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# GLOBAL SCHAUDER DECOMPOSITIONS OF LOCALLY CONVEX SPACES

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## Abstract

We define global Schauder decompositions of locally convex spaces and prove a necessary and sufficient condition for two spaces with global Schauder decompositions to be isomorphic. These results are applied to spaces of entire functions on a locally convex space.


Given two spaces, $E$ and $F$, with Schauder (or even $\mathscr{S}$-absolute) decompositions, the existence of isomorphisms between the spaces forming the decompositions does not imply that $E$ and $F$ are isomorphic. In order to tackle this problem when the underlying decompositions consist of Banach spaces, P. Galindo, M. Maestre and P. Rueda defined in [12] a subclass of $\mathscr{S}$-absolute decompositions of Fréchet spaces: R-Schauder decompositions. To consider the corresponding problem when $E$ and $F$ are locally convex spaces and the underlying decompositions are not necessarily Banach spaces, we were led to define global Schauder decompositions.

## 1. Introduction

In this section we give initial definitions and preliminary results.

First we introduce notation that will be used throughout the article. Let $E$ denote a locally convex space over the complex numbers $\mathbf{C}$, and let $E'$ denote the space of all continuous linear functionals on $E$. When $E'$ is endowed with the strong topology (i.e. the topology of uniform convergence over the bounded subsets of $E$), we denote it by $E'_\beta$.

For $E$ a locally convex space we let $\mathscr{P}(^nE)$ denote the space of all continuous $n$-homogeneous polynomials on $E$. The topology on $\mathscr{P}(^nE)$ of uniform convergence over the compact (respectively bounded) subsets of $E$ is denoted by $\tau_0$ (respectively $\tau_b$). A third topology on $\mathscr{P}(^nE)$ can be defined in the following way. A semi-norm $p$ on $\mathscr{P}(^nE)$ is $\tau_w$-continuous if for every zero

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neighbourhood $V$ in $E$ there exists a positive constant $C(V)$ such that

$$p(P) \leq C(V)\|P\|_V$$

for all $P \in \mathscr{P}(^n E)$. The topology generated by all such semi-norms is denoted by $\tau_w$. When $n = 1$, $E'_i := (\mathscr{P}(^1 E), \tau_w)$ is the inductive dual of $E$, $E'_\beta := (\mathscr{P}(^1 E), \tau_b)$ is the strong dual of $E$ and $E'_c := (\mathscr{P}(^1 E), \tau_0)$. By $\widehat{\bigotimes}_{n,s,\pi} E$ (respectively $\widehat{\bigotimes}_{n,s,\varepsilon} E$) we denote the completed symmetric $n$-fold tensor product of $E$ endowed with the projective tensor topology (resp. the injective tensor topology).

For more definitions and properties of polynomials and holomorphic functions on locally convex spaces we refer the reader to [7] and [8], and for more information on locally convex spaces we refer the reader to [13] and [14].

DEFINITION 1.1. A sequence of subspaces $\{E_n\}_n$ of a locally convex space $E$ is a *Schauder decomposition* of $E$ if:

- For each $x$ in $E$ there exists a unique sequence of vectors $(x_n)_n$, $x_n \in E_n$, such that
$$x = \sum_{n=1}^{\infty} x_n := \lim_{m\to\infty} \sum_{n=1}^{m} x_n.$$
- The projections $(u_n)_{n=1}^{\infty}$ defined by
$$u_m\left(\sum_{n=1}^{\infty} x_n\right) := \sum_{n=1}^{m} x_n$$
are continuous.

The topology on each $E_n$ is induced by the topology on $E$. A Schauder decomposition $\{E_n\}_n$ of a locally convex space $E$ is *absolute* if for each $p \in \mathrm{cs}(E)$,

$$q\left(\sum_{n=1}^{\infty} x_n\right) := \sum_{n=1}^{\infty} p(x_n)$$

defines a continuous semi-norm on $E$.

The following definition is our main tool in this paper.

DEFINITION 1.2. A Schauder decomposition $\{E_n\}_{n=0}^{\infty}$ of a locally convex space $E$ is a *global Schauder decomposition* if for all $r > 0$, all $x = \sum_{n=1}^{\infty} x_n \in E$ with $x_n \in E_n$ for each $n$,

$$r \cdot x := \sum_{n=1}^{\infty} r^n x_n \in E; \tag{1}$$

and for each $p \in \mathrm{cs}(E)$,

$$p_r\left(\sum_{n=1}^{\infty} x_n\right) := \sum_{n=1}^{\infty} r^n p(x_n) \tag{2}$$

defines a continuous semi-norm on $E$.

In particular, taking $r = 1$ we see that global Schauder decompositions are absolute.

REMARK 1.3. If $\{E_n\}_n$ is a global Schauder decomposition for the locally convex space $E$, there is a generating family of semi-norms $p \in \mathrm{cs}(E)$ of the form

$$p\left(\sum_{n=1}^{\infty} x_n\right) = \sum_{n=1}^{\infty} p(x_n). \tag{3}$$

Let $q(x) := \sup_n p(x_n)$ where $p$ is a continuous semi-norm satisfying (3). Since $\sup_n p(x_n) \leq \sum_{n=1}^{\infty} p(x_n)$, the semi-norm $q$ is continuous. Let $q_2(x) := \sup_n(2^n p(x_n))$, from the inequality

$$q(x) \leq p(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} 2^n p(x_n) \leq \sup_n (2^n p(x_n)) \sum_{n=1}^{\infty} \left(\frac{1}{2^n}\right) = q_2(x) \leq p_2(x),$$

it follows that the semi-norms $\{q(x) = \sup_n p(x_n)\}$ generate the topology on $E$. Moreover, condition (2) in Definition 1.2 is equivalent to the condition that for each $q \in \mathrm{cs}(E)$,

$$q_r\left(\sum_{n=1}^{\infty} x_n\right) := \sup_n (r^n q(x_n)) \tag{4}$$

defines a continuous semi-norm on $E$. Thus the locally convex topology of $E$ can be defined both by $l_1$-type or by $c_0$-type norms.

For completeness we will give the definitions for two other types of Schauder decompositions, $\mathscr{S}$-absolute decompositions and R-Schauder decompositions. Let $\mathscr{S}$ denote the set of all sequences $(\alpha_n)_{n=1}^{\infty} \subset \mathbb{C}$ such that $\limsup_{n\to\infty} |\alpha_n|^{1/n} \leq 1$.

DEFINITION 1.4. A Schauder decomposition $\{E_n\}_n$ of a locally convex space $E$ is an *$\mathscr{S}$-absolute decomposition* if for all $\alpha = (\alpha_n)_n \in \mathscr{S}$ and $x = \sum_{n=1}^{\infty} x_n \in E$, with $x_n \in E_n$ for all $n$,

$$\alpha \cdot x := \sum_{n=1}^{\infty} \alpha_n x_n \in E \tag{5}$$

and, for each $p \in \mathrm{cs}(E)$ and each $\alpha = (\alpha_n)_n$ in $\mathscr{S}$,

$$p_\alpha\Big(\sum_{n=1}^{\infty} x_n\Big) := \sum_{n=1}^{\infty} |\alpha_n| p(x_n) \tag{6}$$

defines a continuous semi-norm on $E$.

For results and applications of $\mathscr{S}$-absolute decompositions we refer the reader to [7] and [8]. We will just mention that a Schauder decomposition of a barrelled locally convex space satisfying (5) is an $\mathscr{S}$-absolute decomposition.

Definition 1.5. Let $\{E_n\}_n$ denote an absolute Schauder decomposition of the locally convex space $E$. We say that $\{E_n\}_n$ is a T.S. *(=Taylor series) complete decomposition* if for any sequence $(x_n)_n$ with $x_n \in E_n$ for all $n$, $\sum_{n=1}^{\infty} p(x_n) < \infty$ for all $p \in \mathrm{cs}(E)$ implies $\sum_{n=1}^{\infty} x_n \in E$.

T.S. completeness and conditions (1) and (5) are all completeness conditions with respect to a decomposition. In particular, the $\mathscr{S}$-absolute Schauder decomposition of a sequentially complete locally convex space is T.S. complete. If $\{E_n\}_n$ is a T.S. complete and global Schauder decomposition, then it is an $\mathscr{S}$-absolute decomposition.

Let us now consider the case when $E$ is a Fréchet space such that there is a sequence of Banach spaces $\{E_n\}_{n=0}^{\infty}$ which is a Schauder decomposition of $E$. Let $0 < R \le \infty$. The decomposition $\{E_n\}_{n=0}^{\infty}$ is *R-Schauder* ([12]) if for every sequence $(x_n)_n$, $x_n \in E_n$, the series $x = \sum_{n=1}^{\infty} x_n$ converges in $E$ if and only if $\limsup_n \|x_n\|_n^{1/n} \le 1/R$.

If $E$ is a Fréchet space and $\{E_n\}_{n=0}^{\infty}$ is an $\infty$-Schauder decomposition of $E$ consisting of Banach spaces, then it is a global Schauder decomposition of $E$. Indeed, let $A = \{(r^n)_n : r > 0\}$, consider the Köthe sequence space $\lambda^1(A, (E_n)_n)$. This is the Fréchet space $\big\{(x_n)_n \in \prod_{n=1}^{\infty} E_n : p_r\big(\sum_{n=0}^{\infty} x_n\big) := \sum_{n=0}^{\infty} r^n \|x_n\|_n < \infty \text{ for all } r > 0\big\}$, endowed with the topology generated by the semi-norms $\{p_r\}_{r>0}$. Clearly, $\{E_n\}_{n=0}^{\infty}$ forms a global Schauder decomposition of $\lambda^1(A, (E_n)_n)$. By ([12], Theorem 1) $E$ is topologically isomorphic to $\lambda^1(A, (E_n)_n)$, hence $\{E_n\}_{n=0}^{\infty}$ forms a global Schauder decomposition of $E$.

To show that the converse is not true, consider the Köthe matrix $A^* = \big\{\big((nr)^n\big)_n : r > 0\big\}$ and a sequence of Banach spaces $\{E_n\}_{n=0}^{\infty}$. The corresponding Köthe sequence space $\lambda^1(A^*, (E_n)_n) = \big\{(x_n)_n \in \prod_{n=1}^{\infty} E_n : p_r^*\big(\sum_{n=0}^{\infty} x_n\big) := \sum_{n=0}^{\infty} (nr)^n \|x_n\|_n < \infty \text{ for all } r > 0\big\}$ endowed with the topology generated by the semi-norms $\{p_r^*\}_{r>0}$ is a Fréchet space. It is easy to check that $\{E_n\}_{n=0}^{\infty}$ is a global Schauder decomposition of $\lambda^1(A^*, (E_n)_n)$. Let $x = \sum_{n=1}^{\infty} x_n \in \prod_{n=1}^{\infty} E_n$ such that $\|x_n\|_n = 1/n^n$, then $\limsup_n \|x_n\|_n^{1/n} = 0$. On the other hand, for $r \ge 1$ the series $p_r(x) = \sum_{n=0}^{\infty} (nr)^n \|x_n\|_n$ is divergent hence $x$ does not belong to $\lambda^1(A^*, (E_n)_n)$, i.e. $\{E_n\}_{n=0}^{\infty}$ is not an $\infty$-Schauder decomposition for $\lambda^1(A^*, (E_n)_n)$.

## 2. Application of Global Schauder Decompositions

PROPOSITION 2.1. *Let $E$ and $F$ be locally convex spaces. Let $\{E_n\}_{n=0}^{\infty}$ and $\{F_n\}_{n=0}^{\infty}$ be T.S.-complete global Schauder decompositions for $E$ and $F$ respectively. For each $n$ let*

$$T_n : E_n \longrightarrow F_n$$

*be an isomorphism satisfying the following two conditions:*

(A) *For every $q \in \mathrm{cs}(F)$ there exist $p \in \mathrm{cs}(E)$ and positive numbers $c$ and $t$ such that*

$$q(T_n(x_n)) \leq ct^n p(x_n) \tag{7}$$

*for every $x = \sum_{n=0}^{\infty} x_n$ in $E$ and every positive integer $n$.*

(B) *For every $p \in \mathrm{cs}(E)$ there exist $q \in \mathrm{cs}(F)$ and positive numbers $d$ and $v$ such that*

$$p(T_n^{-1}(y_n)) \leq dv^n q(y_n)$$

*for every $y = \sum_{n=0}^{\infty} y_n$ in $F$ and every positive integer $n$.*

*Then $T = \sum_{n=0}^{\infty} T_n$ is an isomorphism between $E$ and $F$.*

PROOF. Let $q \in \mathrm{cs}(F)$. By condition (A) there exist $p \in \mathrm{cs}(E)$ and positive numbers $c$ and $t$ such that

$$\sum_{n=0}^{\infty} q(T_n(x_n)) \leq c \sum_{n=0}^{\infty} t^n p(x_n) = cp_t(x) < \infty$$

for every $x = \sum_{n=0}^{\infty} x_n \in E$. Thus $T$ is well defined and continuous. Let $y = \sum_{n=0}^{\infty} y_n \in F$, we will prove that $T$ is surjective. Since $T_n$ is an isomorphism for every $n$, there exist $\{x_n\}_n$, $x_n \in E_n$, such that $T(x_n) = y_n$. Let $p \in \mathrm{cs}(E)$. By condition (B) there exist $q \in \mathrm{cs}(F)$ and positive numbers $d$ and $v$ such that

$$\sum_{n=0}^{\infty} p(x_n) = \sum_{n=0}^{\infty} p(T_n^{-1}(y_n)) \leq d \sum_{n=0}^{\infty} v^n q(y_n) = dq_v(y) < \infty.$$

Since $\{E_n\}_{n=0}^{\infty}$ is T.S. complete, $x = \sum_{n=0}^{\infty} x_n \in E$ and $T(x) = y$.

Define $S = \sum_{n=0}^{\infty} T_n^{-1}$. Since the hypotheses are symmetric with respect to $E$ and $F$, the above also proves that $S$ is well defined and continuous. It is easy to check that $S$ is the inverse of $T$.

The converse proposition holds in a more general situation.

PROPOSITION 2.2. *Let $E$ and $F$ be locally convex spaces, and let $\{E_n\}_{n=0}^{\infty}$ and $\{F_n\}_{n=0}^{\infty}$ be their respective Schauder decompositions. Let*

$$T : E \longrightarrow F$$

*be an isomorphism such that $T(E_m) \subseteq F_m$ for every positive integer $m$. Then $T_m := T|_{E_m} \longrightarrow F_m$ is an isomorphism for each $m$ and $T_m$ satisfies conditions (A) and (B) of Proposition 2.1.*

PROOF. Let $y_m \in F_m \subset F$. Since $T$ is surjective there exists $x = \sum_{n=0}^{\infty} x_n$ such that $T(x) = \sum_{n=0}^{\infty} T(x_n) = y_m$. By hypothesis $T(x_n) \in F_n$ for every $n$, hence $T_m(x_n) = 0$ for $m \neq n$ and $y_m = T(x_m) = T_m(x_m)$, i.e. $T_m$ is surjective. Since $T$ is injective $T_m$ is also injective. Thus $T_m$ is a bijective mapping. The continuity of $T_m$ and $T_m^{-1}$ follows from the continuity of $T$ and $T^{-1}$.

We now show that conditions (A) and (B) are satisfied. Let $q \in \mathrm{cs}(F)$. Since $T$ is continuous, there exist $p \in \mathrm{cs}(E)$ and $c > 0$ such that $q(T(x)) \leq cp(x)$ for every $x \in E$. In particular, for $x = x_m \in E_m$ we have

$$q_m(T_m(x_m)) \leq cp_m(x_m).$$

Hence inequality (7) is satisfied for $t = 1$. Condition (B) follows in a similar way from the continuity of $T^{-1}$.

## 3. Stability Properties of Global Schauder Decompositions

The following lemma is an adjustment of ([8], Lemma 3.31).

LEMMA 3.1. *Let $E$ be a barrelled locally convex space and $\{E_n\}_{n=0}^{\infty}$ be a Schauder decomposition of $E$ satisfying condition* (1)*, then $\{E_n\}_{n=0}^{\infty}$ is a global Schauder decomposition of $E$.*

PROOF. Let $p$ be a continuous semi-norm on $E$, and let $r > 0$. The set

$$\left\{x \in E : p_r(x) = \sum_{n=1}^{\infty} r^n p(x_n) \leq 1\right\} = \bigcap_{m=1}^{\infty} \left\{x = \sum_{n=1}^{\infty} x_n \in E : \sum_{i=1}^{m} r^i p(x_i) \leq 1\right\}$$

is a barrel, and consequently a neighbourhood of zero in $E$. Thus $p_r$ is continuous for every $r > 0$.

LEMMA 3.2. *Let $E$ be a sequentially complete locally convex space and $\{E_n\}_{n=0}^{\infty}$ be a Schauder decomposition of $E$ satisfying condition* (2)*, then $\{E_n\}_{n=0}^{\infty}$ is a global Schauder decomposition of $E$.*

PROOF. Let $x = \sum_{n=1}^{\infty} x_n \in E$ and $r > 0$, denote $s_n := \sum_{i=1}^{n} r^i x_i$. If $p \in \mathrm{cs}(E)$ then

$$p(s_n - s_m) = \sum_{i=m}^{n} r^i p(x_i) = p_r\left(\sum_{i=m}^{n} x_i\right) \longrightarrow 0$$

as $m, n \to \infty$. Thus $(s_n)_n$ is a Cauchy sequence, hence $\sum_{n=1}^{\infty} r^i x_i \in E$.

PROPOSITION 3.3. *Let $\{E_n\}_n$ denote a global Schauder decomposition for the locally convex space $E$. Then $\{\overline{E}_n\}_n$ is a global Schauder decomposition for the completion $\overline{E}$.*

PROOF. Let $x \in \overline{E}$, then there exists a net $(x_\beta)_\beta \subset E$ such that $x = \lim_{\beta\to\infty} x_\beta$. Since $x_\beta \in E$ there exist $(x_{\beta,n})_n$ such that $x_\beta = \sum_{n=0}^{\infty} x_{\beta,n}$ for every $\beta$. The nets $(x_{\beta,n})_\beta$ are Cauchy for every $n$, hence there exists $x_n \in \overline{E}_n$ for every $n$ such that $x_n := \lim_{\beta\to\infty} x_{\beta,n}$. Let $p \in \mathrm{cs}(E)$ be from the generating family of continuous semi-norms satisfying (3). Given $\varepsilon > 0$ we can find $\beta_0 > 0$ such that

$$\sum_{n=0}^{\infty} p(x_{\beta,n} - x_{\beta',n}) < \varepsilon$$

for all $\beta$, $\beta' > \beta_0$. By passing to the limit in $\beta'$ and extending $p$ by continuity to the completion we get

$$\sum_{n=0}^{\infty} p(x_{\beta,n} - x_n) \leq \varepsilon$$

when $\beta > \beta_0$. This implies that the series $\sum_{n=0}^{\infty} x_n$ is convergent and $x_\beta \to \sum_{n=0}^{\infty} x_n$. Since $(x_\beta)_\beta$ has a unique limit, $x = \sum_{n=0}^{\infty} x_n$. The projections $(u_n)_n$ defined by

$$u_m\left(\sum_{n=0}^{\infty} y_n\right) := \sum_{n=1}^{m} y_n$$

where $\sum_{n=0}^{\infty} y_n \in E$, are linear and continuous and hence can be extended by uniform continuity to the completion $\overline{E}$. Hence $\{\overline{E}_n\}_n$ is a Schauder decomposition for the completion $\overline{E}$.

Let $r > 0$ and let $\hat{p} \in \mathrm{cs}(\overline{E})$. Since $\{E_n\}_n$ is a global Schauder decomposition for $E$, the mapping

$$\hat{p}_r\left(\sum_{n=0}^{\infty} y_n\right) = \sum_{n=0}^{\infty} r^n \hat{p}(y_n),$$

where $y = \sum_{n=0}^{\infty} y_n \in E$, defines a continuous semi-norm on $E$. Taking $r = 1$ we get

$$\sum_{n=0}^{\infty} \hat{p}(x_n) \leq \lim_{\beta\to\infty} \sum_{n=0}^{\infty} \hat{p}(x_n - x_{\beta,n}) + \lim_{\beta\to\infty} \sum_{n=0}^{\infty} \hat{p}(x_{\beta,n})$$
$$= \lim_{\beta\to\infty} \sum_{n=0}^{\infty} \hat{p}(x_{\beta,n}) = \lim_{\beta\to\infty} \hat{p}_1(x_\beta),$$

hence $\sum_{n=0}^{\infty} \hat{p}(x_n)$ defines a continuous semi-norm on $\overline{E}$. For an arbitrary $r > 0$ we have

$$\sum_{n=0}^{\infty} r^n \hat{p}(x_n) = \sum_{n=0}^{\infty} \hat{p}(r^n x_n) \leq \lim_{\beta\to\infty} \sum_{n=0}^{\infty} \hat{p}(r^n x_{\beta,n}) = \lim_{\beta\to\infty} \hat{p}_r(x_\beta).$$

Since $\hat{p}_r$ is a continuous semi-norm on $E$, the limit exists and is finite. Thus $\sum_{n=0}^{\infty} r^n \hat{p}(x_n)$ defines a continuous semi-norm on $\overline{E}$. An application of Lemma 3.2 completes the proof.

PROPOSITION 3.4. *If $\{E_n\}_n$ is a global Schauder decomposition for the locally convex space $E$ then $\{(E_n)_i{}'\}_n$ is a global Schauder decomposition for the inductive dual of $E$, $E_i{}'$.*

PROOF. By ([14], 10.3) $\overline{E}_i' = E_i'$, and by Proposition 3.3 $\{\overline{E}_n\}_n$ is a global Schauder decomposition for $\overline{E}$. Hence we can assume that $E$ and all $E_n$ are complete.

Let $\varphi \in E'$, we denote $\varphi|_{E_n}$ by $\varphi_n$. By Remark 1.3 there exists a continuous semi-norm $p$ such that $|\varphi(x)| \leq p(x)$ for any $x \in E$, with $p(x_n) \to 0$ as $n \to \infty$, and $p(x) = \sup_n p(x_n)$ for any $x = \sum_{n=1}^{\infty} x_n \in E$. Hence $\varphi \in (E, p)'$ and can be extended to $E_p := \overline{(E, p)/p^{-1}(0)}$. We will denote the extension of $p$ to $E_p$ again by $p$, and let $(E_p)_n := \overline{(E_n, p|_{E_n})/p|_{E_n}{}^{-1}(0)}$. Then

$$E_p = \left\{ \sum_{n=1}^{\infty} x_n : x_n \in (E_p)_n,\ p|_{E_n}(x_n) \to 0 \text{ as } n \to \infty \right\},$$

and $p\big(\sum_{n=1}^{\infty} x_n\big) = \sup_n p|_{E_n}(x_n)$. Let $\bar{\varphi}_n$ denote the extension of $\varphi_n$ in $((E_p)_n)'$. Since $\{E_n\}_n$ is a Schauder decomposition of $E$, $\varphi = \sum_{n=1}^{\infty} \bar{\varphi}_n$ pointwise on $E$. Let $p'$ be the dual semi-norm of $p$ on $E'$ and let $B_p$ be the unit ball of $E_p$. Then

$$p'\left(\varphi - \sum_{n=1}^{m} \bar{\varphi}_n\right) = \sup_{x\in B_p} \left|\varphi(x) - \sum_{n=1}^{m} \bar{\varphi}_n(x)\right| = \sup_{x\in B_p} \left|\sum_{n=m+1}^{\infty} \bar{\varphi}_n(x)\right|.$$

Let $x \in E$ and $\{\lambda_n\}_n \subset \mathbf{C}$, $|\lambda_n| \leq 1$ for all $n \in \mathbf{N}$. Since $\{E_n\}_n$ is an absolute decomposition and $E$ is complete, $\sum_{n=1}^{\infty} \lambda_n x_n \in E$ (see p. 189 of [8]). This allows us to choose $\{\lambda_n\}_n$ so that $\lambda_n \bar{\varphi}_n(x) = |\bar{\varphi}_n(x)|$ for all $n$. Since $\sup_n |\lambda_n| p(x_n) \leq 1$ for all $x \in B_p$, it follows that $\lambda \cdot x \in B_p$. Hence

$$p'\left(\varphi - \sum_{n=1}^{m} \bar{\varphi}_n\right) = \sup_{x \in B_p} \sum_{n=m+1}^{\infty} \left|\bar{\varphi}_n(x)\right|.$$

Suppose $\sup_{x \in B_p} \sum_{n=m+1}^{\infty} \left|\bar{\varphi}_n(x)\right|$ does not tend to zero as $m \to \infty$. Then there exists $\delta > 0$ such that for all $m \in \mathbf{N}$ we can find $x^{(m)} \in B_p$ with

$$\sum_{n=m+1}^{\infty} \left|\bar{\varphi}_n(x^{(m)})\right| \geq \delta.$$

Let $m = 1$ and $x^{(1)}$ be the corresponding element of $B_p$. There exists $m_1 > 1$ such that

$$\sum_{n=1}^{m_1} \left|\bar{\varphi}_n(x_n^{(1)})\right| \geq \frac{\delta}{2}.$$

By induction we can build an increasing sequence $\{m_j\}_{j \in \mathbf{N}} \subset \mathbf{N}$ and a sequence $\{x^{(j)}\} \subset B_p$ such that

$$\sum_{n=m_j+1}^{m_{j+1}} \left|\bar{\varphi}_n(x_n^{(j+1)})\right| \geq \frac{\delta}{2}$$

for all $j$. Let

$$y_n = \begin{cases} 0 & n \leq m_j, \\ \frac{1}{n} x_n^{(j+1)} & m_j + 1 \leq n \leq m_{j+1}, \\ 0 & n > m_{j+1}. \end{cases}$$

Since $p(x^{(j)}) \leq 1$ and $p(x) = \sup_k p(x_k)$, we have that $p(y_n) \leq 1/n$, hence $p(y_n) \to 0$ as $n \to \infty$ and $(y_n)_n \subset B_p$. This implies that $\sum_{n=1}^{\infty} y_n \in E_p$. As before we can choose $\{\lambda_n\}_n \subset \mathbf{C}$, $|\lambda_n| \leq 1$ for all $n \in \mathbf{N}$, so that $\varphi(\sum_{n=1}^{\infty} \lambda_n y_n) = \sum_{n=1}^{\infty} |\varphi(y_n)|$. However

$$\sum_{n=1}^{\infty} |\varphi(y_n)| = \sum_{n=1}^{\infty} \frac{1}{n} \left|\bar{\varphi}_n(x_n^{(j+1)})\right| \geq \frac{\delta}{2} \sum_{n=1}^{\infty} \frac{1}{n},$$

i.e. $\sum_{n=1}^{\infty} |\varphi(y_n)|$ is divergent, a contradiction. Hence $p'\left(\varphi - \sum_{n=1}^{m} \bar{\varphi}_n\right) \to 0$ and $\varphi = \sum_{n=1}^{\infty} \varphi_n \in E'_p$. Since, by definition, $E_i{}' = \operatorname*{ind}_{p \in \mathrm{cs}(E)} \left(\overline{(E, p)/p^{-1}(0)}\right)'$, the mapping $E'_p \longrightarrow E_i{}'$ is continuous, and $\varphi = \sum_{n=1}^{\infty} \varphi_n$ in $E'_i$. Moreover, by ([14], Proposition 10.3.4) the canonical surjection $E_i{}' \longrightarrow (E_n)'_i$ is open and continuous, hence $E'_i$ induces the inductive topology on $(E_n)'$. Thus $\{(E_n)_i{}'\}_n$ is a Schauder decomposition for $E_i{}'$. The above also shows that $\varphi_n(x) = \varphi_n(x_n) = \varphi(x_n)$.

Next we show that $\{(E_n)_i{}'\}_n$ is a global Schauder decomposition for $E_i{}'$. Let $\varphi \in E'_i$, $\varphi = \sum_{n=1}^{\infty} \varphi_n$, and let $r > 0$. If $x \in E$ then $r \cdot x \in E$ and

$$(r \cdot \varphi)(x) := \sum_{n=1}^{\infty} r^n \varphi_n(x_n) = \varphi\left(\sum_{n=1}^{\infty} r^n x_n\right) = \varphi(r \cdot x)$$

is well defined. Since $\varphi$ is continuous there exists a continuous semi-norm $p$ on $E$ such that $|\varphi(x)| \leq p(x)$ for any $x \in E$. Then

$$|(r \cdot \varphi)(x)| \leq \sum_{n=1}^{\infty} r^n \varphi_n(x_n) \leq p_r(x).$$

Since $p_r$ is a continuous semi-norm on $E$, this implies that $r \cdot \varphi \in E'$. An application of Lemma 3.1 completes the proof.

A proof for the following proposition can be obtained by modifying the proof of Proposition 3.4.

PROPOSITION 3.5. *If $\{E_n\}_n$ is an $\mathscr{S}$-absolute decomposition for the locally convex space $E$ then $\{(E_n)'_i\}_n$ is an $\mathscr{S}$-absolute decomposition for $E_i{}'$.*

Next we look at the strong dual of a locally convex space.

PROPOSITION 3.6. *If $\{E_n\}_n$ is a global Schauder decomposition for the locally convex space $E$ then $\{(E_n)'_\beta\}_n$ is a global Schauder decomposition for $E'_\beta$.*

PROOF. Let $\varphi = \sum_{n=0}^{\infty} \varphi_n \in E'$ where $\varphi_n := \varphi|_{E_n}$. By the continuity of $\varphi$ there exists $p \in \mathrm{cs}(E)$ such that $|\varphi(x)| \leq p(x)$ for all $x = \sum_{n=0}^{\infty} x_n \in E$. Let $r > 0$, then

$$\sum_{n=1}^{\infty} r^n \varphi_n(x_n) = \sum_{n=1}^{\infty} \varphi_n(r^n x_n) \leq p_r\left(\sum_{n=1}^{\infty} x_n\right). \tag{8}$$

Let $\left(\sum_{n=1}^{\infty} r^n \varphi_n\right)(x) := \lim_{n\to\infty} \sum_{i=1}^{n} r^i \varphi_i(x_i)$. By (8), $\sum_{n=1}^{\infty} r^n \varphi_n \in E'$. The topology on $E'_\beta$ is generated by all semi-norms of the form

$$s(\varphi) := \sup\{|\varphi(x)| : x \in A\},$$

for all $A$ bounded subsets of $E$. Let $B$ be a bounded set in $E$, $r \cdot B := \left\{\sum_{n=1}^{\infty} r^n x_n : x \in B\right\}$ and $p \in \mathrm{cs}(E)$. Then since $p_r \in \mathrm{cs}(E)$,

$$\sup_{x \in r \cdot B} p(x) = \sup_{x \in B} p\left(\sum_{n=0}^{\infty} r^n x_n\right) = \sup_{x \in B} p_r(x) < \infty.$$

Hence the set $r \cdot B$ is bounded in $E$. Therefore

$$\left\|\sum_{n=m}^{\infty} r^n \varphi_n\right\|_B = \sup_{x \in B}\left|\sum_{n=m}^{\infty} \varphi_n(r^n x_n)\right| = \sup_{x \in r \cdot B}\left|\sum_{n=m}^{\infty} \varphi_n\right| \to 0$$

as $m \to \infty$. Hence $\{(E_n)_\beta{}'\}_n$ is a Schauder decomposition for $E'_\beta$ satisfying (1). It remains to show that condition (2) is satisfied. Let $B$ be a bounded set in $E$ and let

$$\tilde{B} := \left\{\sum_{n=1}^{\infty} \lambda_n x_n : x \in B,\, (\lambda_n)_n \subset \mathbb{C} \text{ such that } |\lambda_n| \leq 1 \text{ for all } n \in \mathbb{N}\right\}.$$

The set $\tilde{B}$ is bounded in $E$. Indeed, let $p \in \mathrm{cs}(E)$ satisfying (3). Then

$$\sup_{x \in \tilde{B}} p(x) = \sup_{x \in B} \sum_{n=1}^{\infty} p(\lambda_n x_n) = \sup_{x \in B} \sum_{n=1}^{\infty} p(x_n) < \infty.$$

This allows us to choose $\{\lambda_n\}_n$ so that $\lambda_n \varphi_n(x) = |\varphi_n(x)|$ for all $n$. Then

$$\sup_{x \in 2r\tilde{B}} |\varphi(x)| = \sup_{x \in \tilde{B}}\left|\sum_{n=1}^{\infty}(2r)^n \varphi_n(x_n)\right| = \sup_{x \in B} \sum_{n=1}^{\infty}(2r)^n |\varphi_n(x_n)| \geq (2r)^n \|\varphi_n\|_B$$

for all $n$. Let $q(\varphi) = \sup_{x \in B}|\varphi(x)|$, then

$$q_r(\varphi) \leq \sum_{n=1}^{\infty} r^n \sup_{x \in B} |\varphi_n(x_n)| \leq \sum_{n=1}^{\infty} \frac{1}{2^n} \sup_{x \in 2r\tilde{B}} |\varphi(x)| = \|\varphi(x)\|_{2r\tilde{B}}$$

is continuous on $E'_\beta$.

## 4. Global Schauder Decompositions of Spaces of Holomorphic Functions

In this section $\mathscr{H}(E)$ denotes the space of entire functions on a locally convex space $E$.

PROPOSITION 4.1. *Let $E$ be a locally convex space. Then*

(1) $\{(\mathcal{P}(^nE), \tau_0)\}_{n=0}^{\infty}$ *is a global Schauder decomposition for* $(\mathcal{H}(E), \tau_0)$.

(2) $\{(\mathcal{P}(^nE), \tau_w)\}_{n=0}^{\infty}$ *is a global Schauder decomposition for* $(\mathcal{H}(E), \tau_\delta)$ *and* $(\mathcal{H}(E), \tau_w)$.

PROOF. By ([8], Proposition 3.36), $\{(\mathcal{P}(^nE), \tau_0)\}_{n=0}^{\infty}$ and $\{(\mathcal{P}(^nE), \tau_w)\}_{n=0}^{\infty}$ are $\mathcal{S}$-absolute Schauder decompositions for $(\mathcal{H}(E), \tau_0)$ and $(\mathcal{H}(E), \tau_\delta)$ respectively.

Let $f = \sum_{n=0}^{\infty} \frac{\widehat{d^n} f(0)}{n!} \in \mathcal{H}(E)$ and $r > 0$. If $K \subset E$ is a compact balanced set, by the local boundedness of $f$ there exists a balanced open $V \subset E$ such that $K \subset V$ and $\sum_{n=0}^{\infty} \left\| \frac{\widehat{d^n} f(0)}{n!} \right\|_V < \infty$. Then

$$\|r \cdot f\|_{1/rV} \leq \sum_{n=0}^{\infty} r^n \left\| \frac{\widehat{d^n} f(0)}{n!} \right\|_{1/rV} = \sum_{n=0}^{\infty} \left\| \frac{\widehat{d^n} f(0)}{n!} \right\|_V \leq \|f\|_V. \tag{9}$$

Hence $r \cdot f \in \mathcal{H}(E)$.

Since $(\mathcal{H}(E), \tau_\delta)$ is barrelled, by Lemma 3.1 $\{(\mathcal{P}(^nE), \tau_w)\}_{n=0}^{\infty}$ is a global Schauder decomposition for $(\mathcal{H}(E), \tau_\delta)$. By replacing $V$ by $K$ in (9) we get $\|r \cdot f\|_K \leq \|f\|_{rK}$ for all $f \in \mathcal{H}(E)$. Hence $\{(\mathcal{P}(^nE), \tau_0)\}_{n=0}^{\infty}$ is a global Schauder decomposition for $(\mathcal{H}(E), \tau_0)$. The proof that $\{(\mathcal{P}(^nE), \tau_w)\}_{n=0}^{\infty}$ is a global Schauder decomposition for $(\mathcal{H}(E), \tau_w)$ is similar (see also Proposition 3.36 of [8]).

Propositions 4.1 and 2.1 imply

COROLLARY 4.2. *Let* $E$ *be a locally convex space. Then* $\tau_\delta = \tau_w$ *on* $\mathcal{H}(E)$ *if and only if for every* $\tau_\delta$*-continuous semi-norm* $q$ *there exist a* $\tau_w$*-continuous semi-norm* $p$ *and positive numbers* $c$ *and* $t$ *such that*

$$q\left(\frac{\widehat{d^n} f(0)}{n!}\right) \leq ct^n p\left(\frac{\widehat{d^n} f(0)}{n!}\right) \tag{10}$$

*for every* $f = \sum_{n=0}^{\infty} \frac{\widehat{d^n} f(0)}{n!} \in \mathcal{H}(E)$ *and every positive integer* $n$.

Let $E$ be a locally convex space, denote

$$\mathcal{H}_b(E) = \{f \in H(E) : \|f\|_A < \infty \text{ for every bounded set } A\}.$$

The functions in $\mathcal{H}_b(E)$ are called *holomorphic functions of bounded type*. When endowed with $\tau_b$, the topology of uniform convergence over the bounded sets of $E$, $\mathcal{H}_b(E)$ becomes a locally convex space. The proof of Proposition 4.1 can easily be modified to show the following:

PROPOSITION 4.3. *Let* $E$ *be a locally convex space. Then* $\{(\mathcal{P}(^nE), \tau_b)\}_{n=0}^{\infty}$ *is a global Schauder decomposition for* $(\mathcal{H}_b(E), \tau_b)$.

In ([12], Examples 2 and 4) are given several examples of spaces of entire functions on a Banach space such that the corresponding polynomial subspaces are their $\infty$-Schauder decompositions – and, thus, their global Schauder decompositions. All our results will apply to these spaces, and in particular Proposition 2.1 reduces in this special case to ([12], Theorem 9(ii)).

Let $E$ be a Banach space with a unit ball $B_E$. In ([6]) the authors defined the space $\mathcal{H}_{\mathrm{bI}}(E)$ of entire functions whose restrictions to $nB_E$ are integral for all $n$. Endowed with the system of semi-norms $\{p_n(f) = \|f|_{nB_E}\|_I\}_{n=1}^{\infty}$, $\mathcal{H}_{\mathrm{bI}}(E)$ is a Fréchet space and $\{(\mathcal{P}_I({}^nE), \|\cdot\|_I)\}_{n=0}^{\infty}$ is an $\infty$-Schauder (and hence global) decomposition for $\mathcal{H}_{\mathrm{bI}}(E)$. Now consider the entire functions of bounded nuclear type on $E$, $H_{\mathrm{Nb}}(E)$ ([8], Definition 4.47). With the topology generated by the semi-norms $\{\pi_n(f) = \|f|_{nB_E}\|_N\}_{n=1}^{\infty}$, $\mathcal{H}_{\mathrm{Nb}}(E)$ is a Fréchet space and a short calculation shows that $\{(\mathcal{P}_N({}^nE), \|\cdot\|_N)\}_{n=0}^{\infty}$ is an $\infty$-Schauder decomposition for $\mathcal{H}_{\mathrm{Nb}}(E)$. By ([5], Theorem 2) if $\ell_1 \not\hookrightarrow \widehat{\bigotimes}_{n,s,\varepsilon} E$ for some integer $n$ then $\mathcal{P}_N({}^nE)$ and $\mathcal{P}_I({}^nE)$ are isometrically isomorphic. By ([12], Corollary 11) we obtain

PROPOSITION 4.4. *Let $E$ be a Banach space such that $\widehat{\bigotimes}_{n,s,\varepsilon} E$ does not contain a copy of $\ell_1$ for any $n \in \mathbb{N}$. Then $\mathcal{H}_{\mathrm{bI}}(E)$ and $\mathcal{H}_{\mathrm{Nb}}(E)$ are isomorphic.*

Furthermore, by ([5], Proposition 3) we can replace "$\widehat{\bigotimes}_{n,s,\varepsilon} E$ does not contain a copy of $\ell_1$ for any $n \in \mathbb{N}$" with the condition that $E'$ has RNP.

Now let $E$ be a locally convex space, let

$$(11)\quad G_b(E) := \{\varphi \in \mathcal{H}_b(E)^* : \varphi \text{ is } \tau_0\text{-continuous on the bounded subsets of } \mathcal{H}_b(E)\}.$$

When endowed with the topology $\tau_g$ of uniform convergence on the bounded subsets of $\mathcal{H}_b(E)$, $G_b(E)$ becomes a complete locally convex space. Let $E$ be a locally convex space such that the $\tau_b$-bounded sets of $\mathcal{H}_b(E)$ are locally bounded. By ([8], Lemma 3.25) if $B$ is a locally bounded $\tau_b$-bounded subset of $\mathcal{H}_b(E)$, then it is relatively compact in $(\mathcal{H}_b(E), \tau_0)$. This allows us to apply ([15], Theorem 1.1) (see also p. 115 of [2]), and we obtain that $G_b(E)'_i = (\mathcal{H}_b(E), \tau_b^{\mathrm{bor}})$. We have proved the following proposition.

PROPOSITION 4.5. *Let $E$ be a locally convex space such that the $\tau_b$-bounded sets of $\mathcal{H}_b(E)$ are locally bounded. Then*

$$G_b(E)'_i = (\mathcal{H}_b(E), \tau_b^{\mathrm{bor}}).$$

If $E$ is a bornological DF space then the $\tau_b$-bounded sets of $\mathcal{H}_b(E)$ are locally bounded by ([10], Proposition 15). By ([10], Theorem 4), $(\mathcal{H}_b(E), \tau_b)$

is Fréchet, and hence ultrabornological, which implies $\tau_b = \tau_b^{\text{bor}}$. Thus if $E$ is a bornological DF space, $G_b(E)'_i = (\mathscr{H}_b(E), \tau_b)$.

PROPOSITION 4.6. *Let $E$ be a locally convex space. Then the sequence $\{(\mathscr{P}(^nE), \tau_b^{\text{bor}})' \bigcap G_b(E)\}_{n=0}^{\infty}$ is a global Schauder decomposition for $G_b(E)$.*

PROOF. The space $G_b(E)$ is a subspace of $(\mathscr{H}_b(E), \tau_b^{\text{bor}})'$ since its elements are $\tau_b$-continuous on the bounded sets of $\mathscr{H}_b(E)$ and hence are $\tau_b^{\text{bor}}$-continuous. Let $(f_\beta)_\beta$ be a bounded net in $H_b(E)$ which tends to 0 uniformly on every compact subset $K$ of $E$, and let $r > 0$. Since $\{(\mathscr{P}(^nE), \tau_b)\}_{n=0}^{\infty}$ is a global Schauder decomposition for $(\mathscr{H}_b(E), \tau_b)$ and $rK$ is also a compact set, we have

$$\sum_{n=0}^{\infty} r^n \left\| \frac{\widehat{d^n} f_\beta(0)}{n!} \right\|_K = \sum_{n=0}^{\infty} \left\| \frac{\widehat{d^n} f_\beta(0)}{n!} \right\|_{rK} \longrightarrow 0$$

as $\beta \to \infty$. Hence $\left\{\sum_{n=0}^{\infty} r^n \frac{\widehat{d^n} f_\beta(0)}{n!}\right\}_\beta$ is also a bounded $\tau_0$-null net in $\mathscr{H}_b(E)$. Let $\vartheta = \sum_{n=0}^{\infty} \vartheta_n \in G_b(E)$ where $\vartheta_n := \vartheta|_{\mathscr{P}(^nE)}$. Then

$$\left(\sum_{n=0}^{\infty} r^n \vartheta_n\right)(f_\beta) = \sum_{n=0}^{\infty} \vartheta_n \left(\sum_{n=0}^{\infty} r^n \frac{\widehat{d^n} f_\beta(0)}{n!}\right) \longrightarrow 0 \tag{12}$$

as $\beta \to \infty$. This implies $\sum_{n=0}^{\infty} r^n \vartheta_n \in G_b(E)$ for every $r > 0$.

Now let $p$ be a $\tau_g$-continuous semi-norm. Without loss of generality we may assume that

$$p(\vartheta) = \sup_{f \in B} |\vartheta(f)|,$$

where $B$ is a bounded subset of $\mathscr{H}_b(E)$. Let

$$B_n := \left\{ \frac{\widehat{d^n} f(0)}{n!} : f \in B \right\},$$

then

$$\sum_{n=0}^{\infty} r^n p_n(\vartheta_n) = \sum_{n=0}^{\infty} r^n \left\| \vartheta_n \left( \frac{\widehat{d^n} f(0)}{n!} \right) \right\|_{B_n} = \sum_{n=0}^{\infty} \left\| \vartheta_n \left( \frac{\widehat{d^n} f(0)}{n!} \right) \right\|_{rB_n}$$
$$= \sup_{f \in rB} |\vartheta(f)|.$$

Since $rB$ is also a bounded subset of $\mathscr{H}_b(E)$, the semi-norm $\sum_{n=0}^{\infty} r^n p_n$ is continuous. This completes the proof.

By ([3], Proposition 1) the space of all linear functionals on $\mathcal{P}(^n E)$ which are $\tau_0$-continuous on its locally bounded subsets is isomorphic to $\widehat{\bigotimes}_{n,s,\pi} E$. This and Proposition 4.6 give us

COROLLARY 4.7. *Let $E$ be a locally convex space such that for every positive integer $n$ the $\tau_b$-bounded sets of $\mathcal{P}(^n E)$ are locally bounded. Then $\{\widehat{\bigotimes}_{n,s,\pi} E\}_{n=0}^{\infty}$ is a global Schauder decomposition for $G_b(E)$.*

The condition in Corollary 4.7 is satisfied for example by all Fréchet spaces and all bornological DF spaces.

The following definition is given in [4].

DEFINITION 4.8. The locally convex space $E$ is *Q-reflexive* if for every positive integer $n$:

(1) The mapping

$$J_n : \bigotimes_{n,s,\pi} E''_e \longrightarrow (\mathcal{P}(^n E), \tau_b)'_i$$

is continuous.

(2) The extension of $J_n$ to the completion of $\bigotimes_{n,s,\pi} E''_e$ is an isomorphism between $\widehat{\bigotimes}_{n,s,\pi} E''_e$ and $\overline{(\mathcal{P}(^n E), \tau_b)'_i}$.

From Propositions 2.1 and 2.2, combined with Propositions 3.4 and 4.3 and Corollary 4.7, we obtain

PROPOSITION 4.9. *Let $E$ be a locally convex space such that the $\tau_b$-bounded sets of $\mathcal{H}(E''_{\beta\beta})$ are locally bounded. If $E$ is Q-reflexive and $(J_n)_n$ satisfy conditions (A) and (B) from Proposition 2.1, then $J := \sum_{n=0}^{\infty} J_n$ is an isomorphism between $G_b(E''_{\beta\beta})$ and $\overline{(\mathcal{H}_b(E), \tau_b)'_i}$.*

We will need the following lemma.

LEMMA 4.10. *Let $E := \prod_{k=1}^{\infty} F$ for some Banach space $F$, $E_m := \underbrace{F \times \cdots \times F}_{m}$ and $E^m := \prod_{j=m+1}^{\infty} F$. Let $B$ be a $\tau_b$-bounded set in $\mathcal{H}_b(E)$. There exists a positive integer $n_0$ such that $f(x+y) = f(x)$ for all $f \in B$, $x \in E_{n_0}$ and $y \in E^{n_0}$.*

PROOF. Suppose our hypothesis is not true. Then for every positive integer $n$ there exist $f_n \in B$, $x_n \in E_n$ and $y_n \in E^n$ such that $f_n(x_n + y_n) - f_n(x_n) \neq 0$. Let $\lambda \in \mathbf{C}$, then

$$g_n : \lambda \longrightarrow f_n(\lambda x_n + y_n) - f_n(\lambda x_n)$$

is a non-zero entire function. For every $n$ there exists $\lambda_n$ such that $\|\lambda_n x_n\| \leq 1/n$ and $g_n(\lambda_n) \neq 0$. Indeed, otherwise there exists a neighbourhood of zero in $\mathbb{C}$ such that $g_n$ is zero on it, and the Identity Principle implies that $g_n$ is identically zero on $\mathbb{C}$. Consider

$$h_n : \mu \longrightarrow f_n(\lambda_n x_n + \mu y_n) - f_n(\lambda_n x_n).$$

The function $h_n(\mu)$ is a non-constant entire function on $\mathbb{C}$, so by Liouville's Theorem is unbounded. Hence there exists $(\mu_n)_n$ in $\mathbb{C}$ such that

$$|h_n(\mu_n)| = |f_n(\lambda_n x_n + \mu_n y_n) - f_n(\lambda_n x_n)| > n + |f_n(\lambda_n x_n)|$$

for every $n$. Then

$$|f_n(\lambda_n x_n + \mu_n y_n)| \geq |f_n(\lambda_n x_n + \mu_n y_n) - f_n(\lambda_n x_n)| - |f_n(\lambda_n x_n)| > n$$

for every $n$. Since $(\mu_n y_n)_n$ tends to zero and $\|\lambda_n x_n\| \leq 1/n$, the sequence $(\lambda_n x_n + \mu_n y_n)_n$ is bounded. Hence the sequence $(f_n)_n \subset B$ is not bounded on bounded sets in contradiction with the $\tau_b$-boundedness of $B$.

Several examples of locally convex Q-reflexive spaces are given in [4]. We will pay special attention to one of them, $\prod_{k=1}^{\infty} T_J{}^*$, where $T_J{}^*$ denotes the Tsirelson-James space.

EXAMPLE 4.11. Let $E := \prod_{k=1}^{\infty} T_J{}^*$. We will show that the spaces $\overline{(\mathscr{H}_b(E), \tau_b)'_\beta}$ and $G_b(E''_{\beta\beta})$ are isomorphic.

The space $E := \prod_{k=1}^{\infty} T_J{}^*$ is a Fréchet space (moreover, a quojection), hence the $\tau_b$-bounded sets of $\mathscr{P}({}^n E''_{\beta\beta})$ are locally bounded. By ([4], Example 3) $E$ is Q-reflexive. According to Proposition 4.9 it suffices to show that conditions (A) and (B) of Proposition 2.1 hold.

Let $B$ be a bounded subset of $(H_b(E), \tau_b)$. By ([11], Theorem 1.5), for every $f \in \mathscr{H}_b(E)$ there exists a function $AB(f)$ in $\mathscr{H}_b(E''_{\beta\beta})$ such that $AB(f)|_E = f$. Let

$$B'' := \{AB(f) : f \in B\},$$

and let $A''$ be a bounded subset of $E''_{\beta\beta}$. Since $E$ is a distinguished Fréchet space there exists a bounded subset of $E$, $A$, such that $A'' \subset A^{\circ\circ}$. Using ([11], Theorem 1.5) we get

$$\sup_{\tilde{f} \in B''} \|\tilde{f}\|_{A''} = \sup_{f \in B} \|AB(f)\|_{A''} \leq \sup_{f \in B} \|AB(f)\|_{A^{\circ\circ}} = \sup_{f \in B} \|f\|_A < \infty.$$

Consequently the set $B''$ is $\tau_b$-bounded in $\mathscr{H}_b(E''_{\beta\beta})$ and $\sup\{|\varphi(f)| : f \in B''\}$ is a continuous semi-norm on $G_b(E''_{\beta\beta})$. Let $\vartheta = \sum_{n=0}^{\infty} \vartheta_n \in G_b(E''_{\beta\beta})$. Since

$E''_{\beta\beta}$ is Fréchet, every $\vartheta_n$ has a representation $\vartheta_n = \sum_{i=1}^{\infty} \lambda_n^i \otimes_n x_n^i$ for some null sequence $(x_n^i)_i \subset E''_{\beta\beta}$ and some $(\lambda_n^i)_i \in \ell_1$. Then

$$\sup_{f\in B}\left|[J_n(\vartheta_n)]\left(\frac{\widehat{d^n f(0)}}{n!}\right)\right| = \sup_{f\in B}\left|\sum_{i=1}^{\infty}\lambda_n^i\left[AB_n\left(\frac{\widehat{d^n f(0)}}{n!}\right)\right](x_n^i)\right|$$
$$= \sup_{\tilde f\in B''}\left|\vartheta_n\left(\frac{\widehat{d^n \tilde f(0)}}{n!}\right)\right|.$$

Hence condition (A) of Proposition 2.1 is satisfied.

Let $B''$ be a bounded subset of $(\mathcal{H}_b(E''_{\beta\beta}), \tau_b)$. By Lemma 4.10 there exists $n_0 \in \mathbf{N}$ such that $f(x+y) = f(x)$ for all $f \in B''$, $x \in E''_{n_0}$ and $y \in (E^{n_0})'' := \prod_{j=n_0+1}^{\infty}(T_J{}^*)''$. The space $E_{n_0} = \underbrace{T_J{}^* \times \cdots \times T_J{}^*}_{n_0}$ is a Q-reflexive Banach space, and since $E''_{n_0}$ has the RNP, $E_{n_0}$ is isometrically Q-reflexive ([8], Proposition 2.48). By ([9], Proposition 2) $(\mathcal{H}_b(E_{n_0}), \tau_b)'' = \mathcal{H}_b(E''_{n_0})$, and hence the set $J^*(B'')$ is contained and bounded in $(\mathcal{H}_b(E_{n_0}), \tau_b)''$. Since the spaces $\{\mathcal{P}({}^n E_{n_0})\}_n$ are Banach, by ([1], Proposition 8) $(H_b(E_{n_0}), \tau_b)$ is a quasinormable and consequently distinguished Fréchet space. Hence there exists a $\tau_b$-bounded set $B$ in $\mathcal{H}_b(E_{n_0})$ such that $B'' \subset B^{\circ\circ}$. Since $E_{n_0}$ is isometrically Q-reflexive $\|J_n^{-1}\| = 1$ for every $n$. Thus

$$\sup_{f\in B''}\left|[J_n^{-1}(\varphi_n)]\left(\frac{\widehat{d^n f(0)}}{n!}\right)\right| \le \sup_{f\in B^{\circ\circ}}\left|[J_n^{-1}(\varphi_n)]\left(\frac{\widehat{d^n f(0)}}{n!}\right)\right|$$
$$= \sup_{f\in B}\left|\varphi_n\left(\frac{\widehat{d^n f(0)}}{n!}\right)\right|.$$

The set $B$ is $\tau_b$-bounded in $\mathcal{H}_b(E_{n_0})$ and hence in $\mathcal{H}_b(E)$. Thus (B) of Proposition 2.1 is satisfied.

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