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IRRATIONALITY MEASURES FOR CERTAIN q-MATHEMATICAL CONSTANTS

PETER BUNDSCHUH and WADIM ZUDILIN

(To our friend Keijo Väänänen on the occasion of his sixtieth birthday)

Abstract

We prove sharp irrationality measures for a q-analogue of π and related q-series, and indicate open problems on linear and algebraic independence of the series that might be viewed as q-analogues of some classical mathematical constants.

1. Introduction and main results

Almost sixty years ago, Banerjee [1] considered the difference

$$E(n) := \#\{d \in \mathsf{N} : d|n,\ d \equiv 1 \pmod 4\} - \#\{d \in \mathsf{N} : d|n,\ d \equiv 3 \pmod 4\}$$

for $n \in \mathsf{N} := \{1, 2, \dots\}$, and proved, inter alia,

$$(1) \qquad \sum_{n=1}^{\infty} \frac{E(n)}{q^n} = \sum_{m=1}^{\infty} \frac{\sin(m\pi/2)}{q^m - 1} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{q^{2n-1} - 1} =: -f_q(1),$$

where $q \in \mathsf{C}$ satisfies $|q| > 1$. The series on the right-hand side of (1) is connected with the following q-analogue of π,

$$(2) \qquad \pi_q := 1 + 4 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{q^{2n-1} - 1},$$

see [5]. Surprisingly, several other q-series appearing as q-analogues of certain mathematical constants also play a rôle of generating series for classical arithmetic functions (see Section 6 below). It is this circumstance that was used in the first proofs of the irrationality of π_q for integral values of q, by Chowla [6] and Erdős [8].

Recently in [5] we deduced, as a particular case of a more general situation, a fairly weak irrationality measure for the value of (2) in the case $q \in \mathsf{Z}\setminus\{0, \pm 1\}$.

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The main aim of the present note is to considerably improve on this measure for π_q and deduce new measures for the q-mathematical constants

$$\lambda_q := \sum_{n=1}^{\infty} \frac{1}{q^{2n-1} - 1} \qquad \text{and} \qquad \beta_q := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{q^n + 1}, \tag{3}$$

again in the case of integers q.

For the sake of completeness, let us note that the *irrationality exponent* $\mu(\omega)$ of a real irrational number ω is defined by

$$\mu(\omega) := \inf\left\{\mu \in \mathsf{R} : \text{the inequality } \left|\omega - \frac{P}{Q}\right| \le Q^{-\mu} \right.$$
$$\left. \text{has only finitely many solutions } (P, Q) \in \mathsf{Z} \times \mathsf{N}\right\}.$$

Hence we have $\mu(\omega) \ge 2$ for every $\omega \in \mathsf{R} \setminus \mathsf{Q}$ and, in these terms, our earlier result in [5] states $\mu(\pi_q) \le 10.31789\ldots$. In contrast to this, our new result reads as

THEOREM 1. *For $q \in \mathsf{Z} \setminus \{0, \pm 1\}$, the irrationality exponent of π_q is at most* 6.50379809....

REMARK. In fact, our method below allows us to prove the existence of an absolute effective constant $\gamma > 0$ such that for every $(P, Q) \in \mathsf{Z}^2$ with $Q \ge 3$ the following inequality holds:

$$\left|\pi_q - \frac{P}{Q}\right| \ge Q^{-6.50379809\ldots - \gamma(\log\log Q)/\sqrt{\log Q}}.$$

It is curious that the new irrationality measure for π_q is sharper than the known one for π due to M. Hata [11].

Since $\pi_q = 1 - 4f_q(1)$, both numbers π_q and $f_q(1)$ obviously have the same irrationality exponent, and we may restrict ourselves from now on to the investigation of $f_q(1)$. To estimate $\mu(f_q(1))$ from above, we proceed as follows. First we analytically construct (in Section 2) good approximations to $f_q(1)$ as 'very small' linear forms in 1 and $f_q(1)$ with rational coefficients. Whereas, in [5], we mostly adopted for this the hypergeometric construction from [13], we now apply Borwein's method [2] using only a few and elementary complex analysis. To transform these linear forms into 'small' linear forms with integer coefficients, we need very careful arithmetic considerations (compare Lemmas 5 and 7 in Section 3) to find a 'sufficiently small' common denominator of the original rational coefficients. Having small linear forms with not too large integer coefficients, we use a Chudnovsky-type lemma (Lemma 8 in Section 4) for our final conclusion.

Our further results for the q-constants in (3) are the following.

THEOREM 2. *For* $q \in \mathsf{Z} \setminus \{0, \pm 1\}$, *the irrationality exponent of* λ_q *is at most* $3.89810036\dots$.

THEOREM 3. *For* $q \in \mathsf{Z} \setminus \{0, \pm 1\}$, *the irrationality exponent of* β_q *is at most* $18\pi^2/(7\pi^2 - 24) = 3.94020382\dots$.

We sketch their proofs in Section 5 and discuss open problems on q-mathematical constants in Section 6.

2. Analytic construction

We define the function

$$f_q(z) := \sum_{n=1}^{\infty} \frac{(-1)^n}{q^{2n-1} - z}, \tag{4}$$

which is meromorphic in the whole complex plane; compare also the definition at $z = 1$ on the right-hand side of (1). Evidently $f_q(z)$ satisfies a simple linear q-functional equation of order 1, which we do not need explicitly, but which is at the bottom of the following formula

$$f_q(q^{-2n}) = (-1)^n q^{2n} \left(f_q(1) - \sum_{\nu=1}^{n} \frac{(-1)^\nu}{q^{2\nu-1} - 1} \right) \qquad \text{for} \quad n \in \mathsf{N}_0 := \mathsf{N} \cup \{0\}. \tag{5}$$

(Empty sums or products should always be interpreted as 0 or 1, respectively.) Furthermore, we will require later the Taylor coefficients of $f_q(z)$ at the origin:

$$\frac{f_q^{(\nu)}(0)}{\nu!} = -\frac{q^{\nu+1}}{q^{2(\nu+1)} + 1}, \qquad \text{where} \quad \nu \in \mathsf{N}_0. \tag{6}$$

We next introduce the following auxiliary integral

$$J(L, M, N) := \frac{1}{2\pi i} \oint_{|z|=1} \frac{\prod_{l=1}^{L} (z - q^{2l-1})}{z^M \prod_{n=1}^{N} (1 - q^{2n} z)} f_q(z)\, \mathrm{d}z, \tag{7}$$

where $L, M, N \in \mathsf{N}$ are parameters to be specified later, and the integration is positively oriented. From the residue theorem we immediately see

$$J(L,M,N) = -\sum_{n=1}^{N} \frac{q^{2Mn}\prod_{l=1}^{L}(q^{-2n}-q^{2l-1})}{\prod_{\substack{\nu=1\\ \nu\neq n}}^{N}(1-q^{2(\nu-n)})}\cdot\frac{f_q(q^{-2n})}{q^{2n}}$$
$$+ \sum_{\substack{(\kappa,\mu,\nu)\in\mathbb{N}_0^3\\ \kappa+\mu+\nu=M-1}} \frac{1}{\kappa!}\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{\kappa}\prod_{l=1}^{L}(z-q^{2l-1})\bigg|_{z=0}$$
$$\times\frac{1}{\mu!}\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{\mu}\prod_{n=1}^{N}(1-q^{2n}z)^{-1}\bigg|_{z=0}\cdot\frac{f^{(\nu)}(0)}{\nu!}.$$

This and (5), (6) yield

$$(8)\quad J(L,M,N) = (-1)^{N+1}\sum_{n=1}^{N}\frac{q^{2(M-L)n+n(n-1)}\prod_{l=1}^{L}(1-q^{2l+2n-1})}{\prod_{\nu=1}^{n-1}(q^{2\nu}-1)\cdot\prod_{\nu=1}^{N-n}(q^{2\nu}-1)}$$
$$\times\left(f_q(1)-\sum_{\nu=1}^{n}\frac{(-1)^{\nu}}{q^{2\nu-1}-1}\right)$$
$$+\sum_{\kappa+\mu+\nu=M-1}\frac{P_{\kappa,\mu,\nu}}{q^{2(\nu+1)}+1}$$

with certain $P_{\kappa,\mu,\nu}\in\mathbb{Z}[q]$ not to be specified in more detail.

Next we would like to control the size of the factor

$$(9)\quad Q^*(L,M,N) := (-1)^{N+1}\sum_{n=1}^{N}\frac{q^{2(M-L)n+n(n-1)}\prod_{l=1}^{L}(1-q^{2l+2n-1})}{\prod_{\nu=1}^{n-1}(q^{2\nu}-1)\cdot\prod_{\nu=1}^{N-n}(q^{2\nu}-1)}$$

appearing in (8) as the coefficient of $f_q(1)$.

LEMMA 1. *For $q\in\mathbb{C}$, $|q|>1$, we have*

$$(10)\qquad |Q^*(L,M,N)| = |q|^{L^2+2MN+O(1)},$$

where the constant in $O(1)$ depends on q at most.

PROOF. The quotient of two successive summands in (9) is absolutely bounded by $\gamma_1|q|^{-2M}$. (The letter γ_1, as well as γ_2, γ_3, … later, depends only on q but not on L, M, N.) Furthermore, the absolute value of the summand in (9) corresponding to $n = N$ is gripped between $\gamma_2|q|^{L^2+2MN}$ and $\gamma_3|q|^{L^2+2MN}$, hence we get the required assertion.

LEMMA 2. *Suppose $M+N>L$. Then, for $q\in\mathbb{C}$, $|q|>1$, we have*

$$|J(L,M,N)|=|q|^{-(2L+1)(M+N-L)-N(N+2)+O(M+N)}, \tag{11}$$

where the constant in $O(\,\cdot\,)$ depends on q only.

PROOF. If $|z|>1$, the integrand in (7) has its poles at the points q^{2n-1}, $n=L+1,L+2,\dots$. For $R\in\mathbb{N}$, $R\ge L$, we see that the difference

$$\frac{1}{2\pi i}\oint_{|z|=|q|^{2R}}\frac{\prod_{l=1}^{L}(z-q^{2l-1})}{z^M\prod_{n=1}^{N}(1-q^{2n}z)}f_q(z)\,\mathrm{d}z-J(L,M,N) \tag{12}$$

is equal to the sum of the residues at q^{2n-1}, where $n=L+1,\dots,R$, of the integrand. Taking the estimate $|f_q(z)|\le\gamma_4R|q|^{-2R}$ on $|z|=|q|^{2R}$ into account, we deduce

$$\left|\frac{1}{2\pi i}\oint_{|z|=|q|^{2R}}\frac{\prod_{l=1}^{L}(z-q^{2l-1})}{z^M\prod_{n=1}^{N}(1-q^{2n}z)}f_q(z)\,\mathrm{d}z\right|\le\gamma_5R|q|^{2R(L-M-N)-N^2-N}.$$

Recalling our assumption $M+N>L$, the integral in (12) tends to 0 as $R\to\infty$ and we get

$$J(L,M,N)=\sum_{n=L+1}^{\infty}(-1)^n\frac{\prod_{l=1}^{L}(q^{2n-1}-q^{2l-1})}{q^{(2n-1)M}\prod_{\nu=1}^{N}(1-q^{2n+2\nu-1})}. \tag{13}$$

Here the quotient of two successive summands is absolutely bounded by $\le\gamma_6|q|^{-2(M+N-L)}$, and the first summand equals absolutely to

$$|q|^{-(2L+1)(M+N-L)-N(N+2)},$$

up to a factor bounded above and below by two γ's. Thus, from (13) we conclude with estimate (11).

REMARK. By (10) and (11), Q^* is large and J is small. Hence

$$P^*(L,M,N):=Q^*(L,M,N)f_q(1)-J(L,M,N)$$

satisfies the same asymptotic equality (10) as $Q^*(L,M,N)$.

3. Arithmetic constituents

As one sees from (9) and (8), both expressions $Q^*(L, M, N)$ and

(14)
$$\begin{aligned} P^*(L, M, N) &= (-1)^N \sum_{n=1}^{N} \frac{q^{2(M-L)n+n(n-1)} \prod_{l=1}^{L}(1-q^{2l+2n-1})}{\prod_{\nu=1}^{n-1}(q^{2\nu}-1)\cdot\prod_{\nu=1}^{N-n}(q^{2\nu}-1)} \sum_{\nu=1}^{n} \frac{(-1)^\nu}{q^{2\nu-1}-1} \\ &\quad - \sum_{\kappa+\mu+\nu=M-1} \frac{P_{\kappa,\mu,\nu}}{q^{2(\nu+1)}+1} \end{aligned}$$

are contained in $\mathsf{Z}(q)$. Our nearest aim is the search of a sufficiently small common denominator of the rational approximants P^* and Q^* constructed above.

Let x be an indeterminate. Recall that cyclotomic polynomials

$$\Phi_l(x) := \prod_{\substack{k=1 \\ (k,l)=1}}^{l} (x - e^{2\pi i k/l}), \qquad \deg_x \Phi_l(x) = \varphi(l) := l \prod_{p\mid l}\left(1 - \frac{1}{p}\right),$$

and only they appear as irreducible (over Q) factors of the polynomials $x^n - 1$:

$$x^n - 1 = \prod_{l\mid n} \Phi_l(x), \qquad n \in \mathsf{N}. \tag{15}$$

One of the 'arithmetic' consequences of formula (15) is the fact that the product $\prod_{l=1}^{n} \Phi_l(x)$ realizes the least common multiple of the polynomials $x-1$, x^2-1, $\dots$, x^n-1, and this multiple is much better than the trivial one $\prod_{l=1}^{n}(x^l-1)$ since

$$\deg_x \prod_{l=1}^{n} \Phi_l(x) = \sum_{l=1}^{n} \varphi(l) = \frac{3}{\pi^2}n^2 + O(n\log n) \qquad \text{as} \quad n \to \infty$$

by classical Mertens' formula. In what follows we will also require its variations (see, e.g., [13]):

(16)
$$\sum_{\mu=1}^{n} \varphi(2\mu-1) = \frac{8}{\pi^2}n^2 + O(n\log n), \quad \sum_{\mu=1}^{n} \varphi(2\mu) = \frac{4}{\pi^2}n^2 + O(n\log n)$$

as $n \to \infty$.

LEMMA 3. *For any $n = 0, 1, \dots, N$ and $j = 1, \dots, n$, we have*

$$\frac{\prod_{l=1}^{L}(x^{2(l+n)-1} - 1)}{x^{2j-1} - 1} \cdot \frac{\prod_{\nu=1}^{N} \Phi_{2\nu-1}(x)}{\prod_{\nu=1}^{L} \Phi_{2\nu-1}(x)^{[L/(2\nu-1)]}} \in \mathsf{Z}[x]. \tag{17}$$

PROOF. It follows from the inclusions

$$\frac{\prod_{l=1}^{L}(x^{2(l+n)-1} - 1)}{\prod_{\nu=1}^{L} \Phi_{2\nu-1}(x)^{[L/(2\nu-1)]}} = \frac{\prod_{\mu=1}^{L+n}(x^{2\mu-1} - 1)}{\prod_{\nu=1}^{L} \Phi_{2\nu-1}(x)^{[L/(2\nu-1)]} \cdot \prod_{\mu=1}^{n}(x^{2\mu-1} - 1)} \in \mathsf{Z}[x]. \tag{18}$$

and the fact that

$$x^{2j-1} - 1 = \prod_{(2\nu-1)|(2j-1)} \Phi_{2\nu-1}(x)$$

divides $\prod_{\nu=1}^{N} \Phi_{2\nu-1}(x)$. In order to demonstrate (18), note that all irreducible divisors of the factors $x^{2\mu-1} - 1$, $\mu = 1, 2, \dots, L+n$, have the form $\Phi_{2\nu-1}(x)$. Since, for any integer K,

$$\operatorname{ord}_{\Phi_{2\nu-1}(x)} \prod_{\mu=1}^{K}(x^{2\mu-1} - 1) = \left[\frac{K + \nu - 1}{2\nu - 1}\right],$$

the polynomial $\Phi_{2\nu-1}(x)$ enters the fraction (18) with exponent

$$\left[\frac{L + n + \nu - 1}{2\nu - 1}\right] - \left[\frac{L}{2\nu - 1}\right] - \left[\frac{n + \nu - 1}{2\nu - 1}\right] \geq 0,$$

and the lemma follows.

LEMMA 4. *Let $N \geq L$. For each $v = 1, 2, \dots, M$, we have*

$$\frac{\prod_{k=1}^{N}(x^{k} - 1)}{\prod_{\nu=1}^{L} \Phi_{2\nu-1}(x)^{[L/(2\nu-1)]}} \cdot \frac{\prod_{k=1}^{N}(x^{k} + 1) \cdot \prod_{\mu=[N/2]+1}^{M} \Phi_{2\mu}(x^{2})}{x^{2v} + 1} \in \mathsf{Z}[x]. \tag{19}$$

PROOF. The assumption $N \geq L$ implies

$$\frac{\prod_{k=1}^{N}(x^{k} - 1)}{\prod_{\nu=1}^{L} \Phi_{2\nu-1}(x)^{[L/(2\nu-1)]}} \in \mathsf{Z}[x]. \tag{20}$$

Indeed,

$$\prod_{k=1}^{N}(x^{k} - 1) = \prod_{\mu=1}^{N} \Phi_{\mu}(x)^{[N/\mu]}$$

and the latter product is divisible by the denominator in (20). This shows that the first factor in (19) lies in $\mathsf{Z}[x]$. Furthermore, all cyclotomic polynomials $\Phi_{2\mu}(x^2)$ for $\mu = 1, \ldots, [N/2]$ divides $\prod_{k=1}^{N}(x^k+1)$, and

$$\frac{\prod_{\mu=1}^{M}\Phi_{2\mu}(x^2)}{x^{2v}+1} \in \mathsf{Z}[x].$$

This completes the proof.

From Lemmas 3, 4 and the estimate

$$-\bigl(2(M-L)n+n(n-1)\bigr) \le e(L,M,N) := \begin{cases} 0 & \text{if } M \ge L, \\ (L-M)(L-M+1) & \text{if } M \le L, \end{cases}$$

for $n = 0, 1, \ldots, N$, it follows that our choice for the denominators of (9) and (14) can be

$$(21)\quad D(L,M,N) := q^{e(L,M,N)} \cdot \frac{\prod_{k=1}^{N}(q^{2k}-1)}{\prod_{\nu=1}^{L}\Phi_{2\nu-1}(q)^{[L/(2\nu-1)]}} \cdot \prod_{\nu=1}^{N}\Phi_{2\nu-1}(q) \cdot \prod_{\mu=[N/2]+1}^{M}\Phi_{2\mu}(q^2),$$

provided $N \ge L$. Namely, we obtain

LEMMA 5. *If $N \ge L$, then with the choice given in (21)*

$$(22)\quad D(L,M,N)Q^*(L,M,N) \in \mathsf{Z}[q] \quad \text{and} \quad D(L,M,N)P^*(L,M,N) \in \mathsf{Z}[q].$$

To compute the asymptotic behaviour of $|D(L,M,N)|$ (equivalently, of the degree of (21)) we will require

LEMMA 6. *For large $N \in \mathsf{N}$ one has*

$$(23)\qquad \sum_{\nu=1}^{N}\left[\frac{N}{2\nu-1}\right]\varphi(2\nu-1) = \frac{N^2}{3} + O(N\log^2 N).$$

PROOF. Define $K = [N/\log^2 N] \in \mathsf{N}$, hence $K^{-1} = o(1)$ and $K = o(N)$

as $N \to \infty$. We clearly have

$$\begin{aligned}
\sum_{\nu=1}^{N}\left[\frac{N}{2\nu-1}\right]\varphi(2\nu-1) &= \sum_{\nu=1}^{\infty}\left[\frac{N}{2\nu-1}\right]\varphi(2\nu-1) \\
&= \sum_{k=1}^{\infty} k \sum_{N/(k+1)<2\nu-1\le N/k} \varphi(2\nu-1) = \sum_{k=1}^{\infty}\sum_{2\nu-1\le N/k}\varphi(2\nu-1) \\
&= \sum_{k=1}^{K}\sum_{2\nu-1\le N/k}\varphi(2\nu-1) + \sum_{k=K+1}^{\infty}\sum_{2\nu-1\le N/k}\varphi(2\nu-1) =: \Sigma_1 + \Sigma_2.
\end{aligned}$$

In Σ_2 we estimate trivially each inner sum by $\sum_{\nu\le N/k}(2\nu-1) \le (N/k)^2$, hence

$$\Sigma_2 \le N^2 \sum_{k=K+1}^{\infty}\frac{1}{k^2} = O\left(\frac{N^2}{K}\right) = O(N\log^2 N) \qquad \text{as} \quad N \to \infty.$$

In contrast to this, Σ_1 produces the main term. Indeed, from the first relation in (16) we deduce

$$\begin{aligned}
\Sigma_1 &= \sum_{k=1}^{K}\sum_{\nu\le(N+k)/(2k)}\varphi(2\nu-1) = \sum_{k=1}^{K}\left(\frac{8}{\pi^2}\left(\frac{N+k}{2k}\right)^2 + O\left(\frac{N}{k}\log\frac{N}{k}\right)\right) \\
&= \frac{2N^2}{\pi^2}\sum_{k=1}^{K}\frac{1}{k^2} + O\big(N(\log K)(\log N)\big) \\
&= \frac{2N^2}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2} + O\left(\frac{N^2}{K}\right) + O\big(N(\log K)(\log N)\big) \\
&= \frac{2N^2}{\pi^2}\zeta(2) + O(N\log^2 N)
\end{aligned}$$

that, in view of evaluation $\zeta(2) = \pi^2/6$, gives the desired result.

LEMMA 7. *Let $L = N$ and $M = [\alpha N]$ for certain real $\alpha > 0$. Then*

$$\begin{aligned}
\deg_q D(L, M, N) &= \deg_q D(N, [\alpha N], N) \\
&= \left(\chi(\alpha) + \frac{2}{3} + \frac{6+8\alpha^2}{\pi^2}\right)N^2 + O(N\log^2 N)
\end{aligned} \tag{24}$$

as $N \to \infty$, where

$$\chi(\alpha) := \begin{cases} 0 & \text{if } \alpha \geq 1, \\ (1-\alpha)^2 & \text{if } \alpha \leq 1. \end{cases}$$

PROOF. From definition (21), Lemma 6 and asymptotic formulae (16) we obtain

$$\begin{aligned} &\deg_q D(N, [\alpha N], N) \\ &\quad = e(N, [\alpha N], N) + N(N+1) - \sum_{\nu=1}^{N} \left[\frac{N}{2\nu - 1} \right] \varphi(2\nu - 1) \\ &\qquad + \sum_{\nu=1}^{N} \varphi(2\nu - 1) + 2 \sum_{\mu=[N/2]+1}^{[\alpha N]} \varphi(2\mu) \\ &\quad = \left(\chi(\alpha) + 1 - \frac{1}{3} + \frac{8}{\pi^2} + \frac{8}{\pi^2}\alpha^2 - \frac{8}{\pi^2} \cdot \frac{1}{4} \right) N^2 + O(N \log^2 N) \end{aligned}$$

that after clear reduction becomes (24).

4. Integer linear forms and irrationality measures

Our next lemma provides us with upper bounds for irrationality exponents. Several such lemmas can be found in the literature (compare, e.g., Chudnovsky [7] or Hata [10]). Of course, it depends highly on the information available in any concrete situation, which one is the most appropriate to be applied. For our purpose, the following lemma is very convenient.

LEMMA 8. *Given $\omega \in \mathsf{R}$, there exists an infinite sequence of pairs $(P(N), Q(N)) \in \mathsf{Z} \times \mathsf{N}$ with*

(25) $$|Q(N)\omega - P(N)| = e^{-\psi(N)}, \qquad N = 1, 2, \ldots,$$

where the function $\psi \colon \mathsf{N} \to \mathsf{R}_+$ satisfies the following conditions:

(i) $\psi(N) \to \infty$ *as* $N \to \infty$*;*

(ii) $\limsup_{N \to \infty} \dfrac{\psi(N+1)}{\psi(N)} \leq 1$*;*

(iii) $\rho := \limsup_{N \to \infty} \dfrac{\log Q(N)}{\psi(N)} > 0$.

Then ω is irrational and $\mu(\omega) \leq 1 + \rho$ holds.

REMARK 1. Clearly, condition (i) is enough to guarantee $\omega \notin \mathsf{Q}$, whereas (ii) and (iii) are needed for the main *quantitative* part of the assertion. Note also that $\mu(\omega) \geq 2$ implies *a posteriori* $\rho \geq 1$ in (iii).

REMARK 2. Most lemmas of this kind proved usually deal with the case, when $\psi(N)$ linearly depends on N. In contrast to this, our function ψ is rather unrestricted, except for condition (ii), which says that it should not increase too fast: everything polynomial-like is right.

PROOF OF LEMMA 8. Let $(P, Q) \in \mathsf{Z} \times \mathsf{N}$ be given with Q large enough. Define N as smallest positive integer satisfying $2Q \leq e^{\psi(N)}$. From

$$|(Q\omega - P)Q(N)| = |Q(Q(N)\omega - P(N)) + (QP(N) - PQ(N))|$$

we see that

$$|(Q\omega - P)Q(N)| \geq \begin{cases} 1 - Qe^{-\psi(N)} & \text{if } QP(N) \neq PQ(N), \\ Qe^{-\psi(N)} & \text{if } QP(N) = PQ(N), \end{cases}$$

where we used (25). Hence in both cases we find

$$\left|\omega - \frac{P}{Q}\right| \geq \frac{1}{Q(N)e^{\psi(N)}} \geq \frac{1}{e^{(1+\rho+\varepsilon/2)\psi(N)}} \geq \frac{1}{e^{(1+\rho+\varepsilon)\psi(N-1)}}$$

using hypotheses (ii) and (iii). These inequalities yield

$$\left|\omega - \frac{P}{Q}\right| \geq (2Q)^{-1-\rho-\varepsilon},$$

hence $\mu(\omega) \leq 1 + \rho + \varepsilon$. But since $\varepsilon \in \mathsf{R}_+$ was arbitrary we have the truth of our claim.

PROOF OF THEOREM 1. As in Lemma 7, we take $L = N$ and $M = [\alpha N]$, and omit the dependence on the arguments L, M for linear forms J and their coefficients Q^* and P^*. (In order to simplify our 'theoretic' considerations we omit the choice $L = [\alpha' N]$ for real α' ranging in the interval $0 < \alpha' < 1$, since it always leads to a worse quantitative result.) Lemmas 1 and 2 become

$$|J(N)| = |q|^{-(1+2\alpha)N^2+O(N)}, \qquad |Q^*(N)| = |q|^{(1+2\alpha)N^2+O(1)}, \tag{26}$$

and we have to transform the linear forms in 1 and $f_q(1)$,

$$J(N) = Q^*(N)f_q(1) - P^*(N),$$

into linear forms with coefficients in $\mathsf{Z}[q]$ by multiplying them by $D(N) := D(N, [\alpha N], N)$ from (21). By Lemma 7 we find that

$$|D(N)| = |q|^{(\chi(\alpha)+2/3+(6+8\alpha^2)/\pi^2)N^2+O(N\log^2 N)}, \tag{27}$$

while Lemma 5 guarantees that $Q(N) := D(N)Q^*(N) \in \mathsf{Z}[q]$ and $P(N) := D(N)P^*(N) \in \mathsf{Z}[q]$. With these rational integers $P(N)$, $Q(N)$ we see from (26) and (27) that

$$|Q(N)| = |q|^{(5/3+2\alpha+\chi(\alpha)+(6+8\alpha^2)\pi^{-2})N^2+O(N\log^2 N)} \tag{28}$$

and

$$\begin{aligned} |Q(N)f_q(1) - P(N)| &= |D(N)J(N)| \\ &= |q|^{-(1/3+2\alpha-\chi(\alpha)-(6+8\alpha^2)\pi^{-2})N^2+O(N\log^2 N)}. \end{aligned} \tag{29}$$

Hence we may apply Lemma 8 with

$$\psi(N) := \left(\frac{1}{3} + 2\alpha - \chi(\alpha) - \frac{6+8\alpha^2}{\pi^2}\right)N^2 \log|q| + O(N\log^2 N)$$

and

$$\rho := \frac{5/3 + 2\alpha + \chi(\alpha) + (6+8\alpha^2)\pi^{-2}}{1/3 + 2\alpha - \chi(\alpha) - (6+8\alpha^2)\pi^{-2}}$$

to prove

$$\mu(f_q(1)) \le \frac{2(1+2\alpha)}{1/3 + 2\alpha - \chi(\alpha) - (6+8\alpha^2)\pi^{-2}}.$$

Choosing simply $\alpha = 1$ we obtain

$$\mu(f_q(1)) \le \frac{18\pi^2}{7(\pi^2-6)} = 6.55854710\ldots,$$

while the optimal choice

$$\alpha = \frac{1}{2}\sqrt{\frac{96+35\pi^2}{3(8+\pi^2)}} - \frac{1}{2} = 0.93478179\ldots$$

leads us to the estimate given in Theorem 1.

Using (28) and (29) more directly we can easily get the assertion indicated in the remark after Theorem 1.

5. Irrationality measures for λ_q and β_q

In this section we sketch our proofs of Theorems 2 and 3.

PROOF OF THEOREM 2. Replace the function $f_q(z)$ in the analytic construction by

$$\widetilde{f}_q(z) := \sum_{n=1}^{\infty} \frac{1}{q^{2n-1} - z}.$$

Since

$$\widetilde{f}_q(q^{-2n}) = q^{2n}\biggl(\widetilde{f}_q(1) - \sum_{\nu=1}^{n} \frac{1}{q^{2\nu-1} - 1}\biggr) \qquad \text{for} \quad n \in \mathsf{N}_0$$

and

$$\frac{\widetilde{f}_q^{(\nu)}(0)}{\nu!} = \frac{q^{\nu+1}}{q^{2(\nu+1)} - 1} \qquad \text{for} \quad \nu \in \mathsf{N}_0,$$

we obtain the linear forms

$$\begin{aligned}\widetilde{Q}^*(L, M, N)\widetilde{f}_q(1) - \widetilde{P}^*(L, M, N) &:= \widetilde{J}(L, M, N) \\ &:= \frac{1}{2\pi i} \oint_{|z|=1} \frac{\prod_{l=1}^{L}(z - q^{2l-1})}{z^M \prod_{n=1}^{N}(1 - q^{2n} z)} \widetilde{f}_q(z)\, \mathrm{d}z\end{aligned}$$

of about the same shapes as before, in (9) and (14). The estimates of Lemmas 1 and 2 remain valid for the tilded objects, but the denominator choice is different (cf. (21)):

$$\begin{aligned}&\widetilde{D}(L, M, N) \\ &\quad := q^{e(L,M,N)} \cdot \frac{\prod_{k=1}^{N}(q^{2k} - 1)}{\prod_{\nu=1}^{L} \Phi_{2\nu-1}(q)^{[L/(2\nu-1)]}} \cdot \prod_{\nu=1}^{N} \Phi_{2\nu-1}(q) \cdot \prod_{\mu=N+1}^{M} \Phi_{2\mu}(q)\end{aligned}$$

provided $L \le N$ and $M \le 2N$. Then

$$\deg_q \widetilde{D}(N, [\alpha N], N) = \biggl(\chi(\alpha) + \frac{2}{3} + \frac{4(1+\alpha^2)}{\pi^2}\biggr)N^2 + O(N \log^2 N)$$

as $N \to \infty$, hence

$$\mu(\lambda_q) = \mu(\widetilde{f}_q(1)) \le 1 + \rho := \frac{2(1 + 2\alpha)}{1/3 + 2\alpha - \chi(\alpha) - 4(1+\alpha^2)\pi^{-2}}.$$

The simplest choice $\alpha = 1$ gives

$$\mu(\lambda_q) \le \frac{18\pi^2}{7\pi^2 - 24} = 3.94020382\ldots,$$

while taking $\alpha = \sqrt{5/4+\pi^2/6} - 1/2 = 1.20145057\ldots$ decreases the estimate to $3.89810036\ldots$.

PROOF OF THEOREM 3. To investigate β_q arithmetically via Borwein's method, we consider the meromorphic function

$$h_q(z) := \sum_{n=1}^{\infty} \frac{(-1)^n}{q^n + z}$$

with the property $h_q(1) = -\beta_q$, and use the integral

$$J(L, M, N) := \frac{1}{2\pi i} \oint_{|z|=1} \frac{\prod_{l=1}^{L}(z+q^l)}{z^M \prod_{n=1}^{N}(1-q^n z)} h_q(z)\, \mathrm{d}z. \tag{30}$$

Then we find as in (8) that

$$\begin{aligned} J(L, M, N) &= (-1)^{N+1} \sum_{n=1}^{N} \frac{q^{(M-L)n+n(n-1)/2} \prod_{l=1}^{L}(q^{l+n}+1)}{\prod_{\nu=1}^{n-1}(q^\nu - 1) \cdot \prod_{\nu=1}^{N-n}(q^\nu - 1)} \left(h_q(1) - \sum_{\nu=1}^{n} \frac{(-1)^\nu}{q^\nu + 1} \right) \\ &\quad + \sum_{\kappa+\mu+\nu=M-1} \frac{P_{\kappa,\mu,\nu}}{q^{\nu+1}+1} \end{aligned} \tag{31}$$

with certain $P_{\kappa,\mu,\nu} \in \mathbb{Z}[q]$. From

$$Q^*(L, M, N) := (-1)^{N+1} \sum_{n=1}^{N} \frac{q^{(M-L)n+n(n-1)/2} \prod_{l=1}^{L}(q^{l+n}+1)}{\prod_{\nu=1}^{n-1}(q^\nu - 1) \cdot \prod_{\nu=1}^{N-n}(q^\nu - 1)} \tag{32}$$

we get, by the usual considerations,

$$|Q^*(L, M, N)| = |q|^{L(L+1)/2+MN+O(1)}. \tag{33}$$

In $|z| > 1$, the integrand in (30) has its (simple) poles exactly at $z = -q^n$ for $n > L$. Hence letting $R \in \mathbb{N}$, $R \geq L$, we find

$$\frac{1}{2\pi i} \oint_{|z|=|q|^{R+1/2}} \cdots - J(L, M, N) = \sum_{n=L+1}^{R} \frac{(-1)^n \prod_{l=1}^{L}(q^l - q^n)}{(-q^n)^M \prod_{\nu=1}^{N}(1+q^{n+\nu})} \tag{34}$$

with the same integrand as in (30). Since on $|z| = |q|^{R+1/2}$ we have $|h_q(z)| \ll R|q|^{-R}$, we estimate

$$\left| \frac{1}{2\pi i} \oint_{|z|=|q|^{R+1/2}} \cdots \right| \ll R|q|^{(R+1/2)(L-M-N)-N(N+1)/2}$$

to deduce from (34) (*assuming* $M+N>L$)

$$J(L,M,N)=(-1)^{M+1}\sum_{n=L+1}^{\infty}\frac{(-1)^n\prod_{l=1}^{L}(q^l-q^n)}{q^{Mn}\prod_{\nu=1}^{N}(1+q^{n+\nu})}$$

yielding

$$|J(L,M,N)|=|q|^{-N(N+1)/2-(L+1)(M+N-L)+O(1)}. \tag{35}$$

With $Q^*(L,M,N)$ defined in (32), formula (31) can be written as

$$J(L,M,N)=Q^*(L,M,N)h_q(1)-P^*(L,M,N),$$

where

(36)

$$P^*(L,M,N)=(-1)^{N+1}\sum_{n=1}^{N}\frac{q^{(M-L)n+n(n-1)/2}\prod_{l=1}^{L}(q^{l+n}+1)}{\prod_{\nu=1}^{n-1}(q^\nu-1)\cdot\prod_{\nu=1}^{N-n}(q^\nu-1)}\sum_{\nu=1}^{n}\frac{(-1)^\nu}{q^\nu+1} - \sum_{\kappa+\mu+\nu=M-1}\frac{P_{\kappa,\mu,\nu}}{q^{\nu+1}+1}.$$

We now do a "denominator search" for $Q^*(L,M,N)$ and $P^*(L,M,N)$.

LEMMA 9. *For each* $t\in\mathbb{N}$ *we have*

$$\prod_{s=1}^{t}(q^s+1)=\prod_{(i,j)\in\mathbb{N}_0\times(2\mathbb{N}_0+1)}\Phi_{2^{1+i}j}(q)^{[(t+2^ij)/(2^{i+1}j)]}.$$

REMARK. Note that $[(t+2^ij)/(2^{i+1}j)]=0$ if and only if $2^ij>t$.

PROOF. Let $e(s)\in\mathbb{N}_0$ denote the multiplicity of 2 in s. Then we have

$$q^s+1=\prod_{\substack{j\mid s\\ s/j\text{ odd}}}\Phi_{2j}(q)=\prod_{j\mid(s/2^{e(s)})}\Phi_{2^{1+e(s)}j}(q).$$

From this, putting $I(t):=[(\log t)/(\log 2)]$, we deduce

$$\prod_{s=1}^{t}(q^s+1)=\prod_{s=1}^{t}\prod_{j:2^{e(s)}j\mid s}\Phi_{2^{1+e(s)}j}(q)=\prod_{i=0}^{I(t)}\prod_{\substack{s=1\\ e(s)=i}}^{t}\prod_{j:2^ij\mid s}\Phi_{2^{1+i}j}(q),$$

where we note that j is automatically odd. This formula can be continued as

$$\prod_{s=1}^{t}(q^s+1)=\prod_{i=0}^{I(t)}\prod_{\substack{j\in 2\mathsf{N}_0+1\\ j\le t/2^i}}\Phi_{2^{1+i}j}(q)^{[t/(2^{1+i}j)+1/2]}$$

since the number of $k\in\mathsf{N}_0$ satisfying $2^i j(2k+1)\le t$ is just $[t/(2^{1+i}j)+1/2]$.

From Lemma 9 we see that

$$\prod_{l=1}^{L}(q^{l+n}+1)=\prod_{(i,j)}\Phi_{2^{1+i}j}(q)^{[(L+n)/(2^{1+i}j)+1/2]-[n/(2^{1+i}j)+1/2]} \tag{37}$$

for $n=1,\dots,N$, and every exponent here is at least $[L/(2^{1+i}j)]$ by $[x]-[y]\ge[x-y]$ for any $x,y\in\mathsf{R}$. On the other hand, we know

$$\begin{aligned}\prod_{\nu=1}^{n-1}(q^\nu-1)\cdot\prod_{\nu=1}^{N-n}(q^\nu-1)&=\prod_{d=1}^{n-1}\Phi_d(q)^{[(n-1)/d]}\cdot\prod_{d=1}^{N-n}\Phi_d(q)^{[(N-n)/d]}\\&=\prod_{d=1}^{N-1}\Phi_d(q)^{[(n-1)/d]+[(N-n)/d]}\end{aligned} \tag{38}$$

for $n=1,\dots,N$. Note that in the last product the dth exponent is at most $[(N-1)/d]$.

Assuming $M\ge L$ (implying our earlier assumption $M+N>L$) and $L\ge N-1$, we therefore see from (32) that we can get rid of the denominators in $Q^*(L,M,N)$ by multiplying with

$$\prod_{\substack{d=1\\ d\text{ odd}}}^{N-1}\Phi_d(q)^{[(N-1)/d]}. \tag{39}$$

Namely, if $d<N$ is even, we can write it uniquely as $d=2^{1+i}j$ with $(i,j)\in\mathsf{N}_0\times(2\mathsf{N}_0+1)$ and see from (37), (38) and the corresponding remarks that all cyclotomic polynomials $\Phi_d(q)$ with even d cancel automatically from the denominator in the summands of $Q^*(L,M,N)$ in (32).

To get rid of all denominators in (36), we see after our last considerations that it is enough to multiply $P^*(L,M,N)$ apart from (39) by the least common multiple of $q^\nu+1$, where $\nu=1,2,\dots,\max(M,N)=M$, which is exactly

$\prod_{j=1}^{M}\Phi_{2j}(q)$. Denoting

$$D(L,M,N) := \prod_{\substack{d=1\\ d\ \mathrm{odd}}}^{N-1}\Phi_d(q)^{[(N-1)/d]}\cdot\prod_{j=1}^{M}\Phi_{2j}(q)$$

we deduce from Lemma 6

$$|D(L,M,N)| = |q|^{(1/3+(2\alpha/\pi)^2)N^2+O(N\log^2 N)} \tag{40}$$

supposing $M=[\alpha N]$ with some fixed real $\alpha\ge 1$.

Assuming finally $L=N$, we obtain from (33) and (40)

$$|Q| := |DQ^*| = |q|^{(1/2+\alpha+1/3+(2\alpha/\pi)^2)N^2+O(N\log^2 N)}$$

and from (35) and (40)

$$|DJ| = |q|^{-(1/2+\alpha-1/3-(2\alpha/\pi)^2)N^2+O(N\log^2 N)}.$$

Lemma 8 leads to

$$\mu(\beta_q)=\mu(h_q(1))\le\frac{1+2\alpha}{1/6+\alpha-(2\alpha/\pi)^2} =: \chi(\alpha),$$

and $\chi(\alpha)$ is strictly increasing in $\alpha\ge 1$, hence Theorem 3 follows with the choice $\alpha=1$.

6. Some open problems

In this section, we will assume that a complex number q satisfies $|q|<1$ (i.e., we replace the old values of q by $1/q$).

The series

$$\zeta_q(k) := \sum_{n=1}^{\infty} n^{k-1}\frac{q^n}{1-q^n} = \sum_{m=1}^{\infty}\sigma_{k-1}(m)q^m, \quad \text{where} \quad \sigma_{k-1}(n) := \sum_{d\mid n} d^{k-1}, \tag{41}$$

which are strongly connected with the modular world for even integers $k\ge 4$, may be considered as natural q-analogues of the values of Riemann's zeta function $\zeta(k)$. This analogy motivates arithmetic investigations of the values of $\zeta_q(k)$, for instance, if $1/q\in\mathsf{Z}\setminus\{0,\pm 1\}$ or $1/q$ is a Pisot or Salem number; several results in this direction may be found in [3], [4], [12]. An interesting problem is to investigate arithmetic properties of *q-zeta values* (41) as functions of q. The last news concerning the problem [14] is the linear independence over $\mathsf{C}(q)$ of all series in (41) ($k = 1, 2, \dots$) and algebraic independence over $\mathsf{C}(q)$ of the series in the following collections: $\{\zeta_q(1), \zeta_q(k)\}$, where $k \geq 2$ is arbitrary, and $\{\zeta_q(1), \zeta_q(2), \zeta_q(4), \zeta_q(6)\}$. Another curious object is the q-analogue of Catalan's constant

$$G_q := \sum_{n=1}^{\infty} \frac{(-1)^{n-1} q^{2n-1}}{(1 - q^{2n-1})^2},$$

on which we are unaware of any arithmetic information for its values at algebraic points q with $0 < |q| < 1$.

It is also interesting to look for linear independence results on the above q-mathematical constants with different values of the parameter q, for instance, to prove linear independence of

$$\sum_{n=1}^{\infty} \frac{1}{2^n - 1} \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{1}{3^n - 1}.$$

Such problems are closely related to *elliptic zeta values* introduced in [9]:

$$\zeta_{q,r}(k) = \sum_{n=1}^{\infty} n^{k-1} \frac{q^n - (-1)^k r^n}{(1 - q^n)(1 - r^n)}, \qquad |q| < 1, \quad |r| < 1,$$

especially for odd $k \geq 1$ (since $\zeta_{q,r}(k) = \zeta_q(k) - \zeta_r(k)$ for even k). These functions admit very nice functional equations. Even getting arithmetic results for the elliptic harmonic series ($k = 1$)

$$\zeta_{q,r}(1) = \sum_{n=1}^{\infty} \frac{q^n + r^n}{(1 - q^n)(1 - r^n)} = \zeta_q(1) + \zeta_r(1) + 2 \sum_{\substack{a,b=1 \\ (a,b)=1}}^{\infty} \zeta_{q^a r^b}(1)$$

is of interest.

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MATHEMATICAL INSTITUTE
UNIVERSITY OF COLOGNE
WEYERTAL 86–90
50931 COLOGNE
GERMANY
E-mail: pb@mi.uni-koeln.de

DEPARTMENT OF MECHANICS AND MATHEMATICS
MOSCOW LOMONOSOV STATE UNIVERSITY
VOROBIOVY GORY, GSP-2
119992 MOSCOW
RUSSIA
E-mail: wadim@ips.ras.ru