APPLICATIONS OF MOMENT PROBLEMS TO THE OVERCOMPLETENESS OF SEQUENCES

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Abstract

This paper introduces the notion of a determining function, which is used in order to apply results on the determination of measures by moments to the theory of overcompleteness of sequences in various function spaces, providing strong generalizations of results of Lin and Too. A further application is given to the determination of probability distributions by means of moments of record values.

1. Introduction

Problems involving the determination of a measure from its moments are of importance in many areas of analysis and probability. (See, for example, [8].) Recently, there has been some work to the effect that under some circumstances the moments of two measures need not be assumed to be equal, since as soon as they are approximately equal (in a sense to be made precise later), this forces the measures to differ on a small set; stronger forms of such results force the measures even to coincide [2], [3], [9], [12]. Such results can be applied in the geometry of Banach spaces and in areas of probability theory. In order to derive some of these as clearly as possible, we shall introduce the key notion of a *determining function*, which is, roughly speaking, a function Φ such that the measure $\Phi(x)\,dx$ is uniquely determined by its moments.

The main application that we have in mind is towards the overcompleteness of sequences in Hardy and Bergman spaces. Recall that a sequence $(x_m)_m$ in a Banach space $\mathscr{X}$ is said to be *complete* if its closed linear span is the whole space $\mathscr{X}$, and *overcomplete* (or sometimes *hypercomplete* or *densely closed*) if every infinite subsequence of $(x_m)_m$ is complete in $\mathscr{X}$ (see, for example, the book of Singer [13], and the very recent article [1]).

A weaker but more useful notion is finite overcompleteness: a sequence is *finitely overcomplete*, when it remains complete on deleting finitely many terms from the sequence. For example, the functions $(f_m)_m$ with $f_m(x) = x^m$ for $x \in (0, 1)$ can be shown to form a finitely overcomplete sequence in $L^2(0, 1)$

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by means of the Stone-Weierstrass theorem. Finitely overcomplete sequences have many applications in signal-processing, for example via the theory of frames [7].

Thus, as an application of the theory of moments, we shall provide extensive generalizations of Lin's result [10] that the sequence of functions $(x^m e^{-\lambda x})_m$ is finitely overcomplete in $L^1(0, \infty)$ for any $\lambda > 0$. Here we work with weighted Lebesgue spaces, Hardy spaces and Bergman spaces, and in addition we give some multivariable extensions.

We adopt the following conventions. Let $\mathsf{N} = \{0, 1, 2, \ldots\}$, $\mathsf{R}_+ = \{x \in \mathsf{R} : x \geq 0\}$ and similarly for N^n and R_+^n. Also $\mathsf{C}_+ = \{z \in \mathsf{C} : \operatorname{Re} z > 0\}$.

For a given vector $x = (x_1, \ldots, x_n) \in \mathsf{R}^n$ and a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathsf{N}^n$, we write x^α to denote $x_1^{\alpha_1} \ldots x_n^{\alpha_n}$.

A function $f : \mathsf{R}_+^n \to \mathsf{C}$ is said to be *of polynomial growth*, if there exist constants $C, d > 0$ such that

$$(1) \qquad |f(x)| \leq C(1 + \|x\|^d), \qquad (x \in \mathsf{R}_+^n).$$

Similarly for functions defined on R^n.

The symbol δ_{x_0} denotes a Dirac mass (atomic probability measure) supported on $\{x_0\}$.

The structure of this paper is as follows. In Section 2 we introduce the notion of a determining function, which enables us to present various results on the characterization of finite complex measures on R_+^n or R^n by means of their moments. The main application of this is provided in Section 3, where we explicitly construct a large class of finitely overcomplete sequences (sequences that remain complete even after the removal of finitely-many terms) in weighted Lebesgue spaces and also Hardy and Bergman spaces on C_+; in the latter case the sequences can be seen as perturbations of a function and its derivatives. A further application is presented in Section 4, to the determination of probability distributions by means of the moments of record values, extending results in [12].

2. Approximate Carleman theorems

DEFINITION 2.1. A function $\Phi : \mathsf{R}_+ \to \mathsf{R}$ is said to be a *determining function*, if Φ is Borel measurable with $\Phi(x) \neq 0$ for $x \neq 0$, and Φ has the property that for each $m \geq 0$ the function ψ_m given by $\psi_m(x) = x^m \Phi(x)$ is bounded and

$$(2) \qquad \sum_{m=0}^{\infty} \|\psi_m\|_\infty^{-1/(2m)} = \infty.$$

Likewise, a function $\Phi : \mathsf{R} \to \mathsf{R}$ is said to be a *determining function*, if Φ is Borel measurable with $\Phi(x) \neq 0$ for $x \neq 0$, and Φ has the property that for each $m \geq 0$ the function ψ_{2m} given by $\psi_{2m}(x) = x^{2m}\Phi(x)$ is bounded and

$$(3) \qquad \sum_{m=0}^{\infty} \|\psi_{2m}\|_{\infty}^{-1/(2m)} = \infty.$$

For example, on $[0, \infty)$, if $\Phi(x) = e^{-\lambda x^{\delta}}$ where $\lambda > 0$ and $\delta \geq \frac{1}{2}$, then $\|\psi_m\|_{\infty}^{-1/(2m)} \sim Cm^{-1/(2\delta)}$, for some constant $C > 0$, and if $\Phi(x) = e^{-(\lambda x/\log(x+2))^{1/2}}$, then $\|\psi_m\|_{\infty}^{-1/(2m)} \sim C(m \log m)^{-1}$.

Similarly, on R, the function $\Phi(x) = e^{-\lambda|x|^{\delta}}$ satisfies $\|\psi_{2m}\|_{\infty}^{-1/(2m)} \sim Cm^{-1/\delta}$, so Φ is a determining function provided that $\delta \geq 1$.

Note that, if Φ is a determining function, then so is the product $\gamma\Phi$, whenever γ is a Borel measurable function of polynomial growth such that $\gamma(x) \neq 0$ for $x \neq 0$. This follows because

$$|x^m \gamma(x)\Phi(x)| \leq C(\|\psi_m\|_{\infty} + \|\psi_{m+d}\|_{\infty}),$$

in the notation of (1).

LEMMA 2.2. *Let μ be a finite Borel measure on R_+, and Φ a determining function on R_+. Then the moments M_m, defined by*

$$M_m = \int_0^{\infty} x^m |\Phi(x)| \, d\mu(x),$$

satisfy the Carleman condition

$$(4) \qquad \sum_{m=0}^{\infty} |M_m|^{-1/(2m)} = \infty.$$

Similarly, for μ a finite Borel measure on R, and Φ a determining function on R, the moments M_m defined by

$$M_m = \int_{-\infty}^{\infty} |x|^m |\Phi(x)| \, d\mu(x),$$

satisfy the Carleman condition

$$(5) \qquad \sum_{m=0}^{\infty} |M_{2m}|^{-1/(2m)} = \infty.$$

PROOF. We have
$$|M_m| \leq \|\mu\| \|\psi_m\|_\infty,$$

from which the result follows easily.

Lin and Too [12] showed that if μ_1 and μ_2 are two absolutely continuous probability measures such that

$$\int_0^\infty x^m e^{-\lambda x} \, d\mu_1(x) = \int_0^\infty x^m e^{-\lambda x} \, d\mu_2(x) + c,$$

for sufficiently large m and for some constants c and $\lambda > 0$, then $\mu_1 = \mu_2$.

This can be put into the wider context of determining functions, and one may state a much more general theorem as follows.

THEOREM 2.3. *Let $\Omega = \mathsf{R}$ or R_+, and let μ_1, μ_2 be two finite complex Borel measures on Ω. Let Φ be a determining function on Ω. Suppose that there exist constants c and α such that for each $m \geq 0$*

$$\int_\Omega x^m \Phi(x) \, d\mu_1(x) = \int_\Omega x^m \Phi(x) \, d\mu_2(x) + c_m,$$

where c_m satisfies $\limsup |c_m - c|^{1/m} = \alpha$. Then, there exists a complex measure ν supported on $[-\alpha, \alpha] \cap \Omega$ such that

$$\mu_1 = \mu_2 + \frac{c}{\Phi(1)} \delta_1 + \nu.$$

PROOF. Set

$$d_m = \int_\Omega x^m \Phi(x) \, d \left(\mu_1 - \mu_2 - \frac{c}{\Phi(1)} \delta_1 \right)(x).$$

Then $\limsup |d_m|^{1/m} = \alpha$. Define a measure ρ by

$$d\rho(x) = \Phi(x) \, d \left(\mu_1 - \mu_2 - \frac{c}{\Phi(1)} \delta_1 \right)(x).$$

By Lemma 2.2, the numbers $M_m = \int_\Omega |x|^m d|\rho|(x)$ satisfy the Carleman condition (4) or (5), as appropriate. When the measures are signed real Borel measures, it follows from [9, Cor. 2.4] and [2, Thm. 3.1] – the general case is in [3, Thm. 3.1] – that ρ is supported on $[-\alpha, \alpha] \cap \Omega$. Since Φ does not vanish on $((-\infty, -\alpha) \cup (\alpha, \infty)) \cap \Omega$, the result now follows.

COROLLARY 2.4. *Let X and Y be two nonnegative random variables. Let Φ be a determining function on R_+. Suppose that there exist constants c and c_m ($m \geq 1$) such that*

$$E(X^m \Phi(X)) = E(Y^m \Phi(Y)) + c_m,$$

where $\limsup |c_m - c|^{1/m} = 0$. Then X and Y have the same distribution apart from atomic measures of size $\frac{c}{\Phi(1)}$ concentrated at 0 and 1. In the particular case where the distributions are purely non-atomic, they are the same.

PROOF. By Theorem 2.3, the probability measures μ_1 and μ_2 associated with X and Y satisfy $\mu_1 = \mu_2 + \frac{c}{\Phi(1)}\delta_1 + \nu$, with ν supported by $\{0\}$. Since μ_1 and μ_2 are probability measures, $\nu = -\frac{c}{\Phi(1)}\delta_0$. The corollary follows.

It is convenient to give multivariable versions of the above results.

THEOREM 2.5. *Let Ω be either R or R_+, let μ_1 and μ_2 be finite complex Borel measures on Ω^n. Let $\tilde{1} := (1, 1, \ldots, 1)$ and $\Phi_1, \ldots, \Phi_n$ be determining functions on Ω. Set $\Phi(x) = \prod_{k=1}^{n} \Phi_k(x_k)$ for $x = (x_1, \ldots, x_n) \in \Omega^n$. Suppose that*

$$\int_{\Omega^n} x^\alpha \Phi(x) \, d\mu_1(x) = \int_{\Omega^n} x^\alpha \Phi(x) \, d\mu_2(x) + c(\alpha),$$

where $c(\alpha)$ satisfies the condition that for each $\epsilon > 0$ there is a number $A_\epsilon > 0$ such that

$$|c(\alpha) - c| \leq A_\epsilon (c_1 + \epsilon)^{\alpha_1} \ldots (c_n + \epsilon)^{\alpha_n}$$

for some fixed number c and constants $c_1, \ldots, c_n$. Then there is a complex measure ν supported on $\prod_{k=1}^{n}[-c_k, c_k] \cap \Omega^n$ such that

$$\mu_1 = \mu_2 + \frac{c}{\Phi(\tilde{1})}\delta_{\tilde{1}} + \nu.$$

PROOF. Set

$$d(\alpha) = \int_{\Omega^n} x^\alpha \Phi(x) \, d\left(\mu_1 - \mu_2 - \frac{c}{\Phi(\tilde{1})}\delta_{\tilde{1}}\right)(x).$$

Then for each $\epsilon > 0$ we have

$$|d(\alpha)| \leq A_\epsilon (c_1 + \epsilon)^{\alpha_1} \ldots (c_n + \epsilon)^{\alpha_n}.$$

Define a measure ρ by

$$d\rho(x) = \Phi(x) \, d\left(\mu_1 - \mu_2 - c\frac{c}{\Phi(\tilde{1})}\delta_{\tilde{1}}\right)(x).$$

As in the proof of Theorem 2.3, consider the numbers

$$M(j, m) = \int_\Omega |x_j|^m d|\rho|(x).$$

Since Φ_j is a determining function, we see, as in the proof of Lemma 2.2, that for each j the sequence $(M(j, m))_m$ satisfies the Carleman condition. The result now follows from [3, Thm. 3.1].

The next corollary follows as in the one-dimensional case.

COROLLARY 2.6. *Given the hypotheses of Theorem 2.5, suppose that*

$$\limsup_{|\alpha| \to \infty} |c(\alpha) - c|^{1/|\alpha|} = 0.$$

Then $\mu_1 = \mu_2 + \frac{c}{\Phi(\tilde{1})}(\delta_{\tilde{1}} - \delta_{\tilde{0}})$, where $\tilde{0} = (0, 0, \ldots, 0)$. In the particular case when μ_1 and μ_2 are purely non-atomic, then they are equal.

Another straightforward corollary holds for radial weights.

COROLLARY 2.7. *Let $\Omega = \mathsf{R}$ or R_+ and let μ_1 and μ_2 be two finite complex measures on Ω^n. Let Φ be a determining function on Ω. Suppose that*

$$\int_{\Omega^n} x^\alpha \Phi(\|x\|)\, d\mu_1(x) = \int_{\Omega^n} x^\alpha \Phi(\|x\|)\, d\mu_2(x) + c(\alpha),$$

where $c(\alpha)$ satisfies the condition that for each $\epsilon > 0$ there is a number $A_\epsilon > 0$ such that
$$|c(\alpha) - c| \le A_\epsilon (c_1 + \epsilon)^{\alpha_1} \ldots (c_n + \epsilon)^{\alpha_n}$$

for some fixed number c and constants $c_1, \ldots, c_n$. Then there is a complex measure ν supported on $\prod_{k=1}^n [-c_k, c_k] \cap \Omega^n$ such that

$$\mu_1 = \mu_2 + \frac{c}{\Phi(\sqrt{n})} \delta_{\tilde{1}} + \nu.$$

PROOF. The proof follows along standard lines.

3. Finitely overcomplete sequences

In this section we derive some applications of moment problems to the theory of Banach spaces.

DEFINITION 3.1. A sequence $(x_m)_m$ in a Banach space is *finitely overcomplete* if it remains complete (that is, its linear span is dense) whenever finitely many terms are removed.

In [10] Lin showed that the sequence of functions $(x^m e^{-\lambda x})_m$ is finitely overcomplete in $L^1(\mathsf{R}_+)$ for any $\lambda > 0$. Much more can be proved using the method of moments.

THEOREM 3.2. *Let Ω be either R or R_+, let $p \in [1, \infty)$, and let Φ be a determining function on Ω such that $x \mapsto \Phi(x)^\epsilon \in L^p(\Omega)$ for some $\epsilon \in (0, 1)$. Suppose that the functions $\Psi_m \in L^p(\Omega)$ satisfy $\lim \|\Psi_m\|_p^{1/m} = 0$. Then the sequence $(f_m)_m$ given by*

$$f_m(x) = x^m \Phi(x) + \Psi_m(x),$$

is finitely overcomplete in $L^p(\Omega)$.

PROOF. Note that

$$\int_\Omega |x|^{mp} \Phi(x)^p \, dx = \int_\Omega |x|^{mp} \Phi(x)^{p(1-\epsilon)} \Phi(x)^{p\epsilon} \, dx$$

$$\leq C_m \int_\Omega \Phi(x)^{p\epsilon} \, dx,$$

where $C_m := \sup\{|x|^{mp} \Phi(x)^{p(1-\epsilon)} : x \in \Omega\} < \infty$, since Φ is determining. So the functions f_m lie in $L^p(\Omega)$. Suppose that $g \in L^q(\Omega)$ with $q = \frac{p}{p-1}$ and $\|g\|_q = 1$; if $\int_\Omega g(x) f_m(x) dx = 0$ for $m \geq m_0$, then

$$\left| \int_\Omega (x^m \Phi(x)^{1-\epsilon})(g(x)\Phi(x)^\epsilon) \, dx \right| \leq \|\Psi_m\|_p.$$

Also $g(x)\Phi(x)^\epsilon dx$ is a finite purely non-atomic measure and $x \mapsto \Phi(x)^{1-\epsilon}$ is also a determining function. Hence, by Theorem 2.3, g is zero almost everywhere. By the Hahn-Banach theorem, $(f_m)_m$ is finitely overcomplete in $L^p(\Omega)$.

On taking some examples of determining functions given after Definition 2.1, the above theorem provides a far-reaching generalization of Lin's result, which we state separately, as follows.

COROLLARY 3.3. *Let $\gamma : \mathsf{R}_+ \to \mathsf{C}$ be a function of polynomial growth such that $\gamma(x) \neq 0$ except possibly at 0, and let (Ψ_m) be an arbitrary sequence in $L^p(\mathsf{R}_+)$ such that $\lim \|\Psi_m\|_p^{1/m} = 0$. Then the functions f_m defined by*

$$f_m(x) = x^m \gamma(x) e^{-\lambda x^\delta} + \Psi_m(x)$$

form a finitely overcomplete set in $L^p(\mathsf{R}_+)$ for every p with $1 \leq p < \infty$, for each $\lambda > 0$ and $\delta \geq \frac{1}{2}$. Similarly, the functions g_m defined by

$$g_m(x) = x^m \gamma(x) e^{-\lambda \left(\frac{x}{\log(x+2)} \right)^{1/2}} + \Psi_m(x)$$

form a finitely overcomplete set in $L^p(\mathbb{R}_+)$ for each $\lambda > 0$.

Alternatively, let $\gamma : \mathbb{R} \to \mathbb{C}$ be function of polynomial growth such that $\gamma(x) \neq 0$ except possibly at 0, and let (Ψ_m) be an arbitrary sequence in $L^p(\mathbb{R})$ such that $\lim \|\Psi_m\|_p^{1/m} = 0$. Then the functions f_m defined by

$$f_m(x) = x^m \gamma(x) e^{-\lambda |x|^\delta} + \Psi_m(x)$$

form a finitely overcomplete set in $L^p(\mathbb{R})$ for every p with $1 \leq p < \infty$, for each $\lambda > 0$ and $\delta \geq 1$.

As before, it is possible to derive multivariable extensions of the above results.

THEOREM 3.4. *Let $\Omega = \mathbb{R}$ or $\mathbb{R}_+$, let $p \in [1, \infty)$ and $\Phi_1, \ldots, \Phi_n$ be determining functions on Ω. Define $\Phi : \Omega^n \to \mathbb{R}_+$ by*

$$\Phi(x) = \Phi_1(x_1) \cdots \Phi_n(x_n) \quad for \quad x = (x_1, \ldots, x_n) \in \Omega^n,$$

and suppose that $x \mapsto \Phi(x)^\epsilon \in L^p(\Omega^n)$ for some $\epsilon \in (0, 1)$. Suppose that the functions $\Psi_\alpha \in L^p(\mathbb{R}^n)$ satisfy $\limsup_{|\alpha| \to \infty} \|\Psi_\alpha\|_p^{1/|\alpha|} = 0$. Then the set $(f_\alpha)_\alpha$ given by

$$f_\alpha(x) = x^\alpha \Phi(x) + \Psi_\alpha(x),$$

is finitely overcomplete in $L^p(\Omega^n)$.

PROOF. As in the proof of Theorem 3.2, it is easily verified that the given functions lie in L^p. Once again we use duality, supposing that $g \in L^q(\Omega^n)$ satisfies

$$\int_{\Omega^n} g(x) f_\alpha(x) \, dx = 0$$

for all but finitely many α. We deduce from Theorem 2.5 that $g = 0$ almost everywhere.

This has the following immediate consequence.

COROLLARY 3.5. *Let $\gamma : \mathbb{R}_+^n \to \mathbb{C}$ be a function of polynomial growth such that $\gamma(x) \neq 0$ except possibly at 0, and let $(\Psi_\alpha)_{\alpha \in \mathbb{N}^n}$ be a family of functions in $L^p(\mathbb{R}_+^n)$ such that $\limsup_{|\alpha| \to \infty} \|\Psi_\alpha\|_p^{1/|\alpha|} = 0$. Then for every p with $1 \leq p < \infty$, and for all $\lambda_1, \ldots, \lambda_n > 0$ and $\delta_1, \ldots, \delta_n \geq \frac{1}{2}$, the functions f_α defined by*

$$f_\alpha(x) = x^\alpha \gamma(x) e^{-(\lambda_1 x_1^{\delta_1} + \cdots + \lambda_n x_n^{\delta_n})} + \Psi_\alpha(x)$$

form a finitely overcomplete set in $L^p(\mathbb{R}_+^n)$.

Alternatively, let $\gamma : \mathsf{R}^n \to \mathsf{C}$ be a function of polynomial growth such that $\gamma(x) \neq 0$ except possibly at 0, and let $(\Psi_\alpha)_{\alpha \in \mathsf{N}^n}$ be a family of functions in $L^p(\mathsf{R}^n)$ such that $\lim\sup_{|\alpha| \to \infty} \|\Psi_\alpha\|_p^{1/|\alpha|} = 0$. Then for every p with $1 \leq p < \infty$, and for all $\lambda_1, \ldots, \lambda_n > 0$ and $\delta_1, \ldots, \delta_n \geq 1$, the functions f_α defined by

$$f_\alpha(x) = x^\alpha \gamma(x) e^{-(\lambda_1 |x_1|^{\delta_1} + \cdots + \lambda_n |x_n|^{\delta_n})} + \Psi_\alpha(x)$$

form a finitely overcomplete set in $L^p(\mathsf{R})$.

Now let $\beta \geq -1$ and define the measure μ_β on R_+ by $d\mu_\beta(t) = dt/t^{\beta+1}$ for $t > 0$. We then have the following result about overcompleteness in $L^2(\mathsf{R}_+, d\mu_\beta)$.

COROLLARY 3.6. *Let Φ be a determining function on R_+ such that the function $x \mapsto \Phi(x)^\epsilon$ lies in $L^2(\mathsf{R}_+)$ for some $\epsilon \in (0, 1)$. Then the sequence of functions $(f_m)_m$ given by*

$$f_m(x) = x^m \Phi(x) + \Psi_m(x), \qquad m \in \mathsf{N}, m > \beta/2$$

is finitely overcomplete in $L^2(\mathsf{R}_+, d\mu_\beta)$ whenever

$$\lim_{m \to \infty} \|\Psi_m\|_{L^2(\mathsf{R}_+, d\mu_\beta)}^{1/m} = 0.$$

PROOF. This follows from Theorem 3.2 using the isometry

$$J : L^2(\mathsf{R}_+) \to L^2(\mathsf{R}_+, d\mu_\beta)$$

given by

$$(Jf)(x) = x^{(\beta+1)/2} f(x),$$

and noting that $x \mapsto x^\delta \Phi(x)$ is a determining function whenever $\delta > 0$ and Φ is a determining function.

By means of a multiple of the Laplace transform, depending only on β, there is an isometric isomorphism between $L^2(\mathsf{R}_+, d\mu_\beta)$ for $\beta > -1$ and the weighted Bergman space $X_\beta = A_\beta^2(\mathsf{C}_+)$ consisting of all analytic functions F on the right half-plane C_+ such that the norm

$$\|F\|_{A_\beta^2(\mathsf{C}_+)} = \left(\frac{1}{\pi} \int_{\mathsf{C}_+} |F(x + iy)|^2 x^\beta \, dx \, dy \right)^{1/2}$$

is finite. The case corresponding to $\beta = -1$ (i.e. $d\mu_\beta(t) = dt$) is the Hardy space $X_\beta = H^2(\mathsf{C}_+)$ where the norm is

$$\|F\|_{H^2(\mathsf{C}_+)} = \left(\sup_{x > 0} \int_{-\infty}^\infty |F(x + iy)|^2 \, dy \right)^{1/2}.$$

This Paley-Wiener type result can be found in [4], [5].

Under these isometric isomorphisms, the function f given by $f(t) = t^{\beta+1}e^{-\lambda t}$ with $\lambda > 0$ corresponds to a multiple of the reproducing kernel k_λ satisfying

$$\langle F, k_\lambda \rangle_{X_\beta} = F(\lambda), \qquad (F \in X_\beta).$$

Likewise $t \mapsto t^m f(t)$ corresponds to $(-1)^m$ times the mth derivative of k_λ.

THEOREM 3.7. *Let Φ be a determining function on R_+ such that $x \mapsto \Phi(x)^\epsilon \in L^2(\mathsf{R}_+)$ for some $\epsilon \in (0, 1)$. For $\beta \geq -1$, define a function k in X_β by*

$$k(z) = \int_0^\infty x^{\beta+1} \Phi(x) e^{-xz} \, dx, \qquad (z \in \mathsf{C}_+).$$

If $(G_m)_m$ is a sequence in X_β with $\lim_{m\to\infty} \|G_m\|_{X_\beta}^{1/m} = 0$, then the functions $(k^{(m)} + G_m)_m$ form a finitely overcomplete set in X_β.

PROOF. This follows from the properties of the Laplace transform as a mapping from $L^2(\mathsf{R}_+, d\mu_\beta)$ onto X_β described above together with Corollary 3.6.

The analogue of Lin's result for R, which is a special case of our Corollary 3.3, is that the functions $(x^m e^{-\lambda|x|})_m$ are finitely overcomplete in $L^2(\mathsf{R})$. By using transform methods similar to those above, we obtain a further corollary of interest.

COROLLARY 3.8. *Let Φ be a determining function on R such that $x \mapsto \Phi(x)^\epsilon \in L^2(\mathsf{R})$ for some $\epsilon \in (0, 1)$. Define a function k in $L^2(\mathsf{R})$ by*

$$k(w) = \int_{-\infty}^\infty \Phi(x) e^{-iwx} \, dx, \qquad (w \in \mathsf{R}).$$

If $(G_m)_m$ is a sequence in $L^2(\mathsf{R})$ with $\lim_{m\to\infty} \|G_m\|^{1/m} = 0$, then the functions $(k^{(m)} + G_m)_m$ form a finitely overcomplete set in $L^2(\mathsf{R})$.

PROOF. Observe that k is the Fourier transform of the function $x \to \Phi(x)$, and $i^m k^{(m)}$ is the transform of $x \mapsto x^m \Phi(x)$. The result now follows from the Plancherel theorem that the Fourier transform is a constant multiple of an isometry on $L^2(\mathsf{R})$.

Probably the most important application of Corollary 3.8 is obtained by taking $\Phi(x) = e^{-\lambda|x|}$ for $\lambda > 0$, which gives

$$k_\lambda(w) = \frac{\lambda}{\lambda^2 + w^2},$$

the Poisson kernel for the right half-plane (cf. [6, p. 123]).

4. Characterizing probability distributions

In [10], [12], Lin and Too gave an application of the moment problem to the characterization of probability distributions via the moments of record values. In view of the results of this paper, it is possible to give more general versions of their results, and this is the aim of the present section.

Suppose $(X_m)_m$ be a sequence of independent, identically-distributed non-negative random variables with continuous distribution F. Let k be a positive integer and suppose that $EX_1^p < \infty$ for some $p > k$. The record times of the X_m, denoted $L(m)$, are defined by $L(0) = 1$ and

$$L(m) = \min\{k : X_k > X_{L(m-1)}\}, \qquad (m \geq 1).$$

Then it follows that $EX_{L(m)}^k < \infty$ [11]. Let $(Y_m)_m$ be another such sequence, with continuous distribution G, and with corresponding record values $Y_{M(m)}$.

THEOREM 4.1. *With the notation as above, suppose that* $EX_{L(m)}^k = EY_{M(m)}^k + c_m/m!$, *and there is a number c such that* $|c_m - c|^{1/m} \to 0$. *Then X and Y have the same distribution.*

PROOF. We use a modification of the argument given in [12]. There it is shown that

$$EX_{L(m)}^k = \frac{1}{m!} \int_0^1 (F^{-1}(t))^k \left(\log \frac{1}{1-t}\right)^m dt,$$

and we have

$$\int_0^1 f(t) \left(\log \frac{1}{1-t}\right)^m dt = \int_0^1 g(t) \left(\log \frac{1}{1-t}\right)^m + c_m,$$

where $f(t) = (F^{-1}(t))^k$ and $g(t) = (G^{-1}(t))^k$. This becomes, by means of a change of variable $x = -\log(1-t)$,

$$\int_0^\infty x^m h(x) dx = \int_0^\infty x^m l(x) dx + c_m,$$

with $h(x) = f(t)e^{-x}$ and $l(x) = g(t)e^{-x}$. The first term can be written

$$\int_0^\infty x^m e^{-(p-1)x/p} (F^{-1}(t))^k e^{-x/p} dx,$$

where $x \mapsto e^{-(p-1)x/p}$ is a determining function and $x \mapsto (F^{-1}(t))^k e^{-x/p}$ lies in $L^1(\mathsf{R}_+)$ by Hölder's inequality. The end of the proof follows from Theorem 2.3.

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