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RINGS WITH FINITE GORENSTEIN GLOBAL DIMENSION

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Abstract

We find new classes of non noetherian rings which have the same homological behavior that Gorenstein rings.

1. Introduction and Preliminaries

In his pivotal article [3] Bass studied Gorenstein rings. Among several characterizations, he called a ring R Gorenstein if it is commutative, Noetherian and every system of parameters in R_p generates an irreducible ideal, for all prime ideals p. In the local case with finite Krull dimension, he characterized Gorenstein rings as those that satisfy a property which corresponds to a geometric property of a point on a variety. He also characterized such rings homologically by showing that these are precisely the R that have finite self-injective dimension.

Auslander (in [1]) seems to be the first who noticed the similarity of module behavior over Bass' Gorenstein rings and that over integral group rings of finite groups. Auslander showed that certain syzygy modules over Gorenstein rings have complete resolutions analogous to those exhibited by Tate and used to define Tate homology and cohomology.

Iwanaga ([12]) showed that if a ring is left and right noetherian then having finite left and right self injective dimension implies strong properties about modules over such rings. He argued that over such a ring a module has finite projective dimension if and only if it has finite injective dimension and he showed that there is a universal finite bound for such dimensions.

It has become increasing clear that the most striking homological property satisfied by the category of modules over these rings is that they have a certain relative finite global dimensions. It is natural that this dimension should be called a Gorenstein global dimension. In this paper we exhibit several classes of rings having finite Gorenstein global dimensions.

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Over this paper R will denote a not necessarily commutative ring with identity. We denote by R-Mod the category of left R-modules.

DEFINITION 1.1 ([8, Definition 2.1, Definition 5.1]). Let $M \in R$-Mod, then M is said to be *Gorenstein projective* if there exists an exact sequence of projective R-modules

$$\cdots \to P^{-1} \to P^0 \to P^1 \to \cdots \tag{1.1}$$

such that $M = \ker(P^0 \to P^1)$ and such that $\mathrm{Hom}(P, -)$ leaves the complex exact for any projective R-module P. The sequence (1.1) is often called a complete projective resolution of M. Gorenstein injective modules (and then complete injective resolutions) are defined dually.

DEFINITION 1.2. Let $M \in R$-Mod. The Gorenstein projective dimension of M ($\mathrm{Gpd}(M)$) is defined as the least integer n such that the n-syzygy of M is Gorenstein projective and ∞ if there is no such syzygy (where the syzygies are taken in a projective resolution of M). Then the Gorenstein global projective dimension, $\mathrm{glGpd}(R)$, is defined by

$$\mathrm{glGpd}(R) = \sup\{\mathrm{Gpd}(M) : \ M \in R\text{-Mod}\}.$$

Global Gorenstein injective dimension is defined dually.

For an R-module M we shall denote by $\mathrm{pd}(M)$ and $\mathrm{id}(M)$ the projective and injective dimension of M, respectively. Then we define the finitistic projective dimension $\mathrm{FPD}(R)$ and the finitistic injective dimension $\mathrm{FID}(R)$ as

$$\mathrm{FPD}(R) = \sup\{\mathrm{pd}(M) : \mathrm{pd}(M) < \infty\}$$

and

$$\mathrm{FID}(R) = \sup\{\mathrm{id}(M) : \ \mathrm{id}(M) < \infty\}.$$

THEOREM 1.3 ([6, Theorem 2.28]). *The following are equivalent for a ring R:*

(i) $\mathrm{glGpd}(R) < \infty$ *and* $\mathrm{glGid}(R) < \infty$.

(ii) *for an R-module M,* $\mathrm{pd}(M) < \infty \Leftrightarrow \mathrm{id}(M) < \infty$ *and* $\mathrm{FPD}(R) < \infty$ *and* $\mathrm{FID}(R) < \infty$.

Moreover if one of these two equivalent conditions hold then

$$\mathrm{FPD}(R) = \mathrm{FID}(R) = \mathrm{glGpd}(R) = \mathrm{glGid}(R).$$

DEFINITION 1.4. If R is a ring satisfying one of the conditions of Theorem 1.3 then $\mathrm{glGdim}(R)$ stands for the common value of $\mathrm{glGid}(R)$, $\mathrm{glGpd}(R)$, $\mathrm{FPD}(R)$ and $\mathrm{FID}(R)$.

In [6] it is shown that for a ring R with both finite Gorenstein global dimensions there is a nice program of the so-called Gorenstein homological algebra. The corresponding categories of R-modules are what is known in the literature as Gorenstein categories. These were first studied in the interesting work of Beligiannis [4] and Beligiannis-Reiten [5]. They are also studied in [6] in the context of categories of quasi-coherent sheaves, that is for Grothendieck categories where there may not be enough projectives.

If $\mathrm{glGdim}(R)$ is finite, by [6, Theorem 2.26], we do have Gorenstein projective precovers and therefore for every R-module M there exists a deleted resolution of M by Gorenstein projectives

$$\mathbf{G}_M = \cdots \to G_2 \to G_1 \to G_0 \to 0$$

which is unique up to homotopy, so it defines right derived functors of Hom (see the introduction of [13] for a full explanation of these results). Those are commonly denoted by $\mathrm{Gext}^i(M, N)$. By [6, Theorem 2.25] there are Gorenstein injective preenvelopes for every R-module N, so right derived functors of Hom can be defined from those. Furthermore in [10] (see also [9] for a version over Gorenstein rings) it is shown that both procedures give the same functors Gext^i, that is, there is balance in this situation.

Now if $\mathrm{glGdim}(R) = n < \infty$ then for an R-module M there is a complete projective resolution $\mathbf{P}_M$ attached to the Gorenstein projective n-syzygy of M. Again this resolution is a homotopy invariant so it defines the so-called Tate cohomology groups $\widehat{\mathrm{Ext}}^i(M, N)$ as the ith-cohomology groups of $\mathrm{Hom}(\mathbf{P}_M, N)$, ($i \in \mathbb{Z}$). These groups can also be computed by using a complete injective resolution of an n-cosyzygy of N, as it was noticed in [11].

As a consequence of the previous comments we get that the Avramov-Martsinkovsky's long exact sequence (see [2, 7.1 Theorem]) connecting the three theories (the Ext^i, Gext^i and $\widehat{\mathrm{Ext}}^i$ functors) still holds for R-Mod where $\mathrm{glGdim}(R)$ is finite.

THEOREM 1.5 (Avramov-Martsinkovsky). *Let R be a ring with* $\mathrm{glGdim}(R) = n$.

There exist natural exact sequences

$$0 \to \mathrm{Gext}^1(M, N) \to \mathrm{Ext}^1(M, N) \to \widehat{\mathrm{Ext}}^1(M, N) \to \mathrm{Gext}^2(M, N) \to \\ \cdots \to \mathrm{Gext}^n(M, N) \to \mathrm{Ext}^n(M, N) \to \widehat{\mathrm{Ext}}^n(M, N) \to 0.$$

As a first examples of rings with finite Gorenstein global dimension we get that every Iwanaga-Gorenstein ring (that is, a ring R noetherian on both sides and with finite self-injective dimension) has finite Gorenstein global dimension equal to the self-injective dimension (see for example [9, Proposition 11.2.5, Proposition 11.5.7 and Theorem 9.1.10]). But there are many non-trivial examples of non-noetherian rings having finite Gorenstein global dimension (by a trivial example we mean a ring R with finite left global dimension).

So this paper is devoted to the study of examples of these rings and to prove that the property of the finiteness of the Gorenstein dimension is inherited by extensions of rings such as polynomial and series rings and quasi-Frobenius extensions. As a remarkable example we prove in Theorem 2.7 that the ring with infinitely many non-commuting indeterminates $R\{X_1, X_2, \cdots\}$ has finite Gorenstein global dimension whenever R has such. For the ring of dual numbers $R[X]/(X^2)$ we explicitly describe the finite Gorenstein injective and projective resolutions of every module (Theorem 3.5). We note that if $R \neq 0$, $R[X]/(X^2)$ always has infinite left global dimension.

2. Polynomial and series rings

Let M be an R-module. We will denote by $M[[X^{-1}]]$ the $R[X]$-module $\mathrm{Hom}_R({}_R R[X]_{R[X]}, {}_R M)$ whose action by X is given by the "shift" operator $X(m_0 + m_1 X^{-1} + m_2 X^{-2} + m_3 X^{-3} + \cdots) = m_1 + m_2 X^{-1} + m_3 X^{-2} + \cdots$. These $R[X]$ modules were first introduced by Macaulay, and later Northcott used this notation in [16].

LEMMA 2.1. *Let M be an $R[X]$-module. Then there is a short exact sequence of $R[X]$-modules*

$$0 \to M \to M[[X^{-1}]] \to M[[X^{-1}]] \to 0.$$

As a consequence, if E is an $R[X]$-module which is injective as R-module then $\mathrm{id}_{R[X]}\, E \leq 1$.

PROOF. We define $\alpha : M \to M[[X^{-1}]]$ to be the (unique) morphism of $R[X]$-modules such that $\beta \circ \alpha = \mathrm{id}_{{}_R M}$ where $\beta : M[[X^{-1}]] \to M$ is the morphism of R-modules $\beta(m_0 + m_1 X^{-1} + m_2 X^{-2} + m_3 X^{-3} + \cdots) = m_0$. So $\alpha(e) = e + XeX^{-1} + X^2 e X^{-2} + \cdots$, $e \in E$. So we have the exact sequence of $R[X]$-modules

$$0 \to M \xrightarrow{\alpha} M[[X^{-1}]] \to \frac{M[[X^{-1}]]}{\alpha(M)} \to 0.$$

Now we define a morphism of $R[X]$-modules $\frac{M[[X^{-1}]]}{\alpha(M)} \to M[[X^{-1}]]$,

$$e_0 + e_1 X^{-1} + e_2 X^{-2} + \cdots + \alpha(E) \\ \mapsto Xe_0 - e_1 + (Xe_1 - e_2)X^{-1} + (Xe_2 - e_3)X^{-2} + \cdots.$$

It is straightforward to check that this is well defined and is an isomorphism of $R[X]$-modules.

COROLLARY 2.2. *If M is an $R[X]$-module and if* $\mathrm{id}_R M \leq n$ *then* $\mathrm{id}_{R[X]} M \leq n+1$.

PROOF. Immediate.

We recall that for an $R[X]$-module M there is an analogous exact sequence $0 \to M[X] \to M[X] \to M \to 0$ of $R[X]$-modules which gives that $\mathrm{pd}_R M \leq n \Rightarrow \mathrm{pd}_{R[X]} \leq n+1$.

THEOREM 2.3. *Let R be a ring with* $\mathrm{glGdim}(R) = n$. *Then* $\mathrm{glGdim}(R[X]) = n+1$.

PROOF. Since $R[X]$ is a projective R-module we get that $\mathrm{pd}_{R[X]} M \leq n \Rightarrow \mathrm{pd}_R M \leq n$ and $\mathrm{id}_{R[X]} M \leq n \Rightarrow \mathrm{id}_R M \leq n$. So now suppose that $\mathrm{pd}_{R[X]} < \infty$. Then $\mathrm{pd}_R M < \infty$ and so $\mathrm{pd}_R M \leq n$. This gives that $\mathrm{id}_R M \leq n$ and so that $\mathrm{pd}_{R[X]} \leq n+1$ and that $\mathrm{id}_{R[X]} \leq n+1$. So $\mathrm{pd}_{R[X]} M < \infty \Rightarrow \mathrm{pd}_{R[X]} M \leq n+1$, $\mathrm{id}_{R[X]} M \leq n+1$. Similarly we get $\mathrm{id}_{R[X]} M < \infty \Rightarrow \mathrm{id}_{R[X]} M \leq n+1$, $\mathrm{pd}_{R[X]} M \leq n+1$ So we get that $\mathrm{glGdim}(R[X]) \leq n+1$. If $N \neq 0$ is an R-module with $\mathrm{pd}_R N = n$ and if we make N into an $R[X]$-module with $XN = 0$ then it is standard that $\mathrm{pd}_R[X]N = n+1$. This gives that we have the equality $\mathrm{glGdim}(R[X]) = n+1$.

PROPOSITION 2.4. *If R is right coherent and if* $\mathrm{glGdim}(R) = n$ *then* $\mathrm{glGdim}(R[[X]]) = n+1$.

PROOF. We have $R[[X]] \cong R^{\mathsf{N}}$ as left R-module. Since R is right coherent the product of flat left R-modules is flat. Hence $R[[X]]$ is a flat left R-module. Consequently any injective left $R[[X]]$-module is injective as an R-module. Now let L be an $R[[X]]$-module with $\mathrm{id}_{R[[X]]} L < \infty$. Then by the above $\mathrm{id}_R L < \infty$ and so $\mathrm{pd}_R L \leq n$. By Theorem 2.3 and the first change of rings Theorem ([18, Theorem 4.3.3]) $\mathrm{pd}_{R[[X]]} L = 1 + \mathrm{pd}_R L \leq n+1$.

If $\mathrm{pd}_{R[[X]]} L < \infty$ then $\mathrm{pd}_R L < \infty$. Then $\mathrm{id}_R L \leq n$.

By the first injective change of rings Theorem ([18, pp. 104]) $\mathrm{id}_{R[[X]]} L = 1 + \mathrm{id}_R L \leq n+1$.

REMARK. Theorem 2.3 can also be proved by using change of rings Theorem. But Lemma 2.1 has independent interest so we have opted for an independent proof of Theorem 2.3 for using it.

COROLLARY 2.5. *Let R be a ring with* $\mathrm{glGdim}(R) = n$. *Then:*

(1) $\mathrm{glGdim}(R[X_1, \ldots, X_k]) = n+k$, *for all* $k \geq 1$.

(2) *If* $R[[X_1, \dots, X_{k-1}]]$ *is right coherent then* $\mathrm{glGdim}(R[[X_1 \dots, X_k]]) = n + k$, *for all* $k \geq 1$.

PROOF. This is consequence of the previous results (Theorem 2.3 and Proposition 2.4) and the known fact that if L is an $R[X_1, \dots, X_k]$-module with finite $\mathrm{pd}_R L$ then

$$\mathrm{pd}_{R[X_1,\dots,X_k]} L = k + \mathrm{pd}_R L, \quad \text{and} \quad \mathrm{id}_{R[X_1,\dots,X_k]} L = k + \mathrm{id}_R L.$$

Let us see now that the ring $R\{X_1, X_2, \dots\}$ in the non-commuting indeterminates $X_1, X_2, \dots$ with coefficients in R has finite Gorenstein global dimension. So we will have an "absolutely non-noetherian" example of a ring with finite global Gorenstein dimension. We start with the following remark.

LEMMA 2.6.

$$\mathrm{lgldim}(R\{X_1, X_2, \dots\}) = 1 + \mathrm{lgldim}(R)$$

for non commuting indeterminates $X_1, X_2, \dots$.

PROOF. This is a consequence of [14, Theorem 14]. Notice that the category $R\{X_1, X_2, \cdots\}$-Mod is equivalent with the functor category $R\text{-Mod}^Q$ where Q is a quiver with one vertex v and a countable number of loops beginning and ending in v.

REMARK. We note that the corresponding result with commuting indeterminates is

$$\mathrm{lgldim}(R[X_1, \dots, X_k]) = k + \mathrm{lgldim}(R).$$

THEOREM 2.7. *Let R be a ring with* $\mathrm{glGdim}(R) = n$. *Then* $\mathrm{glGdim}(R\{X_1, X_2, \dots\}) = n + 1$.

To prove this theorem we will appeal to the following lemma. Given ${}_RN$ denote

$$R\{X_1, X_2, \dots\} \otimes_R N$$

by $N\{X_1, X_2, \dots\}$.

LEMMA 2.8. *For a given* ${}_{R\{X_1,X_2,\dots\}}M$ *there exists a short exact sequence*

$$0 \to M\{X_1, X_2, \dots\}^{(\omega_0)} \to M\{X_1, X_2, \dots\} \to M \to 0.$$

PROOF. By restriction of scalars M is an R-module. Let us consider

$$M\{X_1, X_2, \dots\} \cong R\{X_1, X_2, \dots\} \otimes_R M.$$

Then any R-linear $M \to N$ where N is left $R\{X_1, X_2, \ldots\}$-module (and so an R-module) has a unique extension

$$R\{X_1, X_2, \ldots\} \otimes_R M \cong M\{X_1, X_2, \ldots\} \to M.$$

Applying this to $\mathrm{id} : M \to M$ gives $\varphi : M\{X_1, X_2, \ldots\} \to M$ where, for example, $\varphi(mX_i) = X_i m$ for $m \in M$ and $i \geq 1$ and where $X_i m$ is computed using the original scalar multiplication in M. For all $k \geq 1$, we define a family of $R\{X_1, X_2, \ldots\}$-morphisms $\psi_i : M\{X_1, X_2, \ldots\} \to M\{X_1, X_2, \ldots\}$, from the R-linear maps $\epsilon_i : M \to M\{X_1, X_2, \ldots\}$ defined by $m \overset{\epsilon_i}{\mapsto} X_i m - mX_i$. So hence we get an $R\{X_1, X_2, \ldots\}$-linear map

$$\psi : M\{X_1, X_2, \ldots\}^{(\omega_0)} \longrightarrow M\{X_1, X_2, \ldots\}.$$

Using a natural grading on $M\{X_1, X_2, \ldots\}$ it is not hard to argue that this map is an injection. Since each $X_i m - mX_i \in \mathrm{Ker}(\varphi)$, $i \geq 1$ it is easy to see that the $\mathrm{Im}(\psi) \subseteq \mathrm{Ker}(\varphi)$. We claim that

$$0 \to M\{X_1, X_2, \ldots\}^{(\omega_0)} \overset{\psi}{\to} M\{X_1, X_2, \ldots\} \overset{\varphi}{\to} M \to 0$$

is exact. Let us take $u \in \mathrm{Ker}(\varphi)$. We use Gauss' notation $\equiv \ (\mathrm{Im}(\psi))$. Using the relations

$$X_i m \equiv mX_i \ (\mathrm{Im}(\psi)), \qquad \forall i \geq 1$$

and the ones that follow as a consequence (for example $mX_1X_2X_1 \equiv X_1X_2X_1m \ (\mathrm{Im}(\psi))$) we see that in fact every $v \in M\{X_1, X_2, \ldots\}$ satisfies $v \equiv m' \ (\mathrm{Im}(\psi))$ for some $m' \in M$. So if $u \equiv m \ (\mathrm{Im}(\psi))$ is in $\mathrm{Ker}(\varphi)$, m will be also in $\mathrm{Ker}(\varphi)$. But for all $m' \in M$, $\varphi(m') = m'$ so then $m = 0$ and then $u \equiv 0 \ (\mathrm{Im}(\psi))$, so $u \in \mathrm{Im}(\psi)$.

PROOF OF THEOREM 2.7. Let L be an $R\{X_1, X_2, \ldots\}$-module such that $\mathrm{id}_{R\{X_1, X_2, \ldots\}} L < \infty$. If M is an R-module, we have canonical isomorphisms $R\{X_1, X_2, \ldots\} \otimes_R M \cong M\{X_1, X_2, \ldots\}$ and $L \cong \mathrm{Hom}_{R\{X_1, X_2, \ldots\}}(R\{X_1, X_2, \ldots\}, L)$ so

$$\mathrm{Hom}_R(M, L) \cong \mathrm{Hom}_{R\{X_1, X_2, \ldots\}}(M\{X_1, X_2, \ldots\}, L),$$

for all R-module M and all $R\{X_1, X_2, \ldots\}$-module L. Therefore if E is an injective $R\{X_1, X_2, \ldots\}$-module, it is injective as R-module. So $\mathrm{id}_R L < \infty$. Since $\mathrm{glGdim}(R) = n$, $\mathrm{pd}_R L \leq n$. Let

$$0 \to K \to P_{n-1} \to \cdots \to P_1 \to P_0 \to L \to 0$$

be an exact sequence of $R\{X_1, X_2, \ldots\}$-modules with ${}_{R\{X_1,X_2,\ldots\}}P_i$ projective $0 \le i \le n-1$ and K projective as R-module. By Lemma 2.8 we have the short exact sequence

$$0 \to K\{X_1, X_2, \ldots\}^{(w_0)} \to K\{X_1, X_2, \ldots\} \to K \to 0.$$

Since $K\{X_1, X_2, \ldots\}$ is a projective $R\{X_1, X_2, \ldots\}$-module it follows that $\mathrm{pd}_{R\{X_1,X_2,\ldots\}} L \le n+1$. Conversely assume that ${}_{R\{X_1,X_2,\ldots\}}L$ is such that $\mathrm{pd}_{R\{X_1,X_2,\ldots\}} L < \infty$. Then $\mathrm{pd}_R L < \infty$. Again, since $\mathrm{glGdim}(R) = n$, $\mathrm{id}_R L \le n$. So $\mathrm{Ext}_R^{n+i}(M, L) = 0$, $\forall {}_R M$. Now we follow the usual procedure to compute the Ext_R^j functors by using a resolution of M of projective R-modules and the previous isomorphism to get:

$$\mathrm{Ext}_R^j(M, L) \cong \mathrm{Ext}^j_{R\{X_1,X_2,\ldots\}}(M\{X_1, X_2, \ldots\}, L)$$

$\forall j \ge 0$, so $\mathrm{Ext}^{n+i}_{R\{X_1,X_2,\ldots\}}(M\{X_1, X_2, \ldots\}, L) = 0, \forall i \ge 1$. Now from the short exact sequence of Lemma 2.8 we have the long exact sequence of homology

$$\begin{aligned}\cdots \to 0 = \mathrm{Ext}^{n+1}_{R\{X_1,X_2,\ldots\}}(M\{X_1, X_2, \ldots\}, L)^{\omega_0} \\ \cong \mathrm{Ext}^{n+1}_{R\{X_1,X_2,\ldots\}}(M\{X_1, X_2, \ldots\}^{(\omega_0)}, L) \to \mathrm{Ext}^{n+2}_{R\{X_1,X_2,\ldots\}}(M, L) \\ \to \mathrm{Ext}^{n+2}_{R\{X_1,X_2,\ldots\}}(M\{X_1, X_2, \ldots\}, L) = 0 \to \cdots.\end{aligned}$$

So $\mathrm{Ext}^{n+k}_{R\{X_1,X_2,\ldots\}}(M, L) = 0$, $\forall k \ge 2$ and hence $\mathrm{id}_{R\{X_1,X_2,\ldots\}} L \le n+1$.

From the previous it is clear that $\mathrm{FPD}(R\{X_1, X_2, \ldots\}), \mathrm{FID}(R\{X_1, X_2, \ldots\}) \le n+1$. If $\mathrm{pd}_R N = n$ for an R-module $N \ne 0$, then since $R\{X_1\} = R[X_1]$ we have $\mathrm{pd}_{R\{X_1\}} N = n+1$ where $X_1 N = 0$. So then $\mathrm{pd}_{R\{X_1,X_2,\ldots\}} N \ge n+1$ where $X_i N = 0$ for each i. Since this dimension is finite, we see that we get the desired equality.

3. Quasi-frobenius extensions

In [12] Iwanaga proved that a quasi-Frobenius extension of a Gorenstein ring R is also Gorenstein. In this section we prove the analogous result for rings with finite Gorenstein global dimensions. We first recall from [12, Section 2] the definition of a quasi-Frobenius extension. They first appeared in [15].

DEFINITION 3.1. For rings $R \subseteq T$, T/R is called a left quasi-Frobenius extension if ${}_R T$ is finitely generated projective and ${}_T T_R$ is isomorphic to a direct summand in a direct sum of copies of $\mathrm{Hom}_R({}_R T_R, {}_R R_R)$. A quasi-Frobenius extension is a left and right quasi-Frobenius extension.

THEOREM 3.2. *Let $T \supseteq R$ be a quasi-Frobenius extension of R. Then if* $\mathrm{glGdim}(R) = n$ *then* $\mathrm{glGdim}(T) = n$.

PROOF. We show that $\mathrm{pd}_T M < \infty \Rightarrow \mathrm{id}_T M < \infty$. The class of T-modules of finite injective dimension at most n is resolving (which means that for a short exact sequence of T-modules $0 \to A \to B \to C \to 0$ if $\mathrm{id}_T A \leq n$ and $\mathrm{id}_T B \leq n$ then $\mathrm{id}_T C \leq n$) so, since $\mathrm{pd}_T M < \infty$, it suffices to prove that $\mathrm{id}_T T^{(I)} \leq n$ for an arbitrary set I. By [12, Proposition 5]

$$\mathrm{id}_T(T^{(I)}) = \mathrm{id}_T((T \otimes R)^{(I)}) = \mathrm{id}_T(T \otimes R^{(I)}) \leq \mathrm{id}_R(R^{(I)}) \leq n.$$

Conversely we have to check that $\mathrm{id}_T M < \infty \Rightarrow \mathrm{pd}_T M < \infty$. Since $T^+ = \mathrm{Hom}_{\mathsf{Z}}(T, \mathsf{Q}/\mathsf{Z})$ is an injective cogenerator of T-Mod and now the class of T-modules with finite projective dimension is coresolving (the dual of resolving) it suffices to check that $\mathrm{pd}_T (T^+)^I \leq n$. Since T is a direct summand of $\mathrm{Hom}_R(T, R)$, T^+ will be a direct summand of $\mathrm{Hom}_R(T, R)^+$. So it suffices to prove that $\mathrm{pd}_T (\mathrm{Hom}_R(T, R)^+)^I \leq n$. Since $(R^+)^I$ is injective as R-module and R is $\mathrm{glGdim}(R) = n$, $\mathrm{pd}_R (R^+)^I \leq n$ so there exits an exact sequence

$$0 \to P_n \to \cdots \to P_0 \to (R^+)^I \to 0$$

with P_i projective R-modules $0 \leq i \leq n$. But then

$$0 \to T \otimes_R P_n \to \cdots \to T \otimes_R P_0 \to T \otimes_R (R^+)^I \to 0$$

is a finite projective resolution of T-modules of $T \otimes_R (R^+)^I$ (T is flat as R-module). Now

$$\begin{aligned} T \otimes_R (R^+)^I &\cong {}_T\mathrm{Hom}_R({}_R\mathrm{Hom}({}_T T_R, {}_R R_R)_T, {}_R(R^+)^I) \\ &\cong \mathrm{Hom}_{\mathsf{Z}}(R^{(I)} \otimes_R \mathrm{Hom}_R({}_T T_R, {}_R R_R)_T, \mathsf{Q}/\mathsf{Z}) \\ &\cong \mathrm{Hom}_{\mathsf{Z}}({}_T\mathrm{Hom}_R({}_T T_R, {}_R R_R)_R^{(I)}, \mathsf{Q}/\mathsf{Z}) \\ &\cong \big({}_T\mathrm{Hom}_R({}_R T_R, {}_R R_R)_R^{(I)}\big)^+ \\ &\cong {}_T(\mathrm{Hom}_R(T, R)^+)^I. \end{aligned}$$

As a particular case of Theorem 3.2 we get:

COROLLARY 3.3. *If G is a finite group and R is such that* $\mathrm{glGdim}(R) = n$ *then* $\mathrm{glGdim}(RG) = n$.

PROOF. This is immediate by noticing that $R \subseteq RG$ is a left quasi-Frobenius extension.

3.1. The ring of dual numbers

For any ring R, the ring of dual numbers $R[X]/(X^2)$ is a quasi-Frobenius extension of R. So by Theorem 3.2, if $\mathrm{glGdim}(R)$ is finite, $\mathrm{glGdim}(R[X]/(X^2)) = \mathrm{glGdim}(R)$. In this section we shall describe the finite Gorenstein projective and Gorenstein injective resolutions of an $R[X]/(X^2)$-module, when R is a ring with finite left global dimension. We recall that if $R \neq 0$ the left global dimension of $R[X]/(X^2)$ is infinite.

For the next result we need to remark the following: let $\varphi : R \to S$ be a ring homomorphism, then if ${}_R M$ is a left R-module, $\mathrm{Hom}_R({}_R S_S, {}_R M)$ is a left S-module, and we have the R-linear $\mathrm{Hom}_R(S, M) \to M$ ($\sigma \mapsto \sigma(1)$) with such that for any R-linear $N \to M$ there exists a unique S-linear map such that the diagram

$$\begin{tikzcd} & {}_S N \arrow[dl, dashed, "\exists!\, S\text{-linear}"'] \arrow[d, "R\text{-linear}"] \\ \mathrm{Hom}_R(S, M) \arrow[r] & M \end{tikzcd}$$

is commutative. Dually we have the obvious diagram

$$\begin{tikzcd} {}_R M \arrow[r] \arrow[dr, "R\text{-linear}"'] & S \otimes_R M \arrow[d, dashed, "\exists!\, S\text{-linear}"] \\ & {}_S N \end{tikzcd}$$

Let us denote the ring $R[X]/(X^2)$ simply by $R[x]$, with the understanding that $x^2 = 0$. So the elements are $r_0 + r_1 x$, $r_0, r_1 \in R$. Let M be an $R[x]$-module. Then we can define a map $\beta : M[x] \to M$ given by $m + m'x \mapsto m + xm'$. This map is surjective and $R[x]$-linear. The kernel consists of $\ker(\beta) = \{xm - mx : m \in M\}$. Now we define a new structure of $R[x]$-module on M: for $x \in R[x]$ $x \cdot m = -xm$ and let us denote by M^0 the $R[x]$-module M with the new structure. Then $M^0 \cong \ker(\beta)$ (as $R[x]$-modules) by defining $\alpha : M^0 \to M[x]$, $\alpha(m) = xm - mx$. So we have an exact sequence

$$0 \to M^0 \xrightarrow{\alpha} M[x] \xrightarrow{\beta} M \to 0$$

for any ${}_{R[x]}M$ and also an exact

$$0 \to M \to M^0[x] \to M^0 \to 0.$$

So "pasting" we get an exact complex

$$\cdots \to M[x] \to M^0[x] \to M[x] \to M \to 0$$

Dually, given ${}_RM$ we define $M[x^{-1}] = \{m_0 + m_1x^{-1} : m_0, m_1 \in M\} \cong \mathrm{Hom}_R(R[x], M)$. Then $M[x^{-1}]$ is an $R[x]$-module where $x(m_0 + m_1x^{-1}) = m_1$. We have a monomorphism of left $R[x]$-modules $0 \to M \to M[x^{-1}]$ given by $m \mapsto m + (xm)x^{-1}$. The cokernel is isomorphic to M^0. Notice that, for all ${}_RM$ there is an isomorphism of $R[x]$-modules $M[x] \cong M[x^{-1}]$ given by $m_0 + m_1x \mapsto m_1 + m_0x^{-1}$.

LEMMA 3.4. *Let M be an $R[x]$-module and $\alpha : M \to M[x]$ $\alpha(m) = xm - mx$ the previous monomorphism of $R[x]$-modules. Then for any projective $R[x]$-module Q and any $R[x]$-linear map $\delta : M \to Q$ there exists an $R[x]$-linear map $\gamma : M[x] \to Q$ such that $\gamma\alpha = \delta$.*

PROOF. Clearly it suffices to prove the result for a free $R[x]$ module. First we assume that $Q = R[x]$. It is clear that $M[x] \cong \alpha(M) \oplus M$ and $R[x] \cong R_1 \oplus R_2$ as R-modules ($R_i = R$, $i = 1, 2$). Then $\delta = (\delta_1, \delta_2)$. Then there exists $\eta_1 : M[x] \to R_1$ and $\eta_2 : M[x] \to R_2$ such that $\eta_1\alpha = \delta_1$ and $\eta_2\alpha = \delta_2$ and such that $\eta_1(M) = 0$, $\eta_2(M) = 0$. But then by the universal property described above there exists a unique morphism of $R[x]$-modules $\gamma : M[x] \to R[x]$ such that $\gamma\alpha = \delta$ and $\gamma(M) = 0$.

Now let us take $Q = R[x]^{(I)}$. By proceeding as before we find $\gamma : M[x] \to R[x]^I$ such that $\gamma\alpha = \delta$. Let us see that in fact $\gamma(M[x]) \subseteq R[x]^{(I)}$. But this is easy by noticing that if $S \subseteq M$ is a submodule then

$$\begin{tikzcd}
S \arrow[r, "\alpha|_S"] \arrow[d, hook] & S[x] \arrow[d, hook] \\
M \arrow[r, "\alpha"] \arrow[d] & M[x] \arrow[dl] \\
R[x] &
\end{tikzcd}$$

is commutative and the extension $S[x] \to R[x]$ is the restriction of the extension $M[x] \to R[x]$ and also satisfies that $S \to R[x]$ is 0. But then if $S \subseteq M$ is finitely generated whenever $S \to R[x]^{(I)} \to R[x]$ is 0 (and it will be 0 except for a finite number of $i \in I$) the extension $S[x] \to R[x]$ will be 0. So we do get a map $S[x] \to \mathrm{Hom}(R[x], R)^{(I)}$. Then we see we will also get a map $M[x] \to \mathrm{Hom}(R[x], R)^{(I)}$ for any M (by restrictions to all finitely generated $S \subseteq M$ and using the above).

THEOREM 3.5. *Let R be a ring such that* $\mathrm{lgldim}(R) = n$ *and* $M \in R[x]$-Mod. *There exists a finite Gorenstein projective resolution in* $R[x]$-Mod *of M,*

$$0 \to G_n \to G_{n-1} \to \cdots \to G_1 \to G_0 \to M \to 0$$

where G_n is projective as R-module. Dually M has a finite Gorenstein injective resolution in $R[x]$-Mod*,*

$$0 \to M \to U_0 \to U_1 \to \cdots \to U_{n-1} \to U_n \to 0$$

where U_n is injective as R-module.

PROOF. Let M be any $R[x]$-module and consider the short exact sequence

$$0 \to K_0 \to P_0 \to M \to 0$$

with P_0 a projective $R[x]$-module. Since $\mathrm{lgldim}(R) = n$, $\mathrm{pd}_R K_0 \leq n - 1$. Proceeding in this manner we get an exact sequence of $R[x]$-modules

$$0 \to K \to P_{n-1} \to \cdots \to P_0 \to M \to 0$$

such that ${}_{R[x]}P_i$ is projective $\forall 1 \leq i \leq n-1$ and ${}_{R[x]}K$ is projective as R-module. We show that K is Gorenstein projective. By the preceding, we get a short exact sequence of $R[x]$-modules

$$0 \to K^0 \to K[x] \to K \to 0$$

and then, by pasting, an exact

$$\cdots \to K^0[x] \to K[x] \to K^0[x] \to K[x] \to K \to 0.$$

Then we also get a similar exact

$$0 \to K \to K[x^{-1}] \to K^0[x^{-1}] \to K[x^{-1}] \to \cdots.$$

So we have a complete projective resolution of K,

$$\begin{aligned} \cdots \to K^0[x] \to K[x] \to K^0[x] \to K[x] \\ \to K[x^{-1}] \to K^0[x^{-1}] \to K[x^{-1}] \to \cdots. \end{aligned}$$

By Lemma 3.4, $\mathrm{Hom}_{R[x]}(-, Q)$ remains exact the sequence, for all projective ${}_{R[x]}Q$, so

$$\mathrm{Gpd}_{R[x]} M \leq n.$$

Let $0 \to M \to E^0 \to E^1 \to \ldots \to E^{n-1} \to D \to 0$ be a partial injective resolution of M over $R[x]$. Any injective $R[x]$-module is an injective R-module so this is also a partial injective resolution over R. Since $\operatorname{lgldim}(R) = n$ it follows that D is injective as an R-module. By the above we have an exact complex

$$\begin{aligned}\mathscr{D} : \cdots \to D^0[x] \to D[x] \to D^0[x] \\ \to D[x] \to D[x^{-1}] \to D^0[x^{-1}] \to \cdots\end{aligned}$$

such that $D = \operatorname{Ker}(D[x^{-1}] \to D^0[x^{-1}])$. Since ${}_R D$ is injective and $R[x]$ is a flat R-module we have that $D[x] \cong D[x^{-1}] \cong \operatorname{Hom}_R(R[x], D)$ are injective $R[x]$-modules ([9, Theorem 3.2.9]) therefore also injective over R. The exact sequence $0 \to D \to D[x^{-1}] \to D^0 \to 0$ gives us that D^0 is an injective R-module. Then $D^0[x]$ is an injective $R[x]$-module. So $\mathscr{D}$ is an exact complex of injective $R[x]$-modules. We show that $\operatorname{Hom}_{R[x]}(T, \mathscr{D})$ is exact for any injective ${}_{R[x]}T$. Since $\operatorname{Ext}^1_{R[x]}(T^0[x], D^0) \cong \operatorname{Ext}^1_R(T^0, D^0) = 0$ it follows that the sequence

$$\begin{aligned}(3.1) \quad 0 \to \operatorname{Hom}_{R[x]}(T^0[x], D^0) \\ \to \operatorname{Hom}_{R[x]}(T^0[x], D[x]) \to \operatorname{Hom}_{R[x]}(T^0[x], D) \to 0\end{aligned}$$

is exact.

But T is an injective $R[x]$-module, so the sequence $0 \to T \to T^0[x] \to T^0 \to 0$ is split exact. Since T is a direct summand of $T^0[x]$, (3.1) gives that any diagram

$$\begin{tikzcd} & T \arrow[d] \arrow[dl, dashed] \\ D[x] \arrow[r] & D \end{tikzcd}$$

can be completed to a commutative one. Thus D is a Gorenstein injective $R[x]$-module. So $\operatorname{Gid}_{R[x]} M \leq n$.

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REFERENCES

1. Auslander, M., *Anneaux de Gorenstein et torsion en algèbre commutative*, séminaire d'algèbre commutative, Ecole Normale Supérieure de Jeunes Filles, Paris 1967.
2. Avramov, L., and Martsinkovsky, A., *Absolute, relative, and Tate cohomology of modules of finite Gorenstein dimension*, Proc. London Math. Soc. 85(2) (2002), 393–440.
3. Bass, H., *On the ubiquity of Gorenstein rings*, Math. Z. 82 (1963), 8–28.
4. Beligiannis, A., *The homological theory of contravariantly finite subcategories: Gorenstein categories, Auslander-Buchweitz contexts and (co-)stabilization*, Comm. Algebra. 28 (2000), 4547–4596.
5. Beligiannis, A., and Reiten, I., *Homological and Homotopical Aspects of Torsion Theories*, Mem. Amer. Math. Soc. 188 (2007), no. 883.
6. Enochs, E., Estrada, S., and García Rozas, J. R., *Gorenstein categories and Tate cohomology on projective schemes*, to appear in Math. Nachr.
7. Enochs, E., Estrada, S., García Rozas, J. R., and Iacob, A., *Gorenstein quivers*, Arch. Math. (Basel) 88(3) (2007) 199–206.
8. Enochs, E., and Jenda, O., *Gorenstein injective and projective modules*, Math. Z. 220 (1995), 611–633.
9. Enochs, E., and Jenda, O., *Relative Homological Algebra*, de Gruyter Exp. Math. 30 (2000).
10. Holm, H., *Gorenstein derived functors*, Proc. Amer. Math. Soc. 132(7) (2004), 1913–1923.
11. Iacob, A., *Generalized Tate cohomology*, Tsukuba J. Math. 29(2) (2005), 389–404.
12. Iwanaga, Y., *On rings with finite self-injective dimension II*, Tsukuba J. Math. 4 (1980), 107–113.
13. Jørgensen, P., *Existence of Gorenstein projective resolutions and Tate cohomology*, J. Eur. Math. Soc. 9 (2007), 59–76.
14. Mitchell, B., *Projectives in functor categories*, J. Algebra 69 (1981), 24–31.
15. Müller, B. J., *Quasi-Frobenius-Erweiterungen*, Math. Z. 85 (1964), 345–368.
16. Northcott, D., *Injective envelopes and inverse polynomials*, J. London Math. Soc. 8 (1974), 290–296.
17. Stenström, B., *Rings of Quotients*, Springer-Verlag, 1975.
18. Weibel, C. A., *An Introduction to Homological Algebra*, Cambridge Stud. Adv. Math. 38 (1994).

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