents ef, $e(1-f)$, $(1-e)f$ and $(1-e)(1-f)$.) Observe that $(1-e)fR$ is a central QB-ring, by Corollary 4.4 and that the element α_{12}, viewed in $M_2((1-e)fR)$, satisfies the hypotheses of Lemma 4.5. Therefore there exist a central idempotent g in $(1-e)fR$ and matrices $\beta, \gamma \in M_2((1-e)fR)$ such that $\alpha_{12} = g\beta + (1-g)\gamma$, and $g\beta \in M_2(g(1-e)fR)_r^{-1}$, $(1-g)\gamma \in M_2((1-g)(1-e)fR)_l^{-1}$.

Analogously, applying the transposed version of Lemma 4.5 to α_{21}, we find a central idempotent h in $e(1-f)R$ and matrices $\beta', \gamma' \in M_2(e(1-f)R)$ such that $\alpha_{21} = h\beta' + (1-h)\gamma'$, and $h\beta' \in M_2(he(1-f)R)_r^{-1}$, $(1-h)\gamma' \in M_2((1-h)e(1-f)R)_l^{-1}$.

Now, let

$$e_1 = ef + e(1-f)h + (1-e)fg$$

and

$$e_2 = e(1-f)(1-h) + (1-e)f(1-g) + (1-e)(1-f).$$

Notice that e_1 and e_2 are central orthogonal idempotents of R whose sum equals 1. Set $\alpha_r = \alpha_r' + g\beta + h\beta'$ and $\alpha_l = (1-g)\gamma + (1-h)\gamma' + \alpha_l'$. Then $e_1\alpha_r = \alpha_r$, $e_2\alpha_l = \alpha_l$ and $\alpha = \alpha_r + \alpha_l$. Moreover, $\alpha_r \in M_2(e_1R)_r^{-1}$ and $\alpha_l \in M_2(e_2R)_l^{-1}$, as wanted.

Let R be a unital ring. We say that an R-module C is a *defect R-module* if there is a defect idempotent e in R such that $C \cong eR$.

LEMMA 4.7. *Let R be a central QB-ring, and let e be an idempotent in R. Then e is equivalent to a defect idempotent if and only if $1 \oplus e \sim 1$.*

PROOF. Let u be a quasi-invertible element. Then there is a central idempotent p in R such that $u = pu + (1-p)u$, and pu is right invertible in pR, while $(1-p)u$ is left invertible in $(1-p)R$. Denote by v and w the right and left inverses of pu and $(1-p)u$, respectively. As in the proof of Theorem 4.6, the element $x = pv + (1-p)w$ is a quasi-inverse for u, and $1 - ux = (1-p)(1-uw)$. Writing $e = 1 - ux$, we have that

$$1 = e \oplus (1-e) = e \oplus ux = e \oplus p \oplus (1-p)uw \sim e \oplus p \oplus (1-p) = e \oplus 1.$$

Conversely, suppose that $1 \oplus e \sim 1$. Then $1 = p + q$, where p and q are orthogonal idempotents in R such that $p \sim e$ and $q \sim 1$. Write $q = ux$ and $xu = 1$. Then $u \in R_l^{-1}$ and $1 - ux = p \sim e$.

Observe that the previous Lemma states that if R is a central QB-ring, then a right R-module C is a defect module if and only if $R \oplus C \cong R$.

LEMMA 4.8. *Let R be a central QB-ring. Then every defect idempotent p in $M_n(R)$ is equivalent to an idempotent of the form* $\mathrm{diag}(p_1, 0, \dots, 0)$*, where p_1 is a defect idempotent in R. In particular $p(R^n)$ is a defect R-module.*

PROOF. By the usual reduction process it suffices to deal with the case $n = 2$. Let α be a quasi-invertible element in $M_2(R)$. By using Lemma 4.5 and the proof of Theorem 4.6, we can reduce ourselves to the consideration of six cases.

By symmetry it is enough to consider only the cases where $\alpha = \operatorname{diag}(a, b)$ with a and b left invertible elements in R or

$$\alpha = \begin{pmatrix} a & w \\ 0 & b \end{pmatrix},$$

where $a \in R_l^{-1}$ and $b \in R_r^{-1}$ and $w \in ((1-aa')R(1-b'b))_q^{-1}$. (We denote by a', b', w' the quasi-inverses of a, b, w, respectively.) The former case is easy, because the defect idempotent is $\operatorname{diag}(1-aa', 1-bb')$ and since $aa' \sim a'a = 1$, we have $aa' = q_1 + q_2$ where $q_2 \sim 1 - bb'$ and $q_1 \sim 1$. So the defect idempotent of α is equivalent to $\operatorname{diag}(q_2 + (1-aa'), 0)$, which is clearly a defect idempotent in R.

Now we consider the latter case. The defect idempotent of α is

$$\begin{pmatrix} 0 & 0 \\ 0 & (1-(b'b+w'w)) \end{pmatrix},$$

so we have to see that $1-(b'b+w'w)$ is equivalent to a defect idempotent in R. Since $ww' = 1-aa'$ we have $1 \oplus ww' = 1 \oplus 1-aa' \sim aa' \oplus 1-aa' \sim 1$. Hence, we get

$$\begin{aligned} 1 \oplus (1-b'b-w'w) &\sim 1 \oplus ww' \oplus (1-b'b-w'w) \\ &\sim 1 \oplus w'w \oplus (1-b'b-w'w) \\ &\sim 1 \oplus (1-b'b) \\ &\sim b'b \oplus (1-b'b) \\ &\sim 1. \end{aligned}$$

By Lemma 4.7, $1-b'b-w'w$ is equivalent to a defect idempotent.

PROPOSITION 4.9. *Let R be a central QB-ring and let A be a progenerator in Mod-R. If C is a defect R-module then $A \oplus C \cong A$.*

PROOF. By definition of defect R-module there is a quasi-invertible element u in R with quasi-inverse x such that $C \cong pR$, where $p = 1 - ux$. Let A be a progenerator in Mod$-R$ and set $T = \operatorname{End} A$, which is a central QB-ring, by Corollary 4.4 and Theorem 4.6. There exists $n \geq 1$ such that $A^n = U \oplus V$, where $U \cong R_R$. Let $\rho_1\colon A^n \to U$ and $\rho_2\colon A^n \to V$ denote the projections and $\tau_1\colon U \to A^n$ and $\tau_2\colon V \to A^n$ denote the corresponding injections. Fix an isomorphism $\iota\colon U \to R_R$ and put $\pi = \iota\rho_1$ and $\sigma = \tau_1\iota^{-1}$. Note that $\pi\sigma = 1_R$. Now, if $z \in R$ denote by $\hat{z}$ the element in $M_n(T) \cong \operatorname{End}(A^n)$ corresponding to the endomorphism $(\iota^{-1} \circ L_z \circ \iota) \oplus 1_V$ of A^n, where L_z denotes left multiplication by z on R_R. The map $\hat{}\colon R \to M_n(T)$ defined by $z \mapsto \hat{z}$ is a unital injective ring homomorphism, and the element $\hat{u}$ is quasi-invertible in $M_n(T)$ with quasi-inverse $\hat{x}$. Note that

$$1 - \hat{u}\hat{x} = \tau_1(\iota^{-1} L_{1-ux}\iota)\rho_1,$$

so $(1-\hat{u}\hat{x})(A^n) \cong pR \cong C$. By Lemma 4.8 there exists a defect idempotent p_1 in T such that $1 - \hat{u}\hat{x}$ is equivalent to $\mathrm{diag}(p_1, 0, \dots, 0)$ in $M_n(T)$. Therefore $C \cong (1-\hat{u}\hat{x})(A^n) \cong p_1(A)$. Since $T \oplus p_1 T \cong T$ we get $A \oplus C \cong A \oplus p_1(A) \cong A$, as desired.

If R is a ring and M is a right R-module, we denote by $\mathrm{add}(M_R)$ the category of R-modules which are direct summands of M^n for some $n \geq 1$. If $T = \mathrm{End}(M_R)$, then it is well-known that there is a categorical equivalence between the category $FP(T)$ of finitely generated projective right T-modules and the category $\mathrm{add}(M_R)$. This equivalence is provided by the functors $G = - \otimes_T M$, from $FP(T)$ to $\mathrm{add}(M)$ and $F = \mathrm{Hom}(M, -)$, from $\mathrm{add}(M)$ to $FP(T)$ (see e.g. [20, Theorem 4.7], and also [1, Lemma 29.4]).

THEOREM 4.10. *Let R be a central QB-ring. Then R is separative.*

PROOF. First note that being a central QB-ring is a Morita invariant property. Assume that $X \oplus Z \cong Y \oplus Z$ for some finitely generated projective R-modules X, Y, Z, such that Z is isomorphic to a direct summand of X^n and Y^m respectively, for some positive integers n and m. By substituting Z by $Z \oplus X$ we can assume without loss of generality that X, Y and Z generate the same finitely generated projective modules. In other words, we can and will assume that $\mathrm{add}(X) = \mathrm{add}(Y) = \mathrm{add}(Z)$. Consider the central QB-ring $T = \mathrm{End}\, Z_R$. Applying the “Morita functor” $F = \mathrm{Hom}(Z_R, -)$ to the relation $X \oplus Z \cong Y \oplus Z$ we get

$$F(X) \oplus T_T \cong F(Y) \oplus T_T$$

in $FP(T)$, and $F(X)$ and $F(Y)$ are progenerators of Mod$-T$. In order to conclude that $X \cong Y$, it is therefore enough to prove that $F(X) \cong F(Y)$.

Henceforth we change notation and start with an internal decomposition of R-modules $M = A_1 \oplus H = A_2 \oplus K$, where $A_1 \cong A_2 \cong R_R$ and H and K are progenerators in Mod$-R$. We have to prove that $H \cong K$. By [8, Lemma 8.5] and its proof, there is a decomposition $M = E \oplus B \oplus H = E \oplus C \oplus K$, where $B \cong (1-p)R$ and $C \cong (1-q)R$ for left and right defect idempotents $1-p$ and $1-q$ of $u \in R_q^{-1}$. Since B and C are defect modules and H and K are progenerators of Mod$-R$ we get from Proposition 4.9 that $H \cong H \oplus B$ and $K \cong K \oplus C$. Therefore we conclude that

$$H \cong H \oplus B \cong K \oplus C \cong K,$$

as desired.

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