MATH. SCAND. 102 (2008), 59–72

ALTERNATING GROUP ACTIONS ON SPIN 4-MANIFOLDS

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Abstract

Let X be a smooth, closed, connected spin 4-manifold with $b_1(X) = 0$ and non-positive signature $\sigma(X)$. In this paper we use Seiberg-Witten theory to prove that if X admits a spin alternating A_5 action, then $b_2^+(X) \geq |\sigma(X)|/8 + 3$ under some non-degeneracy conditions.

1. Introduction

Let X be a smooth, closed, connected spin 4-manifold. We denote by $b_2(X)$ the second Betti number and denote by $\sigma(X)$ the signature of X. In [12], Y. Matsumoto conjectured the following inequality

$$b_2(X) \geq \frac{11}{8}|\sigma(X)|. \tag{1}$$

This conjecture is well known and has been called the $\frac{11}{8}$-conjecture (see also [7]).

From the classification of unimodular even integral quadratic forms and the Rochlin's theorem, for the choice of orientation with non-positive signature the intersection form of a closed spin 4-manifold X is

$$-2kE_8 \oplus mH, \qquad k \geq 0,$$

where E_8 is the 8×8 intersection form matrix and H is the hyperbolic matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Thus, $m = b_2^+(X)$ and $k = -\sigma(X)/16$ and so the inequality (1) is equivalent to $m \geq 3k$. Since $K3$ surface satisfies the equality with $k = 1$ and $m = 3$, the coefficient $\frac{11}{8}$ is optimal, if the $\frac{11}{8}$-conjecture is true.

Donaldson has proved that if $k > 0$ then $m \geq 3$ [4]. In early 1995, using the Seiberg-Witten theory introduced by Seiberg and Witten [17], Furuta [8]

* This work is supported by NSFC (no. 10771023) and NCET.
Received May 19, 2006.

proved that

$$b_2(X) \geq \frac{5}{4}|\sigma(X)| + 2. \tag{2}$$

This estimate has been dubbed the $\frac{10}{8}$-theorem. In fact, if the intersection form of X is definite, i.e., $m = 0$, then Donaldson proved that $b_2(X)$ and $\sigma(X)$ are zero [4], [5]. Thus, Furuta assumed that m is not zero. Inequality (2) follows by a surgery argument from the non-positive signature, $b_1(X) = 0$ case:

THEOREM 1.1 (Furuta [8]). *Let X be a smooth spin 4-manifold with $b_1(X) = 0$ with non-positive signature. Let $k = -\sigma(X)/16$ and $m = b_2^+(X)$. Then,*

$$2k + 1 \leq m$$

if $m \neq 0$.

His key idea is to use a finite dimensional approximation of the monopole equations. Later Furuta and Kametani [9] used equivariant e-invariants and improved the above $\frac{10}{8}$-theorem as following.

THEOREM 1.2 (Furuta and Kametani [9]). *Suppose that X is a closed oriented spin 4-manifold. If $\sigma(X) < 0$, then*

$$b_2^+(X) \geq \begin{cases} 2(-\sigma(X)/16) + 1, & -\sigma(X)/16 \equiv 0, 1 \pmod 4, \\ 2(-\sigma(X)/16) + 2, & -\sigma(X)/16 \equiv 2 \pmod 4, \\ 2(-\sigma(X)/16) + 1, & -\sigma(X)/16 \equiv 3 \pmod 4. \end{cases}$$

The above inequality was also proved by N. Minami [13] by using an equivariant join theorem to reduce the inequality to a theorem of Stolz [16].

Throughout this paper we will assume that m is not zero and $b_1(X) = 0$, unless stated otherwise.

A $Z/2^p$-action is called a spin action if the generator of the action $\tau\colon X \to X$ lifts to an action $\hat{\tau}\colon P_{\mathrm{Spin}} \to P_{\mathrm{Spin}}$ of the Spin bundle P_{Spin}. Such an action is of even type if $\hat{\tau}$ has order 2^p and is of odd type if $\hat{\tau}$ has order 2^{p+1}.

In [2], Bryan (see also [6]) used Furuta's technique of finite dimensional approximation and the equivariant K-theory to improve the above bound by p under the assumption that X has a spin odd type $Z/2^p$-action satisfying some non-degeneracy conditions analogous to the condition $m \neq 0$. More precisely, he proved

THEOREM 1.3 (Bryan [2]). *Let X be a smooth, closed, connected spin 4-manifold with $b_1(X) = 0$. Assume that $\tau\colon X \to X$ generates a spin smooth*

$Z/2^p$-action of odd type. Let X_i denote the quotient of X by $Z/2^i \subset Z/2^p$. Then

$$2k + 1 + p \leq m$$

if $m \neq 2k + b_2^+(X_1)$ and $b_2^+(X_i) \neq b_2^+(X_j) > 0$ for $i \neq j$.

In the paper [10], Kim gave the same bound for smooth, spin, even type $Z/2^p$-action on X satisfying some non-degeneracy conditions analogous to Bryan's.

In the paper [11], Kiyono and Liu prove that if a spin 4-manifold X admits a spin alternating group A_4 action, then $b_2^+(X) \geq |\sigma(X)|/8 + 3$ under some non-degeneracy conditions.

In this article, we will use the same techniques to study the spin alternating group A_5 actions spin 4-manifolds, we obtain some improvement of the 10/8-theory as in [11].

We organize the remainder of this paper as follows. In section 2, we give some preliminaries to prove the main theorem. We introduce the representation ring and the character table of alternating group A_5 in section 3. In section 4, we use equivariant K-theory and representation theory to study the G-equivariant properties of the moduli space. In the last section we give our main results.

2. Notations and preliminaries

We assume that we have completed every Banach spaces with suitable Sobolev norms. Let $S = S^+ \oplus S^-$ denote the decomposition of the spinor bundle into the positive and negative spinor bundles. Let $D\colon \Gamma(S^+) \to \Gamma(S^-)$ be the Dirac operator, and $\rho\colon \Lambda_C^* \to \mathrm{End}_C(S)$ be the Clifford multiplication. The Seiberg-Witten equations are for a pair $(a, \phi) \in \Omega^1(X, \sqrt{-1}R) \times \Gamma(S^+)$ and they are

$$D\phi + \rho(a)\phi = 0, \qquad \rho(d^+a) - \phi \otimes \phi^* + \frac{1}{2}|\phi|^2 id = 0, \qquad d^*a = 0.$$

Let

$$V = \Gamma\big(\sqrt{-1}\Lambda^1 \oplus S^+\big),$$
$$W' = \Gamma\big(S^- \oplus \sqrt{-1}\,\mathrm{su}(S^+) \oplus \sqrt{-1}\Lambda^0\big).$$

We can think of the equation as the zero set of a map

$$\mathscr{D} + \mathscr{Q}\colon V \to W,$$

where $\mathscr{D}(a, \phi) = (D\phi, \rho(d^+a), d^*a))$, $\mathscr{Q}(a, \phi) = (\rho(a)\phi, \phi\otimes\phi^* - \frac{1}{2}|\phi|^2 id, 0)$, and W is defined to be the orthogonal complement to the constant functions in W'.

Now it is time to describe the group of symmetries of the equations. Define $\mathrm{Pin}(2) \subset \mathrm{SU}(2)$ to be the normalizer of $S^1 \subset \mathrm{SU}(2)$. Regarding $\mathrm{SU}(2)$ as the group of unit quaternions and taking S^1 to be elements of the form $e^{\sqrt{-1}\theta}$, $\mathrm{Pin}(2)$ then consists of the form $e^{\sqrt{-1}\theta}$ or $e^{\sqrt{-1}\theta}J$. We define the action of $\mathrm{Pin}(2)$ on V and W as follows: Since S^+ and S^- are $\mathrm{SU}(2)$ bundles, $\mathrm{Pin}(2)$ naturally acts on $\Gamma(S^\pm)$ by multiplication on the left. $Z/2$ acts on $\Gamma(\Lambda^*_C)$ by multiplication by ± 1 and this pulls back to an action of $\mathrm{Pin}(2)$ by the natural map $\mathrm{Pin}(2) \to Z/2$. A calculation shows that this pullback also describes the induced action of $\mathrm{Pin}(2)$ on $\sqrt{-1}\,\mathrm{su}(S^+)$. Both $\mathscr{D}$ and $\mathscr{Q}$ are seen to be $\mathrm{Pin}(2)$ equavariant maps.

If X is a smooth closed spin 4-manifold. Suppose that X admits a spin structure preserving action by a compact Lie group G. We may assume a Riemannian metric on X so that G acts by isometries. If the action is of even type, Both $\mathscr{D}$ and $\mathscr{Q}$ are $\tilde{G} = \mathrm{Pin}(2) \times G$ equavariant maps.

Now we define V_λ to be the subspace of V spanned by the eigenspaces of $\mathscr{D}^*\mathscr{D}$ with eigenvalues less than or equal to $\lambda \in R$. Similarly, define W_λ using $\mathscr{D}\mathscr{D}^*$. The virtual G-representation $[V_\lambda \otimes C] - [W_\lambda \otimes C] \in R(\tilde{G})$ is the $\tilde{G}$-index of $\mathscr{D}$ and can be determined by the $\tilde{G}$-index and is independent of $\lambda \in R$, where $R(\tilde{G})$ is the complex representation of $\tilde{G}$. In particular, since $V_0 = \mathrm{Ker}\, D$ and $W_0 = \mathrm{Coker}\, D \oplus \mathrm{Coker}\, d^+$, we have

$$[V_\lambda \otimes C] - [W_\lambda \otimes C] = [V_0 \otimes C] - [W_0 \otimes C] \in R(\tilde{G}).$$

Note that $\mathrm{Coker}\, d^+ = H^2_+(X, R)$.

3. The alternating group A_5

The alternating group A_5 is the group of even permutations of a set $\{a, b, c, d, e\}$ with 5 elements, it has 60 elements which can be divided into the following 5 conjugacy classes:

(1) the identity element 1;
(2) 15 elements of order 2 which is conjugate with $x = (ab)(cd)$;
(3) 20 elements of order 3 which is conjugate with $t = (abc)$;
(4) 12 elements of order 5 which is conjugate with $s = (abcde)$;
(5) 12 elements of order 5 which is conjugate with $s^2 = (abced)$.

Thus we have the following character table for A_5 [15]:

	1	t	x	s	s^2
ρ_0	1	1	1	1	1
ρ_1	3	0	-1	$1+\omega+\omega^4$	$1+\omega^2+\omega^3$
ρ_2	3	0	-1	$1+\omega^2+\omega^3$	$1+\omega+\omega^4$
ρ_3	4	1	0	-1	-1
ρ_4	5	-1	1	0	0

where $\omega = e^{2\pi i/5}$.

Let X be a smooth closed spin 4-manifold. Suppose that X admits a spin structure preserving action by a compact Lie group G. We may assume a Riemannian matric on X so that G acts by isometries. This G-action can always be lifted to $\hat{G}$-actions on the spinor bundles, where $\hat{G}$ is the following extension

$$1 \to Z_2 \to \hat{G} \to G \to 1.$$

Recall that the G-action is of even type if $\hat{G}$ contains a subgroup isomorphic to G, and in turn is of odd type, otherwise. For alternating group A_5, the extension of A_5 by Z_2 is isomorphic to $Z_2 \times A_5$, that is any spin alternating group A_5 action on spin 4-manifolds is of even type.

4. The index of $\mathscr{D}$ and the character formula for the K-theory degree

The virtual representation $[V_{\lambda,C}] - [W_{\lambda,C}] \in R(\tilde{G})$ is the same as $\mathrm{Ind}(\mathscr{D}) = [\ker \mathscr{D}] - [\mathrm{Coker}\,\mathscr{D}]$. Furuta determines $\mathrm{Ind}(\mathscr{D})$ as a $\mathrm{Pin}(2)$ representation; denoting the restriction map $r\colon R(\tilde{G}) \to R(\mathrm{Pin}(2))$, Furuta shows

$$r(\mathrm{Ind}(\mathscr{D})) = 2kh - m\tilde{1}$$

where $k = -\sigma(X)/16$ and $m = b_2^+(X)$. Thus $\mathrm{Ind}(\mathscr{D}) = sh - t\tilde{1}$, where s and t are polynomials such that $s(1) = 2k$ and $t(1) = m$. For a spin A_5 action, $\tilde{G} = \mathrm{Pin}(2) \times A_5$, we can write

$$s(\rho_1, \rho_2, \rho_3, \rho_4) = a_0 + b_0\rho_1 + c_0\rho_2 + d_0\rho_3 + e_0\rho_4,$$

and

$$t(\rho_1, \rho_2, \rho_3, \rho_4) = a_1 + b_1\rho_1 + c_1\rho_2 + d_1\rho_3 + e_1\rho_4,$$

such that $a_0 + 3b_0 + 3c_0 + 4d_0 + 5e_0 = 2k$ and $a_1 + 3b_1 + 3c_1 + 4d_1 + 5e_1 = m = b_2^+(X)$.

For any element $g \in A_5$, denote by $\langle g\rangle$ the subgroup of A_5 generated by g. Then we have

$$\begin{aligned}
\dim(H^+(X)^{A_5}) &= a_1 = b_2^+(X/A_5),\\
\dim(H^+(X)^{\langle(abc)\rangle}) &= a_1 + b_1 + c_1 + 2d_1 + e_1 = b_2^+(X/\langle(abc)\rangle),\\
\dim(H^+(X)^{\langle(ab)(cd)\rangle}) &= a_1 + b_1 + c_1 + 2d_1 + 3e_1 = b_2^+(X/\langle(ab)(cd)\rangle)),\\
\dim(H^+(X)^{\langle(abcde)\rangle}) &= a_1 + b_1 + c_1 + e_1 = b_2^+(X/\langle(abcde)\rangle),\\
\dim(H^+(X)^{\langle(abced)\rangle}) &= a_1 + b_1 + c_1 + e_1 = b_2^+(X/\langle(abced)\rangle).
\end{aligned}$$

The Thom isomorphism theory in equivariant K-theory for a general compact Lie group is a deep theory proved using elliptic operator [1]. The subsequent character formula of this section uses only elementary properties of the Bott class.

Let V and W be complex Γ representations for some compact Lie group Γ. Let BV and BW denote balls in V and W and let $f\colon BV \to BW$ be a Γ-map preserving the boundaries SV and SW. $K_\Gamma(V)$ is by definition $K_\Gamma(BV, SV)$, and by the equivariant Thom isomorphism theorem, $K_\Gamma(V)$ is a free $R(\Gamma)$ module with generator the Bott class $\lambda(V)$. Applying the K-theory functor to f we get a map

$$f^* : K_\Gamma(W) \to K_\Gamma(V)$$

which defines a unique element $\alpha_f \in R(\Gamma)$ by the equation $f^*(\lambda(W)) = \alpha_f \cdot \lambda(V)$. The element α_f is called the K-theory degree of f.

Let V_g and W_g denote the subspaces if V and W fixed by an element $g \in \Gamma$ and let $V_g^\perp$ and $W_g^\perp$ be the orthogonal complements. Let $f^g\colon V_g \to W_g$ be the restriction of f and let $d(f^g)$ denote the ordinary topological degree of f^g (by definition, $d(f^g) = 0$ if $\dim V_g \neq \dim W_g$). For any $\beta \in R(\Gamma)$, let $\lambda_{-1}\beta$ denote the alternating sum $\sum(-1)^i\lambda^i\beta$ of exterior powers.

T. tom Dieck proves the following character formula for the degree α_f:

THEOREM ([3]). *Let $f\colon BV \to BW$ be a Γ-map preserving boundaries and let $\alpha_f \in R(\Gamma)$ be the K-theory degree. Then*

$$\operatorname{tr}_g(\alpha_f) = d(f^g)\operatorname{tr}_g(\lambda_{-1}(W_g^\perp - V_g^\perp))$$

where tr_g *is the trace of the action of an element $g \in \Gamma$.*

This formula is very useful in the case where $\dim V_g \neq \dim W_g$ so that $d(f^g) = 0$.

Recall that $\lambda_{-1}\big(\sum_i a_i r_i\big) = \prod_i(\lambda_{-1}r_i)^{a_i}$ and that for a one dimensional representation r, we have $\lambda_{-1}r = (1 - r)$. A two dimensional representation such as h has $\lambda_{-1}h = (1 - h + \Lambda^2 h)$. In this case, since h comes from an SU(2) representation, $\Lambda^2 h = \det h = 1$ so $\lambda_{-1}h = (2 - h)$.

In the following by using the character formula to examine the K-theory degree α_{f_λ} of the map $f_\lambda\colon BV_{\lambda,C} \to BW_{\lambda,C}$ coming from the Seiberg-Witten equations. We will abbreviate α_{f_λ} as α and $V_{\lambda,C}$ and $W_{\lambda,C}$ as just V and W. Let $\phi \in S^1 \subset \mathrm{Pin}(2) \subset G$ be the element generating a dense subgroup of S^1, and recall that there is the element $J \in \mathrm{Pin}(2)$ coming from the quaternion. Note that the action of J on h has two invariant subspaces on which J acts by multiplication with $\sqrt{-1}$ and $-\sqrt{-1}$.

5. The main results

Consider $\alpha = \alpha_{f_\lambda} \in R(\mathrm{Pin}(2) \times A_5)$, it has the following form

$$\alpha = \alpha_0 + \tilde{\alpha_0}\tilde{1} + \sum_{i=1}^{\infty} \alpha_i h_i,$$

where $\alpha_i = l_i + m_i\rho_1 + n_i\rho_2 + q_i\rho_3 + r_i\rho_4$, $i \geq 0$ and $\tilde{\alpha_0} = \tilde{l_0} + \tilde{m}_0\rho_1 + \tilde{n}_0\rho_2 + \tilde{q}_0\rho_3 + \tilde{r}_0\rho_4$.

Since ϕ acts non-trivially on h and trivially on $\tilde{1}$, so

$$\begin{aligned}\dim(V(\rho_1,\rho_2,\rho_3,\rho_4))_\phi &- \dim(W(\rho_1,\rho_2,\rho_3,\rho_4))_\phi \\ &= -(a_1 + 3b_1 + 3c_1 + 4d_1 + 5e_1) = -m = -b_2^+(X).\end{aligned}$$

So if $b_2^+(X) > 0$, $\mathrm{tr}_\phi\,\alpha = 0$.

Since ϕt acts non-trivially on $V(\rho_1,\rho_2,\rho_3,\rho_4)h$, ϕ acts trivially on $\tilde{1}$ and t acts trivially on a_1 and the actions of t on $b_1\rho_1$, $c_1\rho_2$, $e_1\rho_4$ all have a 1-dimensional invariant subspace while the action of t on $d_1\rho_3$ has a 2-dimensional invariant subspace. So we have

$$\begin{aligned}\dim(V(\rho_1,\rho_2,\rho_3,\rho_4))_{\phi t} &- \dim(W(\rho_1,\rho_2,\rho_3,\rho_4))_{\phi t} \\ &= -(a_1 + b_1 + c_1 + 2d_1 + e_1) = -b_2^+(X/\langle t\rangle).\end{aligned}$$

So if $a_1 + b_1 + c_1 + 2d_1 + e_1 = b_2^+(X/\langle t\rangle) \neq 0$, $\mathrm{tr}_{\phi t}\,\alpha = 0$.

Since ϕx acts non-trivially on $V(\rho_1,\rho_2,\rho_3,\rho_4)h$ and ϕ acts trivially on $\tilde{1}$, on the other hand, ϕx acts trivially on $\tilde{1}$ and the action of x on $b_1\rho_1$, $c_1\rho_2$ both have a 1-dimensional invariant subspace while the action of x on $d_1\rho_3$ with a 2-dimensional invariant subspace and the action of x on $e_1\rho_4$ with a 3-dimensional invariant subspace. So we have

$$\begin{aligned}\dim(V(\rho_1,\rho_2,\rho_3,\rho_4))_{\phi x} &- \dim(W(\rho_1,\rho_2,\rho_3,\rho_4))_{\phi x} \\ &= -(a_1 + b_1 + c_1 + 2d_1 + 3e_1) = -b_2^+(X/\langle x\rangle).\end{aligned}$$

So if $a_1+b_1+c_1+2d_1+3e_1=b_2^+(X/\langle x\rangle)\neq 0$, then $\operatorname{tr}_{\phi x}\alpha=0$.

Since ϕs acts non-trivially on $V(\rho_1,\rho_2,\rho_3,\rho_4)h$ and ϕ acts trivially on $\tilde{1}$ while s acts trivially on a_1 and the actions of s on $b_1\rho_1$, $c_1\rho_2$ and $e_1\rho_4$ all have a 1-dimensional invariant subspace. So we have

$$\begin{aligned}\dim(V(\rho_1,\rho_2,\rho_3,\rho_4))_{\phi s}&-\dim(W(\rho_1,\rho_2,\rho_3,\rho_4))_{\phi s}\\&=-(a_1+b_1+c_1+e_1)=-b_2^+(X/\langle s\rangle).\end{aligned}$$

For the same reason, we have

$$\begin{aligned}\dim(V(\rho_1,\rho_2,\rho_3,\rho_4))_{\phi s^2}&-\dim(W(\rho_1,\rho_2,\rho_3,\rho_4))_{\phi s^2}\\&=-(a_1+b_1+c_1+e_1)=-b_2^+(X/\langle s^2\rangle).\end{aligned}$$

So if $a_1+b_1+c_1+e_1=b_2^+(X/\langle s\rangle)=b_2^+(X/\langle s^2\rangle)\neq 0$, then $\operatorname{tr}_{\phi s}\alpha=\operatorname{tr}_{\phi s^2}\alpha=0$.

In summary, if $a_1+b_1+c_1+e_1=b_2^+(X/\langle s\rangle)\neq 0$, we have $\operatorname{tr}_\phi\alpha=\operatorname{tr}_{\phi t}\alpha=\operatorname{tr}_{\phi x}\alpha=\operatorname{tr}_{\phi s}\alpha=\operatorname{tr}_{\phi s^2}\alpha=0$ which implies that

$$\begin{aligned}0=\operatorname{tr}_\phi\alpha&=\operatorname{tr}_\phi\Big(\alpha_0+\tilde{\alpha_0}\tilde{1}+\sum_{i=1}^{\infty}\alpha_i h_i\Big)\\&=\operatorname{tr}_\phi\alpha_0+\operatorname{tr}_\phi\tilde{\alpha_0}+\sum_{i=1}^{\infty}\operatorname{tr}_\phi\alpha_i(\phi^i+\phi^{-i})\\&=(l_0+3m_0+3n_0+4q_0+5r_0)+(\tilde{l_0}+3\tilde{m_0}+3\tilde{n_0}+4\tilde{q_0}+5\tilde{r_0})\\&\qquad+\sum_{i=1}^{\infty}\operatorname{tr}_\phi\alpha_i(\phi^i+\phi^{-i}),\end{aligned}$$

$$\begin{aligned}0=\operatorname{tr}_{\phi t}\alpha&=\operatorname{tr}_t\Big(\alpha_0+\tilde{\alpha_0}\tilde{1}+\sum_{i=1}^{\infty}\alpha_i(\phi^i+\phi^{-i})\Big)\\&=(l_0+q_0-r_0)+(\tilde{l_0}+\tilde{q_0}-\tilde{r_0})+\sum_{i=1}^{\infty}\operatorname{tr}_t\alpha_i(\phi^i+\phi^{-i}),\end{aligned}$$

$$\begin{aligned}0=\operatorname{tr}_{\phi x}\alpha&=\operatorname{tr}_x\Big(\alpha_0+\tilde{\alpha_0}\tilde{1}+\sum_{i=1}^{\infty}\alpha_i(\phi^i+\phi^{-i})\Big)\\&=(l_0-m_0-n_0+r_0)+(\tilde{l_0}-\tilde{m_0}-\tilde{n_0}+\tilde{r_0})+\sum_{i=1}^{\infty}\operatorname{tr}_t\alpha_i(\phi^i+\phi^{-i}),\end{aligned}$$

and so on. From these equations we have $\alpha_0 = -\tilde{\alpha_0}$ and $\alpha_i = 0, i > 0$, that is $\alpha = \alpha_0(1 - \tilde{1})$.

Next we calculate $\operatorname{tr}_J \alpha$. Since J acts non-trivially on both h and $\tilde{1}$, $\dim V_J = \dim W_J = 0$, so $d(f^J) = 1$ and the character formula gives $\operatorname{tr}_J(\alpha) = \operatorname{tr}_J(\lambda_{-1}(m\tilde{1} - 2kh) = \operatorname{tr}_J((1-\tilde{1})^m(2-h)^{-2k}) = 2^{m-2k}$ using $\operatorname{tr}_J h = 0$ and $\operatorname{tr}_J \tilde{1} = -1$.

Now we calculate $\operatorname{tr}_{Jt} \alpha$. Since Jt acts non-trivially on both $V(\rho_1, \rho_2, \rho_3, \rho_4)h$ and $W(\rho_1, \rho_2, \rho_3, \rho_4)\tilde{1}$, so $d(f^{Jt}) = 1$. By tom Dieck formula, we have

$$\begin{aligned}
\operatorname{tr}_{Jt}(\alpha) &= \operatorname{tr}_{Jt}[\lambda_{-1}(a_1 + b_1\rho_1 + c_1\rho_2 + d_1\rho_3 + e_1\rho_4)\tilde{1} \\
&\qquad - \lambda_{-1}(a_0 + b_0\rho_1 + c_0\rho_2 + d_0\rho_3 + e_0\rho_4)h] \\
&= \frac{2^{a_1}[2(1+\varepsilon)(1+\varepsilon^2)]^{b_1}[2(1+\varepsilon^2)(1+\varepsilon)]^{c_1}[2^2(1+\varepsilon)(1+\varepsilon^2)]^{d_1}[2(1+\varepsilon)^2(1+\varepsilon^2)^2]^{e_1}}{2^{a_0}[2(1+\varepsilon^2)(1+\varepsilon)]^{b_0}[2(1+\varepsilon)(1+\varepsilon^2)]^{c_0}[2^2(1+\varepsilon^2)(1+\varepsilon)]^{d_0}[2(1+\varepsilon^2)^2(1+\varepsilon)^2]^{e_0}} \\
&= 2^{(a_1+b_1+c_1+2d_1+e_1)-(a_0+b_0+c_0+2d_0+e_0)}
\end{aligned}$$

Here the 2-dimensional representation h decomposes into two complex lines on which J acts as $\sqrt{-1}$ and $-\sqrt{-1}$. And the 3-dimensional representation ρ_1 decomposes into three complex lines on which t acts as 1, ε and ε^2 where $\varepsilon = e^{2\pi i/3}$. The 3-dimensional representation ρ_2 decomposes into three complex lines on which t acts as 1, ε^2 and ε. The 4-dimensional representation ρ_3 decomposes into four complex lines on which t acts as 1, 1, ε, ε^2. The 5-dimensional representation ρ_4 decomposes into five complex lines on which t acts as 1, ε, ε, ε^2, ε^2. J acts on $\tilde{1}$ as -1.

Since Jx acts non-trivially on $V(\rho_1, \rho_2, \rho_3, \rho_4)h$ and $\tilde{1}$ while the actions of Jx on $b_1\rho_1, c_1\rho_2, d_1\rho_3$ and $e_1\rho_4$ all have two 1-dimensional invariant subspace. So we have

$$\begin{aligned}
\dim(V(\rho_1, \rho_2, \rho_3, \rho_4))_{Jx} &- \dim(W(\rho_1, \rho_2, \rho_3, \rho_4))_{Jx} \\
&= -(2b_1 + 2c_1 + 2d_1 + 2e_1).
\end{aligned}$$

Then if $b_1 + c_1 + d_1 + e_1 \neq 0$, that is $b_2^+(X/\langle s\rangle) + b_2^+(X/\langle t\rangle) \neq 2b_2^+(X/A_5)$, we have $\operatorname{tr}_{Jx} \alpha = 0$.

Since Js acts non-trivially on both $V(\rho_1, \rho_2, \rho_3, \rho_4)h$ and $W(\rho_1, \rho_2, \rho_3, \rho_4)\tilde{1}$, then

$$\dim(V(\rho_1, \rho_2, \rho_3, \rho_4))_{Js} - \dim(W(\rho_1, \rho_2, \rho_3, \rho_4))_{Js} = 0,$$

and $d(f^{Js}) = 1$. From tom Dieck formula, we have

$$\begin{aligned}
\operatorname{tr}_{Js}(\alpha) &= \operatorname{tr}_{Js}[\lambda_{-1}(a_1 + b_1\rho_1 + c_1\rho_2 + d_1\rho_3 + e_1\rho_4)\tilde{1} \\
&\qquad - \lambda_{-1}(a_0 + b_0\rho_1 + c_0\rho_2 + d_0\rho_3 + e_0\rho_4)h] \\
&= 2^{a_1}[2(1+\omega)(1+\omega^4)]^{b_1}[2(1+\omega^2)(1+\omega^3)]^{c_1} \\
&\quad [(1+\omega)(1+\omega^2)(1+\omega^3)(1+\omega^4)]^{d_1} \\
&\quad [2(1+\omega)(1+\omega^2)(1+\omega^3)(1+\omega^4)]^{e_1} \\
&\quad 2^{-a_0}[2(1+\omega^2)(1+\omega^3)]^{-b_0}[2(1+\omega^4)(1+\omega)]^{-c_0} \\
&\quad [(1+\omega^2)(1+\omega^4)(1+\omega)(1+\omega^3)]^{-d_0} \\
&\quad [2(1+\omega^2)(1+\omega^4)(1+\omega)(1+\omega^3)]^{-e_0} \\
&= 2^{(a_1+b_1+c_1+e_1)-(a_0+b_0+c_0+e_0)}[(1+\omega)(1+\omega^4)]^{b_1-c_0} \\
&\quad [(1+\omega^2)(1+\omega^3)]^{c_1-b_0}.
\end{aligned}$$

For the same reason, we have

$$\begin{aligned}
\operatorname{tr}_{Js^2}(\alpha) &= \frac{2^{a_1+b_1+c_1+e_1}[(1+\omega)(1+\omega^4)]^{c_1}[(1+\omega^2)(1+\omega^3)]^{b_1}}{2^{a_0+b_0+c_0+e_0}[(1+\omega^2)(1+\omega^3)]^{c_0}[(1+\omega^4)(1+\omega)]^{b_0}} \\
&= 2^{(a_1+b_1+c_1+e_1)-(a_0+b_0+c_0+e_0)} \\
&\quad [(1+\omega^4)(1+\omega)]^{c_1-b_0}[(1+\omega^2)(1+\omega^3)]^{b_1-c_0}.
\end{aligned}$$

By direct calculation, we have

$$\operatorname{tr}_J \alpha_0 = l_0 + 3m_0 + 3n_0 + 4q_0 + 5r_0 = 2^{m-2k-1}, \tag{3}$$

$$\operatorname{tr}_t \alpha_0 = l_0 + q_0 - r_0 = 2^{(a_1+b_1+c_1+2d_1+e_1)-(a_0+b_0+c_0+2d_0+e_0)-1}, \tag{4}$$

$$\operatorname{tr}_x \alpha_0 = l_0 - m_0 - n_0 + r_0 = 0. \tag{5}$$

$$\begin{aligned}
\operatorname{tr}_s \alpha_0 &= l_0 + (1+\omega+\omega^4)m_0 + (1+\omega^2+\omega^3)n_0 - q_0 \\
&= 2^{(a_1+b_1+c_1+e_1)-(a_0+b_0+c_0+e_0)-1} \\
&\quad [(1+\omega)(1+\omega^4)]^{b_1-c_0}[(1+\omega^2)(1+\omega^3)]^{c_1-b_0}
\end{aligned} \tag{6}$$

$$\begin{aligned}
\operatorname{tr}_s^2 \alpha_0 &= l_0 + (1+\omega^2+\omega^3)m_0 + (1+\omega+\omega^4)n_0 - q_0 \\
&= 2^{(a_1+b_1+c_1+e_1)-(a_0+b_0+c_0+e_0)-1} \\
&\quad [(1+\omega^2)(1+\omega^3)]^{b_1-c_0}[(1+\omega)(1+\omega^4)]^{c_1-b_0}
\end{aligned} \tag{7}$$

Here we use $0 = \operatorname{tr}_{Jx}\alpha = \operatorname{tr}_x(2\cdot\alpha_0) = 2\cdot\operatorname{tr}_x\alpha_0$ and so on.

From (3) and (5) we get

$$l_0 + q_0 + 2r_0 = 2^{m-2k-3}. \tag{8}$$

So we have the following main result.

THEOREM 5.1. *Let X be a smooth spin 4-manifold with $b_1(X) = 0$ and non-positive signature. Let $k = -\sigma(X)/16$ and $m = b_2^+(X)$. If X admits a spin alternating group A_5 action, then $2k + 3 \leq m$ if $b_2^+(X/\langle s\rangle) + b_2^+(X/\langle t\rangle) \neq 2b_2^+(X/A_5)$ and $b_2^+(X/\langle s\rangle) \neq 0$.*

On the other hand, from (4) and (8) we get

$$3r_0 = 2^{m-2k-3} - 2^{(a_1+b_1+c_1+2d_1+e_1)-(a_0+b_0+c_0+2d_0+e_0)-1} \in 3Z \subset Z$$

Then from Theorem 5.1 we know 2^{m-2k-3}, so

$$2^{(a_1+b_1+c_1+2d_1+e_1)-(a_0+b+0+c_0+2d_0+e_0)-1} \in Z,$$

that is

$$(a_1 + b_1 + c_1 + 2d_1 + e_1) - (a_0 + b + 0 + c_0 + 2d_0 + e_0) - 1 \geq 0.$$

Thus we obtain the following result

PROPOSITION 5.1. *Let X be a smooth spin 4-manifold with $b_1(X) = 0$ and non-positive signature. If X admits a spin alternating group A_5 action, then*

$$\dim((\mathrm{Ind}_{A_5} D)^{\langle t\rangle}) + 1 \leq b_2^+(X/\langle t\rangle)$$

if $b_2^+(X/\langle s\rangle) + b_2^+(X/\langle t\rangle) \neq 2b_2^+(X/A_5)$ and $b_2^+(X/\langle s\rangle) \neq 0$.

Now we suppose $b_2^+(X/A_5) > 0$ and $b_2^+(X/\langle s\rangle) + b_2^+(X/\langle t\rangle) = 2b_2^+(X/A_5)$, then we have $b_2^+(X/\langle s\rangle) > 0$ and $b_1 = c_1 = d_1 = e_1 = 0$. In this case, we have the following equations

$$\mathrm{tr}_J\, \alpha_0 = l_0 + 3m_0 + 3n_0 + 4q_0 + 5r_0 = 2^{m-2k-1}, \tag{9}$$

$$\mathrm{tr}_t\, \alpha_0 = l_0 + q_0 - r_0 = 2^{a_1-(a_0+b_0+c_0+2d_0+e_0)-1}, \tag{10}$$

$$\mathrm{tr}_x\, \alpha_0 = l_0 - m_0 - n_0 + r_0 = 2^{m-2k-1}. \tag{11}$$

From (9) and (11), we get $m_0 + n_0 + q_0 + r_0 = 0$. So we have the following proposition.

PROPOSITION 5.2. *Let X be a smooth spin 4-manifold with $b_1(X) = 0$ and non-positive signature. If X admits a spin alternating group A_5 action, then the*

K-theory degree $\alpha = \alpha_0(1-\tilde{1})$ *for some* $\alpha_0 = l_0+m_0\rho_1+n_0\rho_2+q_0\rho_3-(m_0+n_0+q_0)\rho_4$ *if* $b_2^+(X/A_5) > 0$ *and* $b_2^+(X/\langle s\rangle) + b_2^+(X/\langle t\rangle) = 2b_2^+(X/A_5)$.

Next we assume that $b_2^+(X/A_5) = 0$, $b_2^+(X/\langle s\rangle) = 0$ but $b_2^+(X/\langle t\rangle) \neq 0$, that is $a_1 = b_1 = c_1 = e_1 = 0$ and $d_1 \neq 0$. Considering the action of ϕs, we know the actions of ϕs on h, $\rho_1 h$, $\rho_2 h$, $\rho_3 h$, $\rho_4 h$ and $\rho_3\tilde{1}$ are all non-trivial but it acts on $\tilde{1}$, $\rho_1\tilde{1}$, $\rho_2\tilde{1}$, $\rho_4\tilde{1}$ all with a 1-dimensional invariant subspace. So

$$\begin{aligned}\dim(V(\rho_1,\rho_2,\rho_3,\rho_4))_{\phi s} &- \dim(W(\rho_1,\rho_2,\rho_3,\rho_4))_{\phi s}\\ &= -(a_1+b_1+c_1+e_1) = 0\end{aligned}$$

and $d(f^{\phi s}) = 1$. From tom Dieck formula we have

$$\begin{aligned}&\mathrm{tr}_{\phi s}\,\alpha\\ &= \mathrm{tr}_{\phi s}[\lambda_{-1}(d_1\rho_3)\tilde{1} - \lambda_{-1}(a_0+b_0\rho_1+c_0\rho_2+d_0\rho_3+e_0\rho_4)h]\\ &= [(1-\omega)(1-\omega^2)(1-\omega^3)(1-\omega^4)]^{d_1}[(1-\phi)(1-\phi^{-1})]^{-(a_0+b_0+c_0+e_0)}\\ &\quad [(1-\omega^2\phi)(1-\omega^2\phi^{-1})]^{-(c_0+d_0+e_0)}[(1-\omega^3\phi)(1-\omega^3\phi^{-1})]^{-(c_0+d_0+e_0)}\\ &\quad [(1-\omega\phi)(1-\omega\phi^{-1})]^{-(b_0+d_0+e_0)}[(1-\omega^4\phi)(1-\omega^4\phi^{-1})]^{-(b_0+d_0+e_0)}\end{aligned}$$

Since $\mathrm{tr}_{s_\bullet}\alpha : U(1) \to C$ is a C^0-function, ϕ is a generic element, so $a_0+b_0+c_0+e_0 \le 0$, $c_0+d_0+e_0 \le 0$, $b_0+d_0+e_0 \le 0$. On the other hand, $\mathrm{Ind}\,D = -\sigma/8 \in Z$, but we have $\mathrm{Ind}\,D = a_0+3b_0+3c_0+4d_0+5e_0 \le 0$, so

$$a_0+b_0+c_0+e_0 = c_0+d_0+e_0 = b_0+d_0+e_0 = 0, \tag{12}$$

which means $b_0 = c_0$, $d_0 = a_0+b_0$ and $e_0 = -(a_0+2b_0)$. Besides, X is homotopic to $\sharp_n S^2 \times S^2$ for some integer $n \equiv 0 \pmod 4$.

At last, we suppose $b_2^+(X) = 0$, that is $a_1 = b_1 = c_1 = d_1 = e_1 = 0$. Then from the actions of ϕ on h and $\tilde{1}$, and the actions of x on ρ_1, ρ_2, ρ_3 and ρ_4, we have $\dim(Vh)_{\phi x} - \dim(W\tilde{1})_{\phi x} = 0$, so $d(f^{\phi x}) = 1$. Then from tom Dieck formula, we have

$$\begin{aligned}&\mathrm{tr}_{\phi x}\,\alpha\\ &= \mathrm{tr}_{\phi x}[-\lambda_{-1}(a_0+b_0\rho_1+c_0\rho_2+d_0\rho_3+e_0\rho_4)h]\\ &= [(1-\phi)(1-\phi^{-1})]^{-(a_0+b_0+c_0+2d_0+3e_0)}[(1+\phi)(1+\phi^{-1})]^{-(2b_0+2c_0+2d_0+2e_0)}\end{aligned}$$

Since $\mathrm{tr}_{x_\bullet}\alpha : U(1) \to C$ is a C^0-function, ϕ is a generic element, then $a_0+b_0+c_0+2d_0+3e_0 \le 0$ and $2b_0+2c_0+2d_0+2e_0 \le 0$.

On the other hand, $\mathrm{Ind}\,D = -\sigma/8 \in Z$, but we have $\mathrm{Ind}\,D = a_0+3b_0+3c_0+4d_0+5e_0 \le 0$, so $\mathrm{Ind}\,D = 0$. Moreover, $a_0+b_0+c_0+2d_0+3e_0 = 0$

and $2b_0 + 2c_0 + 2d_0 + 2e_0 = 0$. These two equations along with (12) tell us that $a_0 = d_0 = -e_0$ and $b_0 = c_0 = 0$.

In summary, we have the following result from the above discuss.

PROPOSITION 5.3. *Let X be a smooth spin 4-manifold with $b_1(X) = 0$ and non-positive signature. If X admits a spin alternating group A_5 action, $b_2^+(X/A_5) = 0$ and $b_2^+(X/\langle s\rangle) = 0$ but $b_2^+(X/\langle t\rangle) \neq 0$, then as an element of $R(A_5)$,* $\mathrm{Ind}_{A_5} D$ *is of the form*

$$a_0 + b_0(\rho_1 + \rho_2) + (a_0 + b_0)\rho_3 - (a_0 + 2b_0)\rho_4,$$

and X is homotopic to $\sharp_n S^2 \times S^2$ for some integer $n \equiv 0 \pmod 4$. Moreover, if $b_2^+(X) = 0$ then $\mathrm{Ind}_{A_5} D$ *is a multiple of* $1 + \rho_3 - \rho_4$.

Now we look at a concrete example of A_5-action. Consider the $K3$ surface X defined by equations $\sum_{i=0}^4 z_i^2 = 0$ and $\sum_{i=0}^4 z_i^3 = 0$ in CP^4. By the symmetry of defining equations, the alternating group A_5 of degree 5 acts on X by permutations of variables. Via this action, A_5 acts on X smoothly (in fact, holomorphically). For this action it is easy to get that $\mathrm{Ind}_{A_5} D = 2 \in R(A_5)$, $H_2(X) = 4 + 2\rho_3 + 2\rho_4 \in R(A_5)$. Besides, by Theorem 5.1, we must have $b_2^+(X/\langle s\rangle) + b_2^+(X/\langle t\rangle) = 2b_2^+(X/A_5)$ or $b_2^+(X/\langle s\rangle) = 0$.

On the other hand, applying Proposition 4 of [6] to the above action of A_5 on $K3$ surface, we have $\dim(H_2^+(X)^{A_5}) \geq \dim((\mathrm{Ind}_{A_5} D)^{A_5}) + 1 = 2 + 1 = 3$, but $\dim(H_2^+(X)) = 3$, so $(H_2^+(X))^{A_5} = H_2^+(X)$.

For homotopy $K3$ surface we have

PROPOSITION 5.4. *Let X be a homotopy $K3$ surface. If X admits a spin alternating group A_5 action, then as an element of $R(A_5)$, $H_2^+(X, C) = 3$.*

PROOF. From Theorem 5.1, we have $b_2^+(X/\langle s\rangle) + b_2^+(X/\langle t\rangle) = 2b_2^+(X/A_5)$ or $b_2^+(X/\langle s\rangle) = 0$, that is $b_1 = c_1 = d_1 = e_1 = 0$ or $a_1 = b_1 = c_1 = e_1 = 0$. If $b_1 = c_1 = d_1 = e_1 = 0$ which along with the fact $\dim H_2^+(X, C) = 3$ means that $H_2^+(X, C) = 3$. If $a_1 = b_1 = c_1 = e_1 = 0$, then $H_2^+(X) = d_1\rho_3$. But from $\dim H_2^+(X) = 3$ and the degree of ρ_3 being 4, we know there is no representation of $H_2^+(X)$ satisfying this condition.

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