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REAL RANK ESTIMATE BY HEREDITARY C^*-SUBALGEBRAS BY PROJECTIONS

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(Dedicated to the memory of Gert K. Pedersen)

Abstract

In this paper we estimate the real rank of C^*-algebras by that of their hereditary C^*-subalgebras by projections.

0. Introduction

We first recall that a C^*-subalgebra $\mathfrak{B}$ of a C^*-algebra $\mathfrak{A}$ is hereditary if for any positive elements $a \in \mathfrak{A}$, $b \in \mathfrak{B}$, the inequality (or order) $a \le b$ implies $a \in \mathfrak{B}$. Any closed ideal of a C^*-algebra is a hereditary C^*-subalgebra. If $\mathfrak{B}$ is a separable hereditary C^*-subalgebra of a C^*-algebra $\mathfrak{A}$, then there exists a positive element $a \in \mathfrak{B}$ such that $\mathfrak{B}$ is $a\mathfrak{A}a$ (the closure of the set of all the elements axa for $x \in \mathfrak{A}$). Also, the C^*-subalgebra $a\mathfrak{A}a$ generated by a positive element $a \in \mathfrak{A}$ is always hereditary. We say that $p\mathfrak{A}p$ for p a projection of $\mathfrak{A}$ (or the multiplier algebra of $\mathfrak{A}$) is a hereditary C^*-subalgebra by a projection p. See [6] or [8] for more details about hereditary C^*-subalgebras.

Next recall that a unital C^*-algebra $\mathfrak{A}$ has real rank $\le n$, denoted by $\mathrm{RR}(\mathfrak{A}) \le n$, if any $(x_j)_{j=0}^n \in \mathfrak{A}^{n+1}$ with $x_j = x_j^*$ $(0 \le j \le n)$ can be approximated by elements $(y_j)_{j=0}^n \in \mathfrak{A}^{n+1}$ with $y_j = y_j^*$ $(0 \le j \le n)$ such that $\sum_{j=0}^n y_j^2$ is invertible in $\mathfrak{A}$. For a non-unital C^*-algebra $\mathfrak{A}$, we set $\mathrm{RR}(\mathfrak{A}) = \mathrm{RR}(\mathfrak{A}^+)$, where $\mathfrak{A}^+$ is the unitization of $\mathfrak{A}$ by C. Similarly, a C^*-algebra $\mathfrak{A}$ has stable rank $\le n$, denoted by $\mathrm{sr}(\mathfrak{A}) \le n$, by considering $(x_j)_{j=1}^n$, $(y_j)_{j=1}^n \in \mathfrak{A}^n$ without the assumption for them to be self-adjoint. By definition, $\mathrm{RR}(\mathfrak{A}) \in \{0, 1, 2, \dots, \infty\}$ and $\mathrm{sr}(\mathfrak{A}) \in \{1, 2, \dots, \infty\}$, where if there are no such integers n above, we just set $\mathrm{RR}(\mathfrak{A}) = \infty$ and $\mathrm{sr}(\mathfrak{A}) = \infty$. See Brown-Pedersen [4] and Rieffel [9] for more details about the ranks. See also [2] and [10].

As the title shows, in this paper we consider the real rank estimate of C^*-algebras by their hereditary C^*-subalgebras by projections. It has been

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known by [4, Theorem 2.5] that $\mathrm{RR}(\mathfrak{A}) = 0$ for a C^*-algebra $\mathfrak{A}$ if and only if $\mathrm{RR}(p\mathfrak{A}p) = 0$ and $\mathrm{RR}((1-p)\mathfrak{A}(1-p)) = 0$ for p a projection of the multiplier algebra of $\mathfrak{A}$. Thus, our interest here is whether or not the estimate exists in the case of higher real ranks.

1. Main results

Based on the idea of [4, Theorem 2.5], we first obtain the following:

THEOREM 1.1. *Let $\mathfrak{A}$ be a unital C^*-algebra and $p \in \mathfrak{A}$ a projection. Then*

$$\mathrm{RR}(\mathfrak{A}) \le \max\bigl\{\mathrm{RR}(p\mathfrak{A}p), \mathrm{RR}((1-p)\mathfrak{A}(1-p))\bigr\}.$$

PROOF. Suppose that

$$\max\bigl\{\mathrm{RR}(p\mathfrak{A}p), \mathrm{RR}((1-p)\mathfrak{A}(1-p))\bigr\} = n < \infty.$$

If the maximum is infinite, the estimate in the statement is automatic. Let $(x_j)_{j=0}^n \in \mathfrak{A}^{n+1}$ with $x_j = x_j^*$ $(0 \le j \le n)$. Then we may view x_j $(0 \le j \le n)$ as the following matrix:

$$x_j = \begin{pmatrix} a_j & c_j \\ c_j^* & b_j \end{pmatrix}$$

for $a_j = px_jp \in p\mathfrak{A}p$, $b_j = (1-p)x_j(1-p) \in (1-p)\mathfrak{A}(1-p)$ and $c_j = px_j(1-p)$ and $c_j^* = (1-p)x_jp$. Then we have

$$\begin{aligned} x_0^2 + x_1^2 + \cdots + x_n^2 &= \begin{pmatrix} \sum_{j=0}^n a_j^2 + \sum_{j=0}^n c_jc_j^* & \sum_{j=0}^n (a_jc_j + c_jb_j) \\ \sum_{j=0}^n (c_j^*a_j + b_jc_j^*) & \sum_{j=0}^n b_j^2 + \sum_{j=0}^n c_j^*c_j \end{pmatrix} \\ &\equiv \begin{pmatrix} A & C \\ C^* & B \end{pmatrix} \end{aligned}$$

Since $\mathrm{RR}((1-p)\mathfrak{A}(1-p)) \le n$, there exists $(b_j') \in ((1-p)\mathfrak{A}(1-p))^{n+1}$ such that $b_j' = (b_j')^*$, $\|b_j - b_j'\| < \varepsilon$ for $0 \le j \le n$, and $\sum_{j=0}^n (b_j')^2$ and therefore also $\sum_{j=0}^n (b_j')^2 + \sum_{j=0}^n c_j^*c_j$ are invertible in $(1-p)\mathfrak{A}(1-p)$. Thus, we may assume that B is invertible in $(1-p)\mathfrak{A}(1-p)$ in the matrix above. Then we note that

$$(1) \quad \begin{pmatrix} A & C \\ C^* & B \end{pmatrix} = \begin{pmatrix} p & CB^{-1} \\ 0 & 1-p \end{pmatrix} \begin{pmatrix} A - CB^{-1}C^* & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} p & 0 \\ B^{-1}C^* & 1-p \end{pmatrix}$$

(cf. [4, Lemma 2.3]). Hence the left hand side is invertible if and only if $A - CB^{-1}C^*$ is invertible in $p\mathfrak{A}p$.

Since $\mathrm{RR}(p\mathfrak{A}p) \leq n$, there exists $(a_j') \in (p\mathfrak{A}p)^{n+1}$ such that $a_j' = (a_j')^*$, $\|a_j - a_j'\| < \varepsilon$ for $0 \leq j \leq n$, and $\sum_{j=0}^n (a_j')^2$ and therefore also $\sum_{j=0}^n (a_j')^2 + \sum_{j=0}^n c_j c_j^* - CB^{-1}C^*$ are invertible in $p\mathfrak{A}p$. (The proof in [4] can be used, however that is not at all clear. In [4, Theorem 2.5] they approximate $a - cb_0^{-1}c*$ with an invertible element (and not a), and it is not clear that this can be done in a similar way for the case where the real rank n. Thus, we give another proof for this tricky argument below). Therefore, by using (a_j') and (b_j') above, the element $(x_j) \in \mathfrak{A}^{n+1}$ is approximated by $(x_j') \in \mathfrak{A}^{n+1}$ with $x_j' = (x_j')^*$ such that

$$x_j' = \begin{pmatrix} a_j' & c_j \\ c_j^* & b_j' \end{pmatrix}$$

and $\sum_{j=0}^n (x_j')^2$ is invertible in $\mathfrak{A}$.

Let us start to prove the conclusion above by another way using the setting above. First note that $\sum_{j=0}^n (x_j')^2$ is invertible in $\mathfrak{A}$ if and only if there exists $(l_j) \in \mathfrak{A}^{n+1}$ such that $\sum_{j=0}^n l_j x_j' = 1$ in $\mathfrak{A}$ (or invertible in $\mathfrak{A}$). This is a standard (but nontrivial) fact (see [4, Introduction] or [5, Introduction]). Since $\mathrm{RR}(p\mathfrak{A}p) \leq n$, there exists $(f_j) \in (p\mathfrak{A}p)^{n+1}$ such that $\sum_{j=0}^n f_j a_j' = p$ and $\|a_j - a_j'\| < \varepsilon$ $(0 \leq j \leq n)$. Hence $p + \delta(\sum_{j=0}^n c_j c_j^* - CB^{-1}C^*)$ is invertible in $p\mathfrak{A}p$ for

$$0 \leq \delta < 1 \Big/ \Big(\Big\| \sum_{j=0}^n c_j c_j^* \Big\| + \|CB^{-1}C^*\| \Big) < 1 \Big/ \Big(\Big\| \sum_{j=0}^n c_j c_j^* - CB^{-1}C^* \Big\| \Big).$$

Note that we use a standard fact that the elements which are near invertible elements are also invertible (or the elements which have the norm distance from the identity element less than one are invertible). We may assume $\delta < 1$ in addition. Set

$$y_j = \begin{pmatrix} f_j & \delta c_j \\ \delta c_j^* & \delta b_j \end{pmatrix}, \qquad x_j' = \begin{pmatrix} a_j' & c_j \\ c_j^* & b_j \end{pmatrix}$$

for $0 \leq j \leq n$. Set

$$\sum_{j=0}^n y_j x_j' = \begin{pmatrix} \sum_{j=0}^n f_j a_j' + \delta \sum_{j=0}^n c_j c_j^* & \sum_{j=1}^n (f_j c_j + \delta c_j b_j) \\ \delta \sum_{j=1}^n (c_j^* a_j' + b_j c_j^*) & \delta \big(\sum_{j=1}^n b_j^2 + \sum_{j=1}^n c_j^* c_j \big) \end{pmatrix}$$

$$\equiv \begin{pmatrix} A_\delta & C_\delta' \\ \delta C'' & \delta B \end{pmatrix}.$$

Then it follows that $\sum_{j=0}^n y_j x_j'$ is invertible in $\mathfrak{A}$ by using the same matrix decomposition (1) as in the first part of the proof and considering

$$A_\delta - \delta^2 C'_\delta B^{-1} C'' = p + \delta\left(\sum_{j=0}^{n} c_j c_j^* - \delta C'_\delta B^{-1} C''\right)$$

since $1\big/\left(\left\|\sum_{j=0}^{n} c_j c_j^*\right\| + \|C'_\delta B^{-1} C''\|\right) < 1\big/\left(\left\|\sum_{j=0}^{n} c_j c_j^* - \delta C'_\delta B^{-1} C''\right\|\right)$ and

$$\|C'_\delta B^{-1} C''\| \le \|C'_\delta\| \|B^{-1} C''\| \le \left(\sum_{j=0}^{n} (\|f_j c_j\| + \|c_j b_j\|)\right) \|B^{-1} C''\| \equiv K,$$

and so we may assume $\delta < 1\big/\left(\left\|\sum_{j=0}^{n} c_j c_j^*\right\| + K\right)$, from which it follows that $A_\delta - \delta^2 C'_\delta B^{-1} C''$ is invertible in $p\mathfrak{A}p$, which implies the desired conclusion.

Remark. It would be desirable to show the same estimate by replacing p with positive elements of $\mathfrak{A}$, that is, estimate by hereditary C^*-algebras. However, our method above would not work well in this case.

Theorem 1.1 is extended to the following:

Theorem 1.2. *Let $\mathfrak{A}$ be a C^*-algebra and p a projection of the multiplier algebra of $\mathfrak{A}$. Then*

$$\mathrm{RR}(\mathfrak{A}) \le \max\left\{\mathrm{RR}(p\mathfrak{A}p), \mathrm{RR}((1-p)\mathfrak{A}(1-p))\right\}.$$

Proof. We just combine the proof of Theorem 1.1 with the method of the proof of [4, Theorem 2.5]. In fact, when $\mathfrak{A}$ is non-unital and $p \in \mathfrak{A}$ (or $1-p \in \mathfrak{A}$) we consider the unitization $\mathfrak{A}^+$ of $\mathfrak{A}$. Note that $((1-p)\mathfrak{A}(1-p))^+ = (1-p)\mathfrak{A}(1-p) + \mathsf{C}(1-p)$. When $\mathfrak{A}$ is non-unital, $p \notin \mathfrak{A}$ and $1-p \notin \mathfrak{A}$ we consider self-adjoint elements of $\mathfrak{A}^+$:

$$x_j = \begin{pmatrix} a_j + \lambda_j & c_j \\ c_j^* & b_j + \lambda_j \end{pmatrix}$$

for $0 \le j \le n$.

Moreover, the following is obtained in exactly the same way as above:

Corollary 1.3. *Let $\mathfrak{A}$ be a C^*-algebra and p a projection of the multiplier algebra of $\mathfrak{A}$. Then*

$$\mathrm{sr}(\mathfrak{A}) \le \max\left\{\mathrm{sr}(p\mathfrak{A}p), \mathrm{sr}((1-p)\mathfrak{A}(1-p))\right\}.$$

PROOF. We just consider elements of $\mathfrak{A}$:

$$x_j = \begin{pmatrix} a_j & c_j \\ d_j & b_j \end{pmatrix}$$

for $1 \le j \le n$, where a_j, b_j may not be self-adjoint.

REMARK. Blackadar's conjecture [3, Conjecture 4.2.3] is the estimate $\mathrm{sr}(p\mathfrak{A}p) \ge \mathrm{sr}(\mathfrak{A})$ for p a full projection of $\mathfrak{A}$. This is not the same as our result above, however it suggests that his conjecture should be true.

THEOREM 1.4. *Let $\mathfrak{A}$ be a C^*-algebra and $M_n(\mathfrak{A})$ the $n \times n$ matrix algebra over $\mathfrak{A}$. Then*

$$\mathrm{RR}(M_n(\mathfrak{A})) \le \mathrm{RR}(\mathfrak{A}).$$

PROOF. If $n = 1$, the equality is trivial.

Suppose that $\mathfrak{A}$ is unital, and

$$\mathrm{RR}(M_n(\mathfrak{A})) \le \mathrm{RR}(\mathfrak{A}).$$

Using Theorem 1.1 we obtain

$$\mathrm{RR}(M_{n+1}(\mathfrak{A})) \le \max\bigl\{\mathrm{RR}(M_n(\mathfrak{A})), \mathrm{RR}(\mathfrak{A})\bigr\}$$

since $pM_{n+1}(\mathfrak{A})p = M_n(\mathfrak{A})$ and $(1-p)M_{n+1}(\mathfrak{A})(1-p) = \mathfrak{A}$ for p a standard rank n projection of $M_{n+1}(\mathbf{C})$ in $M_{n+1}(\mathfrak{A})$. Hence $\mathrm{RR}(M_{n+1}(\mathfrak{A})) \le \mathrm{RR}(\mathfrak{A})$. By induction the proof is complete.

We may also apply Theorem 1.1 repeatedly for $1 = p_1 + \cdots + p_n \in M_n(\mathfrak{A})$ where p_j are standard rank 1 projections of $M_n(\mathbf{C})$ in $M_n(\mathfrak{A})$.

If $\mathfrak{A}$ is non-unital, we consider the following exact sequence:

$$0 \longrightarrow M_n(\mathfrak{A}) \longrightarrow M_n(\mathfrak{A}^+) \longrightarrow M_n(\mathbf{C}) \longrightarrow 0.$$

Then it follows from [5, Theorem 1.4] and the unital case above that

$$\mathrm{RR}(M_n(\mathfrak{A})) \le \mathrm{RR}(M_n(\mathfrak{A}^+)) \le \mathrm{RR}(\mathfrak{A}^+) = \mathrm{RR}(\mathfrak{A})$$

since $M_n(\mathfrak{A})$ is a closed ideal of $M_n(\mathfrak{A}^+)$.

REMARK. It is known by [4, Theorem 2.10] if $\mathrm{RR}(\mathfrak{A}) = 0$ for a C^*-algebra $\mathfrak{A}$, then $\mathrm{RR}(M_n(\mathfrak{A})) = 0$ for every n. The estimate of Theorem 1.4 answers Osaka's question [7, Question 1.11], which states whether the estimate of Theorem 1.4 is true or not.

In exactly the same way as above,

COROLLARY 1.5. *Let $\mathfrak{A}$ be a C^*-algebra. Then*

$$\mathrm{sr}(M_n(\mathfrak{A})) \le \mathrm{sr}(\mathfrak{A}).$$

PROOF. Use Corollary 1.3 and the argument of the proof of Theorem 1.4.

REMARK. It is known by [9, Theorem 6.1] that for any C^*-algebra $\mathfrak{A}$,

$$\mathrm{sr}(M_n(\mathfrak{A})) = \lceil(\mathrm{sr}(\mathfrak{A}) - 1)/n\rceil + 1$$

where $\lceil x\rceil$ is the least integer n with $n \ge x$. Also, by [1] we have

$$\mathrm{RR}(M_n(C(X))) = \lceil\dim X/(2n-1)\rceil$$

where $C(X)$ is the C^*-algebra of continuous functions on a compact Hausdorff space X. Therefore,

$$\begin{cases} \mathrm{sr}(M_n(\mathfrak{A})) \le \mathrm{sr}(\mathfrak{A}), \\ \mathrm{RR}(M_n(C(X))) \le \mathrm{RR}(C(X)) = \dim X. \end{cases}$$

However, it has been unknown whether or not the real rank of $M_n(\mathfrak{A})$ for a C^*-algebra $\mathfrak{A}$ in general can be computed in terms of that of $\mathfrak{A}$ as the commutative case or the stable rank case above. But we now have Theorem 1.4.

REMARK. For any C^*-algebra $\mathfrak{A}$, the following estimates:

$$\mathrm{RR}(p\mathfrak{A}p) \le \mathrm{RR}(\mathfrak{A}), \qquad \mathrm{sr}(p\mathfrak{A}p) \le \mathrm{sr}(\mathfrak{A})$$

for p a projection of $\mathfrak{A}$ are not correct in general. In fact, it is known by [1] and [9, Theorem 6.4] that

$$\mathrm{RR}(\mathfrak{B} \otimes \mathsf{K}) \le 1, \qquad \mathrm{sr}(\mathfrak{B} \otimes \mathsf{K}) \le 2$$

for any C^*-algebra $\mathfrak{B}$, where K is the C^*-algebra of compact operators. If the estimates by $p\mathfrak{A}p$ above were correct, we must have

$$\mathrm{RR}(\mathfrak{B}) \le 1, \qquad \mathrm{sr}(\mathfrak{B}) \le 2$$

since $1\otimes q(\mathfrak{B}\otimes\mathsf{K})1\otimes q = \mathfrak{B}$ for q a standard rank 1 projection of K, where $\mathfrak{B}$ is unital. This is impossible in general. However, it is known by [4, Corollary 2.8] that if $\mathrm{RR}(\mathfrak{A}) = 0$ for a C^*-algebra $\mathfrak{A}$, then any hereditary C^*-subalgebra of $\mathfrak{A}$ has real rank zero.

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