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EVANS-KISHIMOTO TYPE ARGUMENT FOR ACTIONS OF DISCRETE AMENABLE GROUPS ON MCDUFF FACTORS

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Abstract

We apply the Evans-Kishimoto type argument to centrally free actions of discrete amenable groups on McDuff factors, and classify them. Especially, we present a different proof that the Connes-Takesaki modules are complete cocycle conjugacy invariants for centrally free actions of discrete amenable groups on injective factors.

1. Introduction

In the theory of operator algebras, the study of automorphism groups is one of the most important subjects. Especially, since Connes succeeded in classifying automorphisms of the approximately finite dimensional (AFD) factor of type II_1 in [4] and [1], classification of actions of discrete amenable groups on injective factors has been solved in [9], [16], [19], [11] and finally in [10].

The strategy of Connes's classification is called the model action splitting argument. At first he constructed tensor product type model automorphisms (or actions) on the AFD type II_1 factor. Then he showed "the model action splitting", i.e., every automorphism contains model automorphisms as tensor product components after an appropriate inner perturbation, and then proved that it is cocycle conjugate to the model automorphism. In his argument, the Rohlin property for automorphisms plays a crucial role. Namely, he showed the non-commutative version of a Rohlin type theorem for a certain class of automorphisms in [1]. By means of the Rohlin type theorem, he proved the stability (or 1-cohomology vanishing theorem) for automorphisms via a Shapiro type argument. Connes's argument has been developed by Jones for finite groups in [9], and by Ocneanu for general discrete amenable groups in [16]. Especially, the Rohlin type theorem was extended to the case of discrete amenable groups by Ocneanu, and he proved several cohomology vanishing theorems in [16].

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On the other hand, another approach has been made in the study of automorphisms of C^*-algebras. In [6], Evans and Kishimoto developed the intertwining argument for classification of automorphisms with the Rohlin property. In their approach, they compare two given automorphisms directly without using model actions. Consequently they obtained classification results for a wide class of automorphisms. (In the C^*-algebra case, the model action splitting argument forces us to make a strict restriction on actions.) Their intertwining argument has been further developed in [15] for automorphisms of purely infinite simple C^*-algebras, and for finite group actions in [8].

In this paper, we apply the Evans-Kishimoto type intertwining argument to actions of discrete amenable groups on McDuff factors based on Ocneanu's Rohlin type theorem. Our main theorem says if two centrally free actions of a discrete amenable group on a McDuff factor differ up to approximately inner automorphisms, then they are cocycle conjugate. As a corollary, we get the complete classification of centrally free actions of discrete amenable groups on injective factors in terms of the Connes-Takesaki module by using the characterization of approximately inner automorphisms in [11]. Hence this is an another proof of the classical classification results in [1], [9], [16], [19], [11] for centrally free actions. However our approach seems to be more unified and simple, and this is an advantage of our theory.

Our result is also applicable to the classification of group actions on subfactors by a suitable modification. For example, we present a different proof of Popa's result in [17, Theorem 3.1]. (We remark that the classification result of strongly amenable subfactors of type II_1 by Popa in [17] is crucial in our argument.)

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2. Preliminaries and notations

Let M be a von Neumann algebra. For $\varphi \in M_*^+$, we let $\|x\|_\varphi = \sqrt{\varphi(x^*x)}$, $\|x\|_\varphi^\# = \sqrt{(\|x\|_\varphi^2 + \|x^*\|_\varphi^2)/2}$, $|x|_\varphi = \varphi(|x|)$. Note that $|x|_\varphi$ is not necessarily a norm unless φ is tracial, since it is not subadditive. For $x \in M$ and $\varphi \in M_*$, $x\varphi, \varphi x \in M_*$ are defined as $x\varphi(y) = \varphi(yx)$ and $\varphi x(y) = \varphi(xy)$ respectively. We set $[x, \varphi] = x\varphi - \varphi x$. To avoid possible confusions, we often denote $x\varphi$ and φx by $x \cdot \varphi$ and $\varphi \cdot x$ respectively. We denote $\phi \circ \alpha^{-1}$ by $\alpha(\phi)$ for $\phi \in M_*$ and $\alpha \in \mathrm{Aut}(M)$.

We use the notation $A \Subset B$ when A is a finite subset in B, and denote the cardinality of A by $|A|$.

Fix a free ultrafilter ω over $\mathbf{N}$. We define M_ω and M^ω as in [16]. Each $\alpha \in \mathrm{Aut}(M)$ gives automorphisms $\alpha^\omega \in \mathrm{Aut}(M^\omega)$, and $\alpha^\omega|_{M_\omega} \in \mathrm{Aut}(M_\omega)$. For $\varphi \in M_*$ and $X = (x_n) \in M^\omega$, $\varphi^\omega(X) := \lim_{n\to\omega} \varphi(x_n)$ is a normal functional on M^ω, which we denote by φ for simplicity. When M is a factor, $\tau_\omega(X) := \lim_{n\to\omega} x_n (\in \mathbf{C})$ always exists in the σ-weak topology for $X = (x_n) \in M_\omega$, and τ_ω is a tracial state on M_ω. We denote by $|X|_1$ the L^1-norm with respect to τ_ω.

We next collect some fundamental and useful inequalities in this paper.

Lemma 2.1. *The following inequalities hold for $\varphi \in M_*^+$, $x \in M$, $x_i \in M^\omega$ and $y_i \in M_\omega$.*

(1) $\|x \cdot \varphi\| \le \sqrt{\|\varphi\|}\|x\|_\varphi$, $\|\varphi \cdot x\| \le \sqrt{\|\varphi\|}\|x^*\|_\varphi$, $\|[x,\varphi]\| \le 2\sqrt{\|\varphi\|}\|x\|_\varphi^\#$.

(2) $\|x\|_\varphi^2 \le \|x\cdot\varphi\|\|x\|$, $\|x^*\|_\varphi^2 \le \|\varphi \cdot x\|\|x\|$.

(3) $\|x\|_\varphi^\# \le \sqrt{\frac{1}{2}(|x|_\varphi + |x^*|_\varphi)\|x\|}$.

(4) $\left|\sum_i x_i y_i\right|_\varphi \le \sum_i \|x_i\||y_i|_1$.

Proof. It is elementary to see (1), (2), (3). See [16, Lemma 7.1] for the proof of (4).

Next we recall Ocneanu's Rohlin type theorem, which is a main tool in the proof of Lemma 3.4 below.

Theorem 2.2 ([16, Theorem 6.1]). *Let M be a McDuff factor, G a discrete amenable group, and α an action of G on M_ω which is strongly free and semiliftable. Let $\varepsilon > 0$, and $\{K_i\}_{i\in I}$ be an ε-paving family. Then there exists a partition of unity $\{E_{i,k}\}_{i\in I, k\in K_i} \subset M_\omega$ such that*

$$\sum_{i\in I} |K_i|^{-1} \sum_{k,l\in K_i} |\alpha_{kl^{-1}}(E_{i,l}) - E_{i,k}|_1 \le 5\varepsilon^{\frac{1}{2}},$$
$$[\alpha_g(E_{i,k}), E_{j,l}] = 0, \qquad \textit{for all} \quad g \in G, i, j \in I, k \in K_i, l \in K_j.$$

See [16] for the notation in the above theorem. Here we briefly explain how to construct an ε-paving family $\{K_i\}$. Fix $N \in \mathbf{N}$ such that $N > \frac{4}{\varepsilon}\log\varepsilon^{-1}$, and set $\delta := (\varepsilon/3)^N$. Let K_{n+1} be a $(\delta|\bar{K}_n|^{-1}, \bar{K}_n)$-invariant finite set, where $\bar{K}_n := \cup_{1\le i\le n} K_i$. (In this paper, we say that K is (ε, F)-invariant if $|K \cap \bigcap_{g\in F} g^{-1}K| \ge (1-\varepsilon)|K|$.) Then $\{K_i\}_{1\le i\le N}$ is shown to be an ε-paving family. In this construction, each K_i can be chosen arbitrarily invariant.

3. Classification

We state the main result in this paper.

THEOREM 3.1. *Let M be a McDuff factor, G a countable discrete amenable group, and α, β centrally free actions of G on M. If $\alpha_g\beta_g^{-1} \in \overline{\mathrm{Int}}(M)$ for every $g \in G$, then there exist an α-cocycle v_g and $\theta \in \overline{\mathrm{Int}}(M)$ such that* $\mathrm{Ad}\, v_g\alpha_g = \theta \circ \beta_g \circ \theta^{-1}$. *Moreover we can choose v_g close to* 1, *i.e., for any given $\varepsilon > 0$, $F \Subset G$ and $\varphi \in M_*^+$, we can choose v_g so that $\|v_g - 1\|_\varphi^\# < \varepsilon$ for any $g \in F$.*

The assumption $\alpha_g\beta_g^{-1} \in \overline{\mathrm{Int}}(M)$ implies $\alpha_g = \lim_{n\to\infty} \mathrm{Ad}\, u_g^n\beta$ for some unitaries $u_g^n \in M$. However $\mathrm{Ad}\, u_g^n\beta_g$ is not necessary an action of G. Hence we need to take u_g^n as β-cocycles at first.

PROPOSITION 3.2. *Let α, β be as in Theorem 3.1. Then there exist β-cocycles u_g^n, $n = 1, 2, \ldots,$ such that $\alpha_g = \lim_{n\to\omega} \mathrm{Ad}\, u_g^n\beta_g$.*

PROOF. Since $\alpha_g\beta_g^{-1} \in \overline{\mathrm{Int}}(M)$, there exist unitaries u_g^n, $g \in G$, $n = 1, 2, \ldots,$ such that $\alpha_g = \lim_{n\to\infty} \mathrm{Ad}\, u_g^n\beta_g$. Set $U_g := (u_g^n) \in M^\omega$. Note that $\alpha_g^\omega = \mathrm{Ad}\, U_g\beta_g^\omega$ holds on $M (\subset M^\omega)$. Set $u(g,h) = U_g\beta_g^\omega(U_h)U_{gh}^*$. Then it is easy to verify that $u(g,h) \in M_\omega$ and $\{\mathrm{Ad}\, U_g\beta_g^\omega|_{M_\omega}, u(g,h)\}$ is a cocycle action on M_ω. Moreover $\mathrm{Ad}\, U_g\beta_g^\omega$ is strongly free in the sense of [16, Definition 5.6] by [16, Lemma 5.7]. By Ocneanu's 2-cohomology vanishing theorem [16, Proposition 7.4], we get $c_g \in U(M_\omega)$ such that $c_g\, \mathrm{Ad}\, U_g\beta_g^\omega(c_h)u(g,h)c_{gh}^* = 1$, which yields $c_gU_g\beta_g^\omega(c_hU_h) = c_{gh}U_{gh}$. Hence $W_g := c_gU_g$ becomes a β^ω-cocycle. Let $c_g = (c_g^n)$ be a representing sequence consisting of unitaries. Since (c_g^n) is a centralising sequence, $\mathrm{Ad}\, c_g^n$ converges to id_M. Hence $\alpha_g = \lim_{n\to\omega} \mathrm{Ad}\, c_g^nu_g^n\beta_g$. Set $w_g^n = c_g^nu_g^n$ and $w_n'(g,h) := w_g^n\beta_g(w_h^n)w_{gh}^{n*}$. Then $(\mathrm{Ad}\, w_g^n\beta_g, w_n'(g,h))$ is a cocycle action, and the cocycle identity $W_g\beta_g^\omega(W_h) = W_{gh}$ yields $\lim_{n\to\omega} w_n'(g,h) = 1$ in the σ-strong* topology. By Ocneanu's 2-cohomology vanishing theorem [16, Theorem 7.6], we have $d_g^n \in U(M)$ such that $d_g^n\, \mathrm{Ad}\, w_g^n\beta_g(d_h^n)w_n'(g,h)d_{gh}^{n*} = 1$ and $\lim_{n\to\omega} d_g^n = 1$ in the σ-strong* topology. Then $d_g^nw_g^n$ is a β_g-cocycle and $(d_g^nw_g^n) = (w_g^n)$ in M^ω. It is easy to see $\alpha_g\beta_g^{-1} = \lim_{n\to\omega} \mathrm{Ad}\, d_g^nw_g^n$.

We get the following corollary immediately.

COROLLARY 3.3. *Let α, β be as in Theorem 3.1. Then for any $\varepsilon > 0$, $\Phi \Subset M_*$ and $F \Subset G$, there exists a β-cocycle u_g such that $\|\alpha_g(\phi) - \mathrm{Ad}\, u_g\beta_g(\phi)\| < \varepsilon$ for every $g \in F$ and $\phi \in \Phi$.*

Next we show an approximate 1-cohomology vanishing theorem, which plays a central role in our argument.

LEMMA 3.4. *Let α be a centrally free action of G on M. For any $\varepsilon > 0$, $F = F^{-1} \Subset G$, $\Phi^+ \Subset M_*^+$, and $\Phi \Subset M_*$, there exist $\delta > 0$ and $\Psi \Subset M_*$ with the following property; for any α-cocycle $\{v_g\}$ with $\|[v_g, \psi]\| < \delta$, $g \in F$, $\psi \in \Psi$, we can find $w \in U(M)$ such that $\|[w, \varphi]\| < \varepsilon$ for every $\varphi \in \Phi$, and $\|v_g\alpha_g(w^*)w - 1\|_\phi^\# < \varepsilon$ for every $g \in F$ and $\phi \in \Phi^+$. When $\Phi = \emptyset$, $\Psi = \emptyset$ is possible.*

PROOF. Since every $\phi \in M_*$ is decomposed as $\phi = \phi_1 - \phi_2 + i(\phi_3 - \phi_4)$, $\phi_i \in M_*^+$, it suffices to show the lemma in the case $\Phi = \Phi^+ \Subset M_*^+$. We may assume $0 < \varepsilon < 1$ and $\|\phi\| \le 1$, $\phi \in \Phi^+$. Fix $\varepsilon' > 0$ with $\varepsilon' < (\varepsilon/8)^4$. Let $\{K_i\}_{i\in I}$ be an ε'-paving family such that each K_i is (ε', F)-invariant. We may assume that all K_i are in a subgroup of G generated by F. Define $\mathrm{Length}(g) := \min\{n \mid g = h_1h_2\cdots h_n, h_i \in F\}$, and set L by

$$L := \max\{\mathrm{Length}(g) \mid g \in K_i, i \in I\}.$$

Fix $\delta > 0$ such that $\sum_i |K_i|(L+1)\delta < \varepsilon/3$. Define Ψ by

$$\Psi := \Phi^+ \cup \bigcup_{\substack{1\le k\le L-1,\\ g_i\in F}} \alpha^{-1}_{g_1g_2\dots g_k}(\Phi^+).$$

By Ocneanu's Rohlin type Theorem, there exists a partition of unity $\{E_{i,k}\}_{i\in I, k\in K_i} \subset M_\omega$ such that

$$\sum_{i\in I} |K_i|^{-1} \sum_{k,l\in K_i} |\alpha_{kl^{-1}}(E_{i,l}) - E_{i,k}|_1 < 5\varepsilon'^{\frac{1}{2}},$$

$$[\alpha_g(E_{i,k}), E_{j,l}] = 0 \qquad \text{for all} \quad g \in G, i, j \in I, k \in K_i, l \in K_j.$$

Then by [16, Corollary 6.1], we have

$$\sum_{i\in I}\sum_{k\in K_i\cap g^{-1}K_i} |\alpha_g(E_{i,k}) - E_{i,gk}|_1 \le 10\varepsilon'^{\frac{1}{2}}$$

and

$$\sum_{i\in I}\sum_{k\in K_i\setminus g^{-1}K_i} |E_{i,k}|_1 \le \varepsilon' + 5\varepsilon'^{\frac{1}{2}}$$

for any $g \in F$.

We show that δ and Ψ defined above are the desired ones. Let v_g be an α-cocycle with $\|[v_g, \psi]\| < \delta$, $g \in F$, $\psi \in \Psi$. Set $W := \sum_{i,k} v_k^* E_{i,k}$. First we estimate $\|v_g\alpha_g(W^*)W - 1\|_\phi^\#$ for $g \in F$ as in the proof of [16, Proposition 7.2]. To this end, we investigate $|v_g\alpha_g(W^*)W - 1|_\phi$ and $|(v_g\alpha_g(W^*)W - 1)^*|_\phi$, and then use Lemma 2.1.

We divide

$$v_g\alpha_g(W^*)W - 1 = \sum_{i\in I,k\in K_i}\sum_{j\in I,l\in K_j}(v_g\alpha_g(v_k)v_l^* - 1)\alpha_g(E_{i,k})E_{j,l}$$

into three parts as follows.

$$\begin{aligned}
&\sum_{i\in I,k\in K_i}\sum_{j\in I,l\in K_j}(*)\\
&\quad= \sum_{j\in I,l\in K_j}\sum_{i,k\in K_i\cap g^{-1}K_i}(*) + \sum_{j\in I,l\in K_j}\sum_{i\in I,k\in K_i\backslash g^{-1}K_i}(*)\\
&\quad= \sum_{j\in I,l\in K_j}\sum_{\substack{j\neq i\in I,\\ k\in K_i\cap g^{-1}K_i}}(*) + \sum_{\substack{j\in I,l\in K_j,\\ k\in K_j\cap g^{-1}K_j}}(*) + \sum_{j\in I,l\in K_j}\sum_{i,k\in K_i\backslash g^{-1}K_i}(*)\\
&\quad= \sum_{\substack{j\in I,l\in K_j,\\ k\in K_j\cap g^{-1}K_j,gk=l}}(*) + \sum_{\substack{j\in I,l\in K_j,\\ k\in K_j\cap g^{-1}K_j,gk\neq l}}(*)\\
&\qquad\quad + \sum_{j\in I,l\in K_j}\sum_{\substack{j\neq i\in I,\\ k\in K_i\cap g^{-1}K_i}}(*) + \sum_{j\in I,l\in K_j}\sum_{i,k\in K_i\backslash g^{-1}K_i}(*)\\
&\quad= \sum_1(*) + \sum_2(*) + \sum_3(*).
\end{aligned}$$

In $\sum_1$ we sum for $i = j, k \in K_i \cap g^{-1}K_i, gk = l$, in $\sum_2$ we sum for $i = j, k \in K_i \cap g^{-1}K_i, gk \neq l$, or $i \neq j, k \in K_i \cap g^{-1}K_i, l \in K_j$, and in $\sum_3$ we sum for $j \in I, l \in K_j, k \in K_i\backslash g^{-1}K_i$. Due to the cocycle identity, we have $\sum_1(v_g\alpha_g(v_k)v_l^* - 1)\alpha_g(E_{i,k})E_{j,l} = \sum_j(v_g\alpha_g(v_k)v_{gk}^* - 1)\alpha_g(E_{i,k})E_{j,gk} = 0$, and $\sum_1$ part vanishes. Hence

$$\begin{aligned}
&\left|v_g\alpha_g(W^*)W - 1\right|_\phi\\
&= \left|\sum_2(v_g\alpha_g(v_k)v_l^* - 1)\alpha_g(E_{i,k})E_{j,l} + \sum_3(v_g\alpha_g(v_k)v_l^* - 1)\alpha_g(E_{i,k})E_{j,l}\right|_\phi\\
&\leq 2\sum_2\left|\alpha_g(E_{i,k})E_{j,l}\right|_1 + 2\sum_3\left|\alpha_g(E_{i,k})E_{j,l}\right|_1
\end{aligned}$$

holds by Lemma 2.1(4). Similarly we have

$$\left|(v_g\alpha_g(W^*)W - 1)^*\right|_\phi \leq 2\sum_2\left|\alpha_g(E_{i,k})E_{j,l}\right|_1 + 2\sum_3\left|\alpha_g(E_{i,k})E_{j,l}\right|_1.$$

We estimate the $\sum_2$ part. Then

$$\begin{aligned}
&\sum_2 \left|\alpha_g(E_{i,k})E_{j,l}\right|_1 \\
&= \sum_{\substack{j\in I,\\ l\in K_j}} \sum_{\substack{i\neq j,\\ k\in K_i\cap g^{-1}K_i}} \left|\alpha_g(E_{i,k})E_{j,l}\right|_1 + \sum_{\substack{i\in I, l\in K_i,\\ k\in K_i\cap g^{-1}K_i, gk\neq l}} \left|\alpha_g(E_{i,k})E_{i,l}\right|_1 \\
&= \sum_{\substack{i\in I,\\ k\in K_i\cap g^{-1}K_i}} \left|\alpha(E_{i,k})\left(1-\sum_l E_{i,l}\right)\right|_1 + \sum_{\substack{i\in I,\\ k\in K_i\cap g^{-1}K_i}} \left|\alpha_g(E_{i,k}) \sum_{l\in K_i, l\neq gk} E_{i,l}\right|_1 \\
&= \sum_{i\in I, k\in K_i\cap g^{-1}K_i} \left|\alpha_g(E_{i,k})(1-E_{i,gk})\right|_1 \\
&= \sum_{i\in I, k\in K_i\cap g^{-1}K_i} \left|(\alpha_g(E_{i,k})-E_{i,gk})(1-E_{i,gk})\right|_1 \\
&\leq \sum_{i\in I, k\in K_i\cap g^{-1}K_i} \left|\alpha_g(E_{i,k})-E_{i,gk}\right|_1 \\
&\leq 10\varepsilon'^{\frac{1}{2}}
\end{aligned}$$

holds.

The estimate of the $\sum_3$ part is given as follows.

$$\begin{aligned}
\sum_3 |\alpha_g(E_{i,k})E_{j,l}|_1 = \sum_{j\in I, l\in K_j} \sum_{\substack{i\in I,\\ k\in K_i\setminus g^{-1}K_i}} |\alpha_g(E_{i,k})E_{j,l}|_1 = \sum_{\substack{i\in I,\\ k\in K_i\setminus g^{-1}K_i}} |E_{i,k}|_1 \\
\leq \varepsilon' + 5\varepsilon'^{\frac{1}{2}} \leq 6\varepsilon'^{\frac{1}{2}}.
\end{aligned}$$

Summing up, we get $|v_g\alpha_g(W^*)W - 1|_\phi \leq 32\varepsilon'^{\frac{1}{2}}$ and $|(v_g\alpha(W^*)W - 1)^*|_\phi \leq 32\varepsilon'^{\frac{1}{2}}$. By Lemma 2.1(3), we have

$$\|v_g\alpha_g(W^*)W - 1\|_\phi^\# \leq 8\varepsilon'^{\frac{1}{4}} < \varepsilon.$$

We choose representing sequences $W = (w_n)$ consisting of unitaries, and $E_{i,k} = (e_{i,k}^n)$ such that $\{e_{i,k}^n\}_{i\in I, k\in K_i}$ is a partition of unity for each n. Set $a_n := \sum_{i,k} v_k^* e_{i,k}^n$. (Note that a_n is not necessary a unitary.) Since $W = (w_n) = (a_n)$ in M^ω, $\{w_n - a_n\}$ converges to 0 σ-strongly$*$. Choose a sufficiently large n such that

$$\begin{aligned}
\|v_g\alpha_g(w_n^*)w_n - 1\|_\phi^\# &< \varepsilon, & g\in F, \phi\in\Phi^+, \\
\|[\phi, e_{i,k}^n]\| &< \delta, & \phi\in\Phi^+, i\in I, k\in K_i, \\
\|w_n - a_n\|_\phi^\# &< \varepsilon/3, & \phi\in\Phi^+.
\end{aligned}$$

Set $w := w_n$, $a := a_n$, $e_{i,k} := e^n_{i,k}$. (Note that we never use the assumption $\|[v_g, \psi]\| < \delta$, $\psi \in \Psi$, in the estimation of $\|v_g\alpha_g(w^*)w - 1\|^\#_\phi$.) Next we show $\|[a, \varphi]\| < \varepsilon/3$, $\varphi \in \Phi^+$. To show this, we estimate $\|[v_g, \varphi]\|$, $g \in K_i$, $\varphi \in \Phi^+$ at first. We express g as $g = g_1 g_2 \dots g_k$, $g_i \in F$, $k \le L$. Then it is easy to show $\|[v_g, \varphi]\| \le \sum_{i=1}^k \|[v_{g_i}, \alpha^{-1}_{g_1 \dots g_{i-1}}(\varphi)]\|$. (When $i = 1$, $\alpha_{g_1\dots g_{i-1}}(\varphi)$ means φ.) Since each $\alpha^{-1}_{g_1\dots g_i}(\varphi)$ is in Ψ, $\|[v_{g_i}, \alpha^{-1}_{g_1\dots g_{i-1}}(\varphi)]\| < \delta$ follows by the assumption on v_g. Hence we have $\|[v_g, \varphi]\| \le L\delta$. Finally,

$$\begin{aligned}\|[a,\varphi]\| &\le \sum_{i\in I, k\in K_i} \|[v_k^* e_{i,k}, \varphi]\| \le \sum_{i\in I, k\in K_i} \|[v_k^*, \varphi] e_{i,k}\| + \|v_k^*[e_{i,k}, \varphi]\| \\ &\le \sum_{i\in I, k\in K_i} (L+1)\delta = \sum_{i\in I} |K_i|(L+1)\delta < \varepsilon/3\end{aligned}$$

holds for $\varphi \in \Phi^+$.

By Lemma 2.1(1),

$$\|[w,\varphi]\| \le \|[a,\varphi]\| + \|[w-a,\varphi]\| < \varepsilon/3 + 2\|w-a\|^\#_\varphi < \varepsilon$$

holds for $\varphi \in \Phi^+$, and w is a desired unitary.

REMARK. If we replace $v_g\alpha_g(W^*)W$ with $Wv_g\alpha_g(W^*)$ in the above proof, we then conclude $\|wv_g\alpha_g(w^*) - 1\|^\#_\phi < \varepsilon$.

Now we present a proof of Theorem 3.1 by means of Lemma 3.4.

PROOF OF THEOREM 3.1. Fix a faithful normal state φ_0. Let $\Phi = \{\varphi_i\}_{i=0}^\infty$ be a countable dense subset in M_*, and set $\Phi_n := \{\varphi_i\}_{i=0}^n$. Fix $G_n \Subset G$ such that $G_n \subset G_{n+1}$, $G_n^{-1} = G_n$ and $\bigcup_n G_n = G$.

We construct $w_n, v_g^n, \bar{v}_g^n \in U(M)$, $\Phi'_n, \Psi_n \Subset M_*$, $\Phi_n^+ \Subset M_*^+$, $\delta_n > 0$, and actions $\alpha_g^{(2n)}, \beta_g^{(2n-1)}$ of G satisfying the below conditions inductively. (We set $\alpha^{(0)} := \alpha$, $\beta^{(-1)} := \beta$.)

(1.2n) $\quad \|\beta_g^{(2n-1)}(\varphi) - \alpha_g^{(2n)}(\varphi)\| < 1/2^{2n}, \qquad g \in G_{2n}, \varphi \in \Phi'_{2n}, (n \ge 1).$

(1.2$n+1$) $\quad \|\alpha_g^{(2n)}(\varphi) - \beta_g^{(2n+1)}(\varphi)\| < 1/2^{2n+1},$

$$g \in G_{2n+1}, \varphi \in \Phi'_{2n+1}, (n \ge 0).$$

(2.2n) $\quad \|\beta_g^{(2n-1)}(\psi) - \alpha_g^{(2n)}(\psi)\| < \dfrac{\delta_{2n-1}}{2},$

$$g \in G_{2n-1}, \psi \in \bigcup_{g\in G_{2n-1}} \beta_{g^{-1}}^{(2n-1)}(\Psi_{2n-1}), (n \ge 1).$$

$$(2.2n+1) \quad \|\alpha_g^{(2n)}(\psi) - \beta_g^{(2n+1)}(\psi)\| < \frac{\delta_{2n}}{2}, \qquad g \in G_{2n}, \psi \in \bigcup_{g \in G_{2n}} \alpha_{g^{-1}}^{(2n)}(\Psi_{2n}), \ (n \geq 1).$$

$$(3.n) \quad \|v_g^n - 1\|_\phi^\# < 1/4^n, \qquad g \in G_{n-2}, \phi \in \Phi_{n-2}^+, \ (G_{-1} = G_0 = G_1, \Phi_{-1}^+ = \Phi_0^+ = \{\varphi_0\}),$$

$$(4.n) \quad \|[w_n, \varphi]\| < 1/4^n, \qquad \varphi \in \Phi'_{n-1}, \ (n \geq 3).$$

$$\bar{v}_g^n := v_g^n \operatorname{Ad} w_n(\bar{v}_g^{n-2}), \qquad (\bar{v}_g^1 = v_g^1, \bar{v}_g^2 = v_g^2),$$

$$\alpha_g^{(2n)} := \operatorname{Ad} v_g^{2n} \circ \operatorname{Ad} w_{2n}^* \circ \alpha_g^{(2n-2)} \circ \operatorname{Ad} w_{2n},$$

$$\beta_g^{(2n-1)} := \operatorname{Ad} v_g^{2n-1} \circ \operatorname{Ad} w_{2n-1}^* \circ \beta_g^{(2n-3)} \circ \operatorname{Ad} w_{2n-1},$$

$$\Phi'_{2n} := \Phi_{2n} \cup \operatorname{Ad} w_{2n-1}^* w_{2n-3}^* \cdots w_1^*(\Phi_{2n}) \cup \{\bar{v}_g^{2n-1}\varphi_0, \varphi_0 \bar{v}_g^{2n-1}\}_{g \in G_{2n-1}},$$

$$\Phi'_{2n+1} := \Phi_{2n+1} \cup \operatorname{Ad} w_{2n}^* w_{2n-2}^* \cdots w_2^*(\Phi_{2n+1}) \cup \{\bar{v}_g^{2n}\varphi_0, \varphi_0 \bar{v}_g^{2n}\}_{g \in G_{2n}}, \ (n \geq 1)$$

$$\Phi_n^+ := \{\operatorname{Ad} \bar{v}_g^n(\varphi_0) \mid g \in G_n\}.$$

Here δ_{2n} and Ψ_{2n} (δ_{2n-1} and Ψ_{2n-1}) are chosen as in Lemma 3.4 for $\alpha^{(2n)}$, $1/4^{2n+2}$, G_{2n}, Φ_{2n}^+ and Φ'_{2n+1} (resp. for $\beta^{(2n-1)}$, $1/4^{2n+1}$, G_{2n-1}, Φ_{2n-1}^+, and Φ'_{2n}).

At first set $\Phi'_1 := \Phi_1$ and fix a $\beta^{(-1)}$-cocycle u_g^1 such that

$$\|\alpha_g^{(0)}(\varphi) - \operatorname{Ad} u_g^1 \beta_g^{(-1)}(\varphi)\| < 1/2, \ g \in G_1, \varphi \in \Phi'_1.$$

By Lemma 3.4, we get a unitary w_1 such that $\|u_g \beta_g^{(-1)}(w_1^*) w_1 - 1\|_{\varphi_0}^\# < 1/4$, $g \in G_1$. Set $v_g^1 := u_g^1 \beta_g^{(-1)}(w_1^*) w_1$, and $\beta_g^{(1)} := \operatorname{Ad} u_g^1 \circ \beta_g^{(-1)} = \operatorname{Ad} v_g^1 \circ \operatorname{Ad} w_1^* \circ \beta_g^{(-1)} \circ \operatorname{Ad} w_1$. Then we have

$$(1.1) \qquad \|\alpha_g^{(0)}(\varphi) - \beta_g^{(1)}(\varphi)\| < 1/2$$

and

$$(3.1) \qquad \|v_g^1 - 1\|_{\varphi_0}^\# < 1/4$$

for $g \in G_1$. Set $\bar{v}_g^1 := v_g^1$, $\Phi'_2 := \Phi_2 \cup \operatorname{Ad} w_1^*(\Phi_2) \cup \{\bar{v}_g^1\varphi_0, \varphi_0\bar{v}_g^1\}$, and $\Phi_1^+ := \{\operatorname{Ad} \bar{v}_g^1(\varphi_0)\}_{g \in G_1}$. By Lemma 3.4, we choose Ψ_1 and δ_1 for $\beta_g^{(1)}$, $1/4^3$, G_1, Φ_1^+ and Φ'_2, and the first step is finished.

Next we take an $\alpha^{(0)}$-cocycle u_g^2 such that

$$(a.2)\quad \|\beta_g^{(1)}(\varphi) - \operatorname{Ad} u_g^2 \alpha_g^{(0)}(\varphi)\| < \frac{1}{2^2}, \qquad g \in G_2, \varphi \in \Phi'_2,$$

$$(b.2)\quad \|\beta_g^{(1)}(\psi) - \operatorname{Ad} u_g^2 \alpha_g^{(0)}(\psi)\| < \frac{\delta_1}{2}, \qquad g \in G_1, \psi \in \bigcup_{g \in G_1} \beta_{g^{-1}}^{(1)}(\Psi_1).$$

By Lemma 3.4, we get $w_2 \in U(M)$ such that $\|u_g^2 \alpha_g^{(0)}(w_2^*) w_2 - 1\|_{\varphi_0}^\# < 1/4^2$ for $g \in G_2$. Set $v_g^2 = \bar{v}_g^2 := u_g^2 \alpha_g^{(0)}(w_2^*) w_2$ and $\alpha_g^{(2)} := \operatorname{Ad} u_g^2 \alpha_g^{(0)} = \operatorname{Ad} v_g^2 \circ \operatorname{Ad} w_2^* \circ \alpha_g^{(0)} \circ \operatorname{Ad} w_2$. Then we get

$$(3.2)\qquad \|v_g^2 - 1\|_{\varphi_0}^\# < \frac{1}{4^2}, \qquad g \in G_2.$$

By $(a.2)$ and $(b.2)$,

$$(1.2)\qquad \|\beta_g^{(1)}(\phi) - \alpha_g^{(2)}(\phi)\| < \frac{1}{2^2}, \qquad g \in G_2, \varphi \in \Phi'_2,$$

$$(2.2)\qquad \|\beta_g^{(1)}(\psi) - \alpha_g^{(2)}(\psi)\| < \frac{\delta_1}{2}, \qquad g \in G_1, \psi \in \bigcup_{g \in G_1} \beta_{g^{-1}}^{(1)}(\Psi_1).$$

Set $\Phi'_3 := \Phi_3 \cup \operatorname{Ad} w_2^*(\Phi_3) \cup \{\bar{v}_g^2 \varphi_0, \varphi_0 \bar{v}_g^2\}_{g \in G_2}$ and $\Phi_2^+ := \{\operatorname{Ad} \bar{v}_g^2 \varphi_0 \mid g \in G_2\}$. By Lemma 3.4, we choose δ_2 and Ψ_2 for $\alpha^{(2)}$, G_2, Φ'_3, Φ_2^+ and $1/4^4$, and the second step is finished.

Suppose that we have constructed $\alpha_g^{(2n)}$, $\beta_g^{(2n-1)}$, w_{2n}, v_g^{2n}, $\bar{v}_g^{2n}$, Φ'_{2n+1}, δ_{2n} and Ψ_{2n}.

We choose a $\beta^{(2n-1)}$-cocycle $u_g^{2n+1} \in U(M)$ such that

$$(a.2n+1)\quad \|\alpha_g^{(2n)}(\varphi) - \operatorname{Ad} u_g^{2n+1} \beta_g^{(2n-1)}(\varphi)\| < \frac{1}{2^{2n+1}}, \qquad g \in G_{2n+1}, \varphi \in \Phi'_{2n+1},$$

$$(b.2n+1)\quad \|\alpha_g^{(2n)}(\psi) - \operatorname{Ad} u_g^{2n+1} \beta_g^{(2n-1)}(\psi)\| < \frac{\delta_{2n}}{2}, \qquad g \in G_{2n}, \psi \in \bigcup_{g \in G_{2n}} \alpha_{g^{-1}}^{(2n)}(\Psi_{2n}),$$

$$(c.2n+1)\quad \|\alpha_g^{(2n)}(\psi) - \operatorname{Ad} u_g^{2n+1} \beta_g^{(2n-1)}(\psi)\| < \frac{\delta_{2n-1}}{2}, \qquad g \in G_{2n-1}, \psi \in \bigcup_{g \in G_{2n-1}} \beta_{g^{-1}}^{(2n-1)}(\Psi_{2n-1}).$$

Then by (2.2n) and (c.2$n+1$) we get

$$\|\beta_g^{(2n-1)}(\psi) - \operatorname{Ad} u_g^{2n+1}\beta_g^{(2n-1)}(\psi)\| < \delta_{2n-1}, \qquad g \in G_{2n-1}, \psi \in \beta_{g^{-1}}^{(2n-1)}(\Psi_{2n-1}),$$

which yields $\|[u_g^{2n+1}, \psi]\| < \delta_{2n-1}$, $g \in G_{2n-1}$, $\psi \in \Psi_{2n-1}$. By the choice of δ_{2n-1} and Ψ_{2n-1}, there exists a unitary w_{2n+1} such that

$$\|u_g^{2n+1}\beta_g^{(2n-1)}(w_{2n+1}^*)w_{2n+1} - 1\|_\phi^\# < 1/4^{2n+1}, \qquad g \in G_{2n-1}, \phi \in \Phi_{2n-1}^+,$$

and

$$\text{(4.2}n+1) \qquad \|[w_{2n+1}, \varphi]\| < 1/4^{2n+1}, \qquad \varphi \in \Phi'_{2n}.$$

Set $v_g^{2n+1} := u_g^{2n+1}\beta_g^{(2n-1)}(w_{2n+1}^*)w_{2n+1}$ and $\beta_g^{(2n+1)} := \operatorname{Ad} u_g^{(2n+1)}\beta_g^{(2n-1)} = \operatorname{Ad} v_g^{(2n+1)} \circ \operatorname{Ad} w_{2n+1}^* \circ \beta_g^{(2n-1)} \circ \operatorname{Ad} w_{2n+1}$. Then

$$\text{(3.2}n+1) \qquad \|v_g^{2n+1} - 1\|_\phi^\# < \frac{1}{4^{2n+1}}, \qquad g \in G_{2n-1}, \phi \in \Phi_{2n-1}^+.$$

By (a.2$n+1$) and (b.2$n+1$), we get

$$\text{(1.2}n+1) \quad \|\alpha_g^{(2n)}(\varphi) - \beta_g^{(2n+1)}(\varphi)\| < \frac{1}{2^{2n+1}}, \qquad g \in G_{2n+1}, \varphi \in \Phi'_{2n+1}.$$

$$\text{(2.2}n+1) \quad \|\alpha_g^{(2n)}(\psi) - \beta_g^{(2n+1)}(\psi)\| < \frac{\delta_{2n}}{2}, \qquad g \in G_{2n}, \psi \in \bigcup_{g \in G_{2n}} \alpha_{g^{-1}}^{(2n)}(\Psi_{2n}).$$

Set $\bar{v}_g^{2n+1} := v_g^{2n+1} \operatorname{Ad} w_{2n+1}^*(\bar{v}_g^{2n-1})$, and define

$$\Phi'_{2n+2} := \Phi_{2n+2} \cup \operatorname{Ad} w_{2n+1}^* w_{2n-1}^* \cdots w_1^*(\Phi_{2n+2}) \cup \{\bar{v}_g^{2n+1}(\varphi_0), \varphi_0\bar{v}_g^{2n+1}\}_{g \in G_{2n+1}},$$

$$\Phi_{2n+1}^+ := \{\operatorname{Ad} \bar{v}_g^{2n+1}\varphi_0 \mid g \in G_{2n+1}\}.$$

By Lemma 3.4, we choose $\delta_{2n+1} > 0$ and $\Psi_{2n+1} \Subset M_*$ for $\beta^{(2n+1)}$, $1/4^{2n+3}$, G_{2n+1}, Φ_{2n+1}^+ and Φ'_{2n+2}, and the $(2n+1)$-st step is finished.

Next we choose an $\alpha^{(2n)}$-cocycle u_g^{2n+2} such that

$$(a.2n+2) \quad \|\beta_g^{(2n+1)}(\varphi) - \operatorname{Ad} u_g^{2n+2}\alpha_g^{(2n)}(\varphi)\| < \frac{1}{2^{2n}}, \ g \in G_{2n+2}, \varphi \in \Phi'_{2n+2},$$

$$(b.2n+2)\quad \|\beta_g^{(2n+1)}(\psi) - \operatorname{Ad} u_g^{2n+2}\alpha_g^{(2n)}(\psi)\| < \frac{\delta_{2n+1}}{2},$$
$$g \in G_{2n+1},\ \psi \in \bigcup_{g \in G_{2n+1}} \beta_{g^{-1}}^{(2n+1)}(\Psi_{2n+1}),$$

$$(c.2n+2)\quad \|\beta_g^{(2n+1)}(\psi) - \operatorname{Ad} u_g^{2n+2}\alpha_g^{(2n)}(\psi)\| < \frac{\delta_{2n}}{2},$$
$$g \in G_{2n},\ \psi \in \bigcup_{g \in G_{2n}} \alpha_{g^{-1}}^{(2n)}(\Psi_{2n}).$$

By $(c.2n+2)$ and $(2.2n+1)$, we get

$$\|\alpha_g^{(2n)}(\psi) - \operatorname{Ad} u_g^{2n+2}\alpha_g^{(2n)}(\psi)\| < \delta_{2n}, \qquad g \in G_{2n},\ \psi \in \bigcup_{g \in G_{2n}} \alpha_{g^{-1}}^{(2n)}(\Psi_{2n}).$$

We thus have $\|[u_g^{2n+2}, \psi]\| < \delta_{2n}$ for $g \in G_{2n}$ and $\psi \in \Psi_{2n}$. By the choice of δ_{2n} and Ψ_{2n}, we can find $w_{2n+2} \in U(M)$ such that

$$\|u_g^{2n+2}\alpha_g^{(2n)}(w_{2n+2}^*)w_{2n+2} - 1\|_\phi^\# < \frac{1}{4^{2n+2}}, \qquad g \in G_{2n},\ \phi \in \Phi_{2n}^+$$

and

$$(4.2n+2)\qquad \|[w_{2n+2}, \varphi]\| < \frac{1}{4^{2n+2}}, \qquad \varphi \in \Phi'_{2n+1}.$$

Set $v_g^{2n+2} := u_g^{2n+2}\alpha_g^{(2n)}(w_{2n+2}^*)w_{2n+2}$ and $\alpha_g^{(2n+2)} := \operatorname{Ad} u_g^{2n+2}\alpha_g^{(2n)} = \operatorname{Ad} v_g^{2n+2} \circ \operatorname{Ad} w_{2n+2}^* \circ \alpha_g^{(2n)} \circ \operatorname{Ad} w_{2n+2}$. Then

$$(3.2n+2)\qquad \|v_g^{2n+2} - 1\|_\phi^\# < \frac{1}{4^{2n+2}}, \qquad g \in G_{2n},\ \phi \in \Phi_{2n}^+,$$

and by $(a.2n+2)$ and $(b.2n+2)$, we get

$$(1.2n+2)\quad \|\beta_g^{(2n+1)}(\varphi) - \alpha_g^{(2n+2)}(\varphi)\| < \frac{1}{2^{2n}},$$
$$g \in G_{2n},\ \varphi \in \Phi'_{2n+2}.$$

$$(2.2n+2)\quad \|\beta_g^{(2n+1)}(\psi) - \alpha_g^{(2n+2)}(\psi)\| < \frac{\delta_{2n+1}}{2},$$
$$g \in G_{2n+1},\ \psi \in \bigcup_{g \in G_{2n+1}} \beta_{g^{-1}}^{(2n+1)}(\Psi_{2n+1}).$$

Set $\bar{v}_g^{2n+2} := v_g^{2n+2} \operatorname{Ad} w_{2n+2}^*(\bar{v}_g^{2n})$ and

$$\Phi'_{2n+3} := \Phi_{2n+3} \cup \operatorname{Ad} w_{2n+2}^* w_{2n}^* \cdots w_2^*(\Phi_{2n+3}) \cup \{\bar{v}_g^{2n+2}\varphi_0, \varphi_0 \bar{v}_g^{2n+2}\},$$

$$\Phi_{2n+2}^+ := \{\operatorname{Ad} \bar{v}_g^{2n+2}\varphi_0 \mid g \in G_{2n+2}\}.$$

We choose $\delta_{2n+2} > 0$, $\Psi_{2n+2} \Subset M_*$ for $\alpha_g^{(2n)}$, $1/4^{2n+4}$, G_{2n+2}, Φ_{2n+2}^+ and Φ'_{2n+3} by Lemma 3.4. Then the $(2n+2)$-nd step is finished, and thus we complete induction.

Set $\theta_{2n} := \operatorname{Ad} w_{2n}^* w_{2n-2}^* \cdots w_2^*$. Then we have $\alpha_g^{(2n)} = \operatorname{Ad} \bar{v}_g^{2n} \circ \theta_{2n} \circ \alpha_g \circ \theta_{2n}^{-1}$. We will verify $\{\theta_{2n}\}$ converges to some $\theta \in \operatorname{Aut}(M)$. To this end, we will prove that $\{\theta_{2n}(\varphi)\}$ and $\{\theta_{2n}^{-1}(\varphi)\}$ are Cauchy sequences for $\varphi \in M_*$. Suppose $\varphi \in \Phi_k$. For any n with $k \leq 2n+1$, φ and $\theta_{2n}(\varphi)$ are in Φ'_{2n+1}. By $(4.2n+2)$, we have

$$\|\theta_{2n+2}(\varphi) - \theta_{2n}(\varphi)\| = \|[w_{2n+2}, \theta_{2n}(\varphi)]\| < \frac{1}{4^{2n+2}},$$

and

$$\|\theta_{2n+2}^{-1}(\varphi) - \theta_{2n}^{-1}(\varphi)\| = \|w_{2n+2}^* \varphi w_{2n+2} - \varphi\| < \frac{1}{4^{2n+2}}.$$

It follows that $\{\theta_{2n}(\varphi)\}$ and $\{\theta_{2n}^{-1}(\varphi)\}$ are Cauchy sequences for $\varphi \in \Phi$. Then so are $\{\theta_{2n}(\varphi)\}$ and $\{\theta_{2n}^{-1}(\varphi)\}$ for every $\varphi \in M_*$, since Φ is dense in M_*. Hence $\{\theta_{2n}\}$ converges to some $\theta \in \operatorname{Aut}(M)$.

Next we will verify that $\{\bar{v}_g^{2n}\}$ is a Cauchy sequence with respect to $\|\cdot\|_{\varphi_0}^\#$. Since we have

$$\begin{aligned}\|\bar{v}_g^{2n+2} - \bar{v}_g^{2n}\|_{\varphi_0}^\# \leq \|(v_g^{2n+2} - 1)\bar{v}_g^{2n}\|_{\varphi_0}^\# + \|(w_{2n+2}^* \bar{v}_g^{2n} w_{2n+2} - \bar{v}_g^{2n})\|_{\varphi_0}^\# \\ + \|(v_g^{2n+2} - 1)(w_{2n+2}^* \bar{v}_g^{2n} w_{2n+2} - \bar{v}_g^{2n})\|_{\varphi_0}^\#,\end{aligned}$$

we will estimate the above three terms.

Suppose $g \in G_k$. Then for any n with $2n \geq k$, φ_0, $\operatorname{Ad} \bar{v}_g^{2n}(\varphi_0) \in \Phi_{2n}^+$, and hence $\|v_g^{2n+2} - 1\|_{\operatorname{Ad} \bar{v}_g^{2n}(\varphi_0)}^\# < 1/4^{2n+2}$ and $\|v_g^{2n+2} - 1\|_{\varphi_0}^\# < 1/4^{2n+2}$ hold by $(3.2n+2)$.

We have

$$\begin{aligned}\|(v_g^{2n+2} - 1)\bar{v}_g^{2n}\|_{\varphi_0}^{\#2} &= \frac{1}{2}(\|(v_g^{2n+2} - 1)\bar{v}_g^{2n}\|_{\varphi_0}^2 + \|\bar{v}_g^{2n*}(v_g^{2n+2*} - 1)\|_{\varphi_0}^2) \\ &= \frac{1}{2}(\|v_g^{2n+2} - 1\|_{\operatorname{Ad} \bar{v}_g^{2n}(\varphi_0)}^2 + \|v_g^{2n+2*} - 1\|_{\varphi_0}^2) \\ &\leq \|v_g^{2n+2} - 1\|_{\operatorname{Ad} \bar{v}_g^{2n}(\varphi_0)}^{\#2} + \|v_g^{2n+2} - 1\|_{\varphi_0}^{\#2} \\ &< \frac{2}{16^{2n+2}}.\end{aligned}$$

Hence we get $\|(v_g^{2n+2}-1)\bar{v}_g^{2n}\|_{\varphi_0}^{\#} < \sqrt{2}/4^{2n+2} < 1/2^{2n+1}$.

We next estimate $\|w_{2n+2}^*\bar{v}_g^{2n}w_{2n+2}-\bar{v}_{2n}\|_{\varphi_0}^{\#}$. Since $\varphi_0, \bar{v}_g^{2n}\varphi_0, \varphi_0\bar{v}_g^{2n} \in \Phi'_{2n+1}$, we have $\|[w_{2n+2},\varphi_0]\| < 1/4^{2n+2}$, $\|[w_{2n+2},\bar{v}_g^{2n}\varphi_0]\| < 1/4^{2n+2}$ and $\|[w_{2n+2},\varphi_0\bar{v}_g^{2n}]\| < 1/4^{2n+2}$ by $(4.2n+2)$. Then

$$
\begin{aligned}
&\|(w_{2n+2}^*\bar{v}_g^{2n}w_{2n+2}-\bar{v}_g^{2n})\cdot\varphi_0\| \\
&\quad= \|(w_{2n+2}^*\bar{v}_g^{2n}-\bar{v}_g^{2n}w_{2n+2}^*)w_{2n+2}\cdot\varphi_0\| \\
&\quad\le \|(w_{2n+2}^*\bar{v}_g^{2n}-\bar{v}_g^{2n}w_{2n+2}^*)\cdot\varphi_0\cdot w_{2n+2}\| \\
&\qquad\quad + \|(w_{2n+2}^*\bar{v}_g^{2n}-\bar{v}_g^{2n}w_{2n+2}^*)\cdot[w_{2n+2},\varphi_0]\| \\
&\quad\le \|w_{2n+2}^*\bar{v}_g^{2n}\varphi_0 w_{2n+2}-\bar{v}_g^{2n}w_{2n+2}^*\varphi_0 w_{2n+2}\| + 2/4^{2n+2} \\
&\quad\le \|[w_{2n+2}^*,\bar{v}_g^{2n}\varphi_0]w_{2n+2}\| + \|\bar{v}_g^{2n}\varphi_0-\bar{v}_g^{2n}w_{2n+2}^*\varphi_0 w_{2n+2}\| + 2/4^{2n+2} \\
&\quad\le 1/4^{2n+2} + \|\bar{v}_g^{2n}[\varphi_0,w_{2n+2}^*]\| + 2/4^{2n+2} \\
&\quad\le 1/4^{2n+1}
\end{aligned}
$$

holds. Hence

$$
\begin{aligned}
&\|w_{2n+2}^*\bar{v}_g^{2n}w_{2n+2}-\bar{v}_g^{2n}\|_{\varphi_0}^2 \\
&\qquad\le \|w_{2n+2}^*\bar{v}_g^{2n}w_{2n+2}-\bar{v}_g^{2n}\|\|(w_{2n+2}^*\bar{v}_g^{2n}w_{2n+2}-\bar{v}_g^{2n})\cdot\varphi_0\| \\
&\qquad\le 2\|(w_{2n+2}^*\bar{v}_g^{2n}w_{2n+2}-\bar{v}_g^{2n})\cdot\varphi_0\| \\
&\qquad\le \frac{2}{4^{2n+1}}
\end{aligned}
$$

holds by Lemma 2.1(2).

In a similar way, we can show $\|(w_{2n+2}^*\bar{v}_g^{2n}w_{2n+2}-\bar{v}_g^{2n})^*\|_{\varphi_0}^2 \le 2/4^{2n+1}$. Hence we get $\|w_{2n+2}^*\bar{v}_g^{2n}w_{2n+2}-\bar{v}_g^{2n}\|_{\varphi_0}^{\#} \le \sqrt{2/4^{2n+1}} = \sqrt{2}/2^{2n+1} < 1/2^{2n}$.

The third term $\|(v_g^{2n+2}-1)(w_{2n+2}^*\bar{v}_g^{2n}w_{2n+2}-\bar{v}_g^{2n})\|_{\varphi_0}^{\#}$ is estimated as follows.

$$
\begin{aligned}
&\|(v_g^{2n+2}-1)(w_{2n+2}^*\bar{v}_g^{2n}w_{2n+2}-\bar{v}_g^{2n})\|_{\varphi_0}^{\#2} \\
&\quad= \frac{1}{2}(\|(v_g^{2n+2}-1)(w_{2n+2}^*\bar{v}_g^{2n}w_{2n+2}-\bar{v}_g^{2n})\|_{\varphi_0}^2 \\
&\qquad\quad + \|(w_{2n+2}^*\bar{v}_g^{2n*}w_{2n+2}-\bar{v}_g^{2n*})(v_g^{2n+2*}-1)\|_{\varphi_0}^2) \\
&\quad\le 2\|w_{2n+2}^*\bar{v}_g^{2n}w_{2n+2}-\bar{v}_g^{2n}\|_{\varphi_0}^2 + 2\|v_g^{2n+2*}-1\|_{\varphi_0}^2 \\
&\quad\le 4\|w_{2n+2}^*\bar{v}_g^{2n}w_{2n+2}-\bar{v}_g^{2n}\|_{\varphi_0}^{\#2} + 4\|v_g^{2n+2*}-1\|_{\varphi_0}^{\#2} \\
&\quad\le \frac{4}{4^{2n}} + \frac{4}{4^{2n+2}} \\
&\quad\le \frac{2}{4^{2n-1}}.
\end{aligned}
$$

Hence $\|(v_g^{2n+2}-1)(w_{2n+2}^*\bar{v}_g^{2n}w_{2n+2}-\bar{v}_g^{2n})\|_{\varphi_0}^\# \leq \sqrt{2}/2^{2n-1} < 1/2^{2n-2}$. Summing up, we have the following.

$$\|\bar{v}_g^{2n+2}-\bar{v}_g^{2n}\|_{\varphi_0}^\# \leq \frac{1}{2^{2n+1}}+\frac{1}{2^{2n}}+\frac{1}{2^{2n-2}} \leq \frac{1}{2^{2n-3}}.$$

It follows that $\{\bar{v}_g^{2n}\}$ is a Cauchy sequence and converges to some unitary $\hat{v}_g^0$.

In the same way, we can show that $\operatorname{Ad} w_{2n+1}^* w_{2n-1}^* \cdots w_1^*$ and $\bar{v}_g^{2n+1}$ converge to some $\sigma \in \operatorname{Aut}(M)$ and $\hat{v}_g^1 \in U(M)$ respectively. By $(1.n)$ we get $\operatorname{Ad} \hat{v}_g^0 \circ \theta \circ \alpha_g \circ \theta^{-1} = \operatorname{Ad} \hat{v}_g^1 \circ \sigma \circ \beta_g \circ \sigma^{-1}$, and hence α and β are cocycle conjugate. By construction, θ and σ are approximately inner.

We will choose a cocycle close to 1. Suppose $\operatorname{Ad} v_g\alpha = \theta \circ \beta_g \circ \theta^{-1}$, $\theta \in \overline{\operatorname{Int}}(M)$. Fix $F \Subset G$ and $\varepsilon > 0$. Then there exists a unitary w such that $\|wv_g\alpha_g(w^*)-1\|_{\varphi_0}^\# < \varepsilon$ for each $g \in F$. (See the remark after Lemma 3.4.) Define a new α-cocycle v_g' by $v_g' := wv_g\alpha_g(w^*)$. We then have $\|v_g'-1\|_{\varphi_0}^\# < \varepsilon$ for $g \in F$, and

$$\begin{aligned}\operatorname{Ad} v_g'\alpha_g &= \operatorname{Ad}(wv_g\alpha_g(w^*)) \circ \alpha_g \\ &= \operatorname{Ad} w \circ \operatorname{Ad} v_g \circ \alpha_g \circ \operatorname{Ad} w^* \\ &= \operatorname{Ad} w \circ \theta \circ \beta_g \circ \theta^{-1} \circ \operatorname{Ad} w^*.\end{aligned}$$

Put $\sigma := \operatorname{Ad} w \circ \theta$. Then $\sigma \in \overline{\operatorname{Int}}(M)$, and we get $\operatorname{Ad} v_g'\alpha_g = \sigma \circ \beta_g \circ \sigma^{-1}$.

We present applications of Theorem 3.1. Let M be an injective factor. By the Connes-Krieger-Haagerup classification of injective factors [2], [12], [5], [7], M is a McDuff factor. Since $\operatorname{Ker}(\operatorname{mod}) = \overline{\operatorname{Int}}(M)$ by [3] and [11], we get the following corollary.

COROLLARY 3.5. *Let M be an injective factor, G a discrete amenable group, and α, β centrally free actions of G on M. Then* $\operatorname{Ad} v_g\alpha_g = \theta \circ \beta_g \circ \theta^{-1}$ *for some α-cocycle v_g and $\theta \in \overline{\operatorname{Int}}(M)$ if and only if* $\operatorname{mod}(\alpha) = \operatorname{mod}(\beta)$. *(In the type II_1 case, we regard* $\operatorname{mod}(\alpha)$ *as trivial for $\alpha \in \operatorname{Aut}(M)$.)*

Theorem 3.1 can be modified for a relative McDuff subfactor $N \subset M$ by appropriate changes. Indeed in the proof we only have to replace M_ω with $M_\omega \cap N^\omega$, which is a subfactor version of a central sequence algebra. Especially if $N \subset M$ is a strongly amenable subfactor of type II_1 in the sense of Popa [18], then it is relatively McDuff thanks to Popa's classification theorem for strongly amenable subfactors of type II_1 [18]. We also have $\overline{\operatorname{Int}}(M, N) = \operatorname{Ker} \Phi$ by [13], where $\Phi(\alpha)$ is the Loi invariant for $\alpha \in \operatorname{Aut}(M, N)$, and the equivalence between strong outerness and central freeness by [17]. (Also see [14] for the latter fact.) Hence Theorem 3.1 gives an alternative proof of the main theorem in [17].

COROLLARY 3.6. *Let $N \subset M$ be a strongly amenable subfactor of type II_1, G a discrete amenable group, and α, β strongly outer actions of G on $N \subset M$. Then* $\mathrm{Ad}\, v_g \alpha_g = \theta \circ \beta_g \circ \theta^{-1}$ *for some α-cocycle $v_g \in U(N)$ and $\theta \in \overline{\mathrm{Int}}(M, N)$ if and only if* $\Phi(\alpha) = \Phi(\beta)$.

When $N \subset M$ is a strongly amenable subfactor of type II_∞, we have $\overline{\mathrm{Int}}(M, N) = \mathrm{Ker}\,\Phi \cap \mathrm{Ker}(\mathrm{mod})$. Hence we have the following corollary.

COROLLARY 3.7. *Let $N \subset M$ be a strongly amenable subfactor of type II_∞, G a discrete amenable group, and α, β strongly outer actions of G on $N \subset M$. Then* $\mathrm{Ad}\, v_g \alpha_g = \theta \circ \beta_g \circ \theta^{-1}$ *for some α-cocycle $v_g \in U(N)$ and $\theta \in \overline{\mathrm{Int}}(M, N)$ if and only if* $\Phi(\alpha) = \Phi(\beta)$ *and* $\mathrm{mod}(\alpha) = \mathrm{mod}(\beta)$.

It is worth noting that Corollary 3.7 yields the classification of strongly amenable subfactor of type III_λ, $0 < \lambda < 1$. See [13], [17].

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