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WEAK LINEAR CONVEXITY AND A RELATED NOTION OF CONCAVITY

LARS HÖRMANDER

1. Introduction

Linear convexity is a property of open sets in $\mathbf{C}^n$ which is stronger than pseudoconvexity but weaker than convexity:

DEFINITION 1.1. An open set $\Omega \subset \mathbf{C}^n$, $n > 1$, is called *linearly convex* if every $\zeta \in \complement\Omega$ is contained in an affine complex hyperplane $\Pi_\zeta \subset \complement\Omega$, and Ω is called *weakly linearly convex* if this is true when $\zeta \in \partial\Omega$.

The notion was first studied by Behnke and Peschl [2] when $n = 2$, long before the geometric conditions for pseudoconvexity were fully understood. A renewed interest has been created by the study of analytic functionals.

Weak linear convexity implies pseudoconvexity. If $R(z)$ denotes say the euclidean distance from $z \in \Omega$ to $\partial\Omega$ then Ω is pseudoconvex if and only if $-\log R$ is plurisubharmonic in Ω, or equivalently,

$$(1.1)\quad \sum_{j,k=1}^{n} t_j\bar t_k\partial^2 R(z)/\partial z_j\partial\bar z_k \le \Big|\sum_1^n t_j\partial R(z)/\partial z_j\Big|^2/R(z) \quad \text{in } \Omega,\quad t\in\mathbf{C}^n,$$

in the sense of distribution theory. Note that R is Lipschitz continuous with Lipschitz constant 1. By Rademacher's theorem $\partial R/\partial z_j$ is defined almost everywhere and bounded, so the right-hand side of (1.1) is a well defined function in $L^\infty_{\text{loc}}(\Omega)$. If $\partial\Omega \in C^2$, then $R \in C^2$ in a neighborhood of $\partial\Omega$ if defined with a negative sign outside Ω, and (1.1) implies that $\sum t_j\bar t_k\partial^2 R(z)/\partial z_j\partial\bar z_k \le 0$ when $z \in \partial\Omega$ and $\langle R'_z(z), t\rangle = 0$. If ϱ is a C^2 *defining function* of Ω, thus $\varrho < 0$ in Ω, $\varrho = 0$ and $d\varrho \ne 0$ on $\partial\Omega$, this means that

$$(1.2)\quad \sum_{j,k=1}^{n} t_j\bar t_k\partial^2\varrho(z)/\partial z_j\partial\bar z_k \ge 0 \quad \text{if } z\in\partial\Omega,\quad t\in\mathbf{C}^n,\quad \sum_1^n t_j\partial\varrho(z)/\partial z_j = 0.$$

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Conversely (1.2) implies (1.1) and thus pseudoconvexity.

The primary aim of this paper is to study similar conditions for weak linear convexity. Let us first recall two well-known basic results. (See e.g. [3, Proposition 4.6.4 and Corollary 4.6.5].)

PROPOSITION 1.2. *Let $\Omega \subset \mathbf{C}^n$ be a bounded connected open set with C^1 boundary, and assume that Ω is locally weakly linearly convex in the sense that every $\zeta \in \partial\Omega$ has a neighborhood ω such that $\omega \cap T_{\mathbf{C}}(\zeta) \cap \Omega = \emptyset$; here $T_{\mathbf{C}}(\zeta)$ is the complex tangent plane of Ω at ζ, that is, the affine complex hyperplane through ζ contained in the tangent plane. Then Ω is weakly linearly convex.*

PROPOSITION 1.3. *Let $\Omega \subset \mathbf{C}^n$ be a (locally) weakly linearly convex open set with a C^2 boundary, and choose a defining function $\varrho \in C^2(\mathbf{C}^n)$. Then it follows that the second differential $d^2\varrho$ of ϱ is a positive semidefinite quadratic form in the complex tangent plane $T_{\mathbf{C}}(\zeta)$ at $\zeta \in \partial\Omega$. Conversely, if Ω is open, bounded and connected, and $d^2\varrho$ is positive definite in $T_{\mathbf{C}}(\zeta)$ for every $\zeta \in \partial\Omega$, then Ω is weakly linearly convex.*

Proposition 1.3 is in fact a very easy corollary of Proposition 1.2. – In two more recent papers [4], [5], Kiselman has proved that in the last statement it suffices to assume that $d^2\varrho$ is positive semidefinite, which gives a characterization of bounded connected weakly linearly convex open sets with a C^2 boundary in terms of a pointwise convexity condition on the boundary. However, since the localization principle in Proposition 1.2 is valid when $\partial\Omega$ is just in C^1, it is natural to ask if Kiselman's result can be extended to this case. The necessary "Behnke-Peschl condition" in Proposition 1.3 has a natural analogue

$$\varliminf_{T_{\mathbf{C}}(\zeta)\ni w\to\zeta} \varrho(w)/|w-\zeta|^2 \geq 0, \tag{1.3}$$

where $\zeta \in \partial\Omega$ and ϱ is a C^1 defining function for Ω. Since the ratio between two C^1 defining functions is bounded, this condition is independent of the choice of defining function. If $\varrho \in C^2$ then (1.3) means precisely that $d^2\varrho$ is positive semidefinite in $T_{\mathbf{C}}(\zeta)$, so (1.3) is an extension of the usual Behnke-Peschl condition to the case where $\partial\Omega \in C^1$. The necessity of (1.3) is obvious when $\partial\Omega \in C^1$ but we have only been able to prove the sufficiency under stronger regularity conditions. Before stating the result we shall recall some simple facts concerning open sets $\Omega \subset \mathbf{R}^N$ with C^1 boundary. (It is convenient to use real notation for a moment.)

As above let $R(x)$ be the distance from $x \in \Omega$ to $\partial\Omega$, and let $h = R^2$, thus

$$R(x) = \inf_{y\in\partial\Omega} |x-y|, \quad h(x) = \inf_{y\in\partial\Omega} |x-y|^2, \quad x \in \Omega. \tag{1.4}$$

The functions R and h are Gateau differentiable everywhere,

$$h(x+\tilde{x}) = h(x) + h'(x;\tilde{x}) + o(|\tilde{x}|), \quad \tilde{x} \to 0;$$
$$h'(x;\tilde{x}) = \sup\{2\langle \tilde{x}, x-y\rangle;\ y \in \partial\Omega, |x-y|^2 = h(x)\}.$$

(See e.g. [3, Lemma 2.1.29].) Thus h is differentiable at x if and only if there is only one $y \in \partial\Omega$ with minimum distance to x. Since R is Lipschitz continuous by the triangle inequality it follows from Rademacher's theorem that R, hence also $h = R^2$, is differentiable for almost every $x \in \Omega$. If $\Omega_\delta = \{x \in \Omega;\ R(x) < \delta\}$ and $\partial\Omega \in C^1$, then $R \in C^1(\Omega_\delta)$ if and only if every $x \in \Omega_\delta$ is contained in an open ball $\subset \Omega$ with radius δ. If this is true for some $\delta > 0$ we shall say that Ω satisfies the *interior ball condition*. If $\nu(y)$ is the interior unit normal of $\partial\Omega$ at $y \in \partial\Omega$, then $\partial\Omega \times (0,\delta) \ni (y,t) \mapsto y + t\nu(y) \in \Omega_\delta$ is a homeomorphism.

We can now state an improvement of the results of Kiselman [5]:

THEOREM 1.4. *Assume that $\Omega \subset \mathbf{C}^n$ is open, bounded and connected, that $\partial\Omega \in C^1$ and that the interior ball condition is fulfilled. Let ϱ be a C^1 defining function of Ω. If* (1.3) *is valid for every $\zeta \in \partial\Omega$, and for some constant C*

$$\varrho(w) \geq -C|w-\zeta|^2, \quad \zeta \in \partial\Omega, \ w \in T_{\mathbf{C}}(\zeta), \tag{1.5}$$

then Ω is weakly linearly convex.

The interior ball condition is fairly strong. In fact, $\partial\Omega$ is in $C^{1,1}$ (that is, there is a defining function with Lipschitz continuous first derivatives) if and only if both the interior and the exterior ball condition are fulfilled. The exterior ball condition is stronger than (1.5) but closely related.

The proof of Theorem 1.4 given at the end of Section 2 will be based on a study of the function h in Ω defined by (1.4). As a preparation we rephrase the conclusion and the hypotheses of Theorem 1.4 in terms of h:

PROPOSITION 1.5. *An open set $\Omega \subset \mathbf{C}^n$ is weakly linearly convex if and only if*

$$h(w) \leq h(z) + 2\Re\langle w-z, h'_z(z)\rangle + |\langle w-z, h'_z(z)\rangle|^2/h(z), \quad w \in \Omega, \tag{1.6}$$

when h is differentiable at $z \in \Omega$.

We shall always use the notation $\langle\cdot,\cdot\rangle$ for the bilinear scalar product in $\mathbf{C}^n$ and $(\cdot,\cdot)$ for the sesquilinear one.

PROOF. If Ω is weakly linearly convex and h is differentiable at $z \in \Omega$, then $h'_z(z) = \bar{z} - \bar{\zeta}$ where ζ is the point in $\partial\Omega$ where $|z-\zeta|^2 = h(z)$. The plane Π_ζ of Definition 1.1 must be the plane through ζ with normal $z-\zeta$ for it does not intersect the open ball with center z and ζ on the boundary. The right-hand side of (1.6) is equal to

$$\begin{aligned}|z-\zeta|^2 + 2\Re\langle w-z, \bar z-\bar\zeta\rangle + |\langle w-z,\bar z-\bar\zeta\rangle|^2/|z-\zeta|^2 \\ = |w-\zeta|^2 - \big(|w-z|^2 - |(w-z, z-\zeta)|^2/|z-\zeta|^2\big).\end{aligned}$$

The parenthesis is the square of the distance from w to the complex line through z and ζ, so *the right-hand side of* (1.6) *is the square of the distance from* w *to* Π_ζ, which is $\geq h(w)$ since $\Pi_\zeta \subset \complement\Omega$. This proves (1.6); if h is not differentiable at z then (1.6) remains valid with $h'_z(z)$ replaced by $\bar z - \bar\zeta$ for any $\zeta \in \partial\Omega$ with $|z-\zeta|^2 = h(z)$.

Now assume instead that (1.6) is valid. Since $h(w) > 0$ when $w \in \Omega$ it follows from the interpretation of the right-hand side just given that $\Pi_\zeta = \{w; (w-\zeta, z-\zeta) = 0\}$ does not intersect Ω. For an arbitrary $\zeta \in \partial\Omega$ and every $\varepsilon > 0$ we can choose $z \in \Omega$ with $|z-\zeta| < \varepsilon$ so that h is differentiable at z. If $\tilde\zeta$ is the point in $\partial\Omega$ with $|\tilde\zeta - z|^2 = h(z)$, then $|\tilde\zeta-\zeta| \leq |\tilde\zeta - z| + |z-\zeta| < 2\varepsilon$, and we have a plane $\Pi_{\tilde\zeta} \ni \tilde\zeta$ with $\Pi_{\tilde\zeta} \subset \complement\Omega$. When $\varepsilon \to 0$ it follows that there is such a plane through ζ, which completes the proof.

Using the first part of the proof we can easily convert the conditions (1.3) and (1.5) to properties of h:

PROPOSITION 1.6. *If $\partial\Omega \in C^1$ and h is differentiable at $z \in \Omega$, then*

$$\overline{\lim_{w\to 0}} \big(h(z+w) - h(z) - 2\Re\langle w, h'_z(z)\rangle - |\langle w, h'_z(z)\rangle|^2/h(z)\big)/|w|^2 \leq 0 \tag{1.7}$$

if (1.3) *is valid at the point $\zeta \in \partial\Omega$ with $h(z) = |z-\zeta|^2$. If* (1.5) *is valid, then*

$$h(z+w) \leq h(z) + 2\Re\langle w, h'_z(z)\rangle + C'|w|^2, \quad z+w \in \Omega, \tag{1.8}$$

for some constant C' independent of z and w when z and w are bounded.

PROOF. By the interpretation of the right-hand side of (1.6) given in the proof of Proposition 1.5, the meaning of (1.7) is that

$$\overline{\lim_{w\to 0}} (h(z+w) - |z+w-\zeta-w'|^2)/|w|^2 \leq 0,$$

where $\zeta + w'$ is the orthogonal projection of $z+w$ on $T_{\mathbb C}(\zeta)$, thus $|w'| \leq |w|$. If $\zeta + w' \notin \Omega$ then $h(z+w) - |z+w-\zeta-w'|^2 \leq 0$ so there is nothing to prove. If $\zeta + w' \in \Omega$, it follows from (1.3) that $R(\zeta+w') = o(|w'|^2)$, hence

$$R(z+w) \leq |z+w-\zeta-w'| + R(\zeta+w') \leq |z+w-\zeta-w'| + o(|w'|^2),$$

and if we square it follows that

$$h(z+w) \le |z+w-\zeta-w'|^2 + o(|w|^2)$$

as claimed in (1.7). Using (1.5) instead of (1.3) we obtain (1.8) in the same way.

The condition (1.7) is evidently an infinitesimal version of the desired inequality (1.6). The proof of Theorem 1.4 will be achieved by bridging the gap between them in Section 2. This is analogous to the characterisation of concave functions as functions with second derivative ≤ 0, but the continuous differentiability of h near $\partial\Omega$ assumed in Theorem 1.4 will be important then. The proof of Theorem 1.4 is completed in Section 2, but in Section 3 we pursue the study of Lipschitz continuous functions with the properties in Proposition 1.6 further as a possible step toward reducing the regularity assumptions in Theorem 1.4. Just as the study of plurisubharmonic functions only relies on subharmonic functions of one complex (i.e. two real variables), it would suffice to discuss functions of two variables in Sections 2 and 3. However, we shall take the opportunity to do so for functions of $N \ge 2$ real variables which will require a substantial modification of the somewhat indirect proofs given in [5] when $N = 2$.

2. Quadratically concave functions

Proposition 1.5 is an explicit version of the fact that if Ω is weakly linearly convex, then the minimum $h(z)$ of the squared distance from $z \in \Omega$ to $\partial\Omega$ is the infimum of the squared distance to the planes Π_ζ, $\zeta \in \partial\Omega$, in Definition 1.1. Restricted to a complex line the distance to a complex hyperplane Π is either a constant or else a multiple of the distance to the intersection of Π with the line. We take this as motivation for the following definition:

DEFINITION 2.1. A positive function h defined in an open set $\Omega \subset \mathsf{R}^N$ will be called quadratically concave in Ω if for some set $M \subset \mathsf{R}_+ \times \mathsf{R}^N$, $\mathsf{R}_+ = \{t \in \mathsf{R}; t \ge 0\}$,

$$h(x) = \inf_{(a,b)\in M} |ax+b|^2, \quad x \in \Omega. \tag{2.1}$$

The term is new and tentative but it will be convenient here to have a name for this property. The following theorem confirms the close connection with weak linear convexity as expressed in Proposition 1.5.

THEOREM 2.2. *If h is a positive quadratically concave function in an open set $\Omega \subset \mathsf{R}^N$, then h is locally Lipschitz continuous in Ω, thus differentiable* *almost everywhere with derivatives in* $L^\infty_{\mathrm{loc}}(\Omega)$*, and for every point* $x \in \Omega$ *where h is differentiable*

$$h(y) \le h(x) + \langle y-x, h'(x)\rangle + \tfrac14|y-x|^2|h'(x)|^2/h(x), \quad x, y \in \Omega, \tag{2.2}$$

$$\sqrt{h(y)}\sqrt{h(x)} \le \left|\frac{h'(x)}{|h'(x)|}h(x) + \tfrac12(y-x)|h'(x)|\right|, \quad x, y \in \Omega, \tag{2.3}$$

where $|h'(x)|/|h'(x)|$ *should be read as* 1 *if* $h'(x) = 0$*. Conversely, these equivalent conditions imply that h is quadratically concave. A bound C for h at one point* $x \in \Omega$ *implies a uniform bound and uniform Lipschitz continuity in terms of x and C on every compact subset of* Ω*. For every compact set* $K \subset \Omega \times \Omega$ *there is a constant* C_K *such that when h is differentiable at x then*

$$h(y) - h(x) - \langle y-x, h'(x)\rangle \le C_K|y-x|^2, \ \ (x, y) \in K, \tag{2.4}$$

$$\varlimsup_{y\to x} \big(h(y)-h(x)-\langle y-x, h'(x)\rangle\big)/|y-x|^2 \le \tfrac14|h'(x)|^2/h(x), \ \ x \in \Omega. \tag{2.5}$$

From (2.4) *and* (2.5) *it follows that the second derivatives of h are measures and that*

$$\langle h''t, t\rangle \le \tfrac12|t|^2|h'|^2/h \quad \text{in } \Omega, \ \ t \in \mathsf{R}^N. \tag{2.6}$$

PROOF. Assume that (2.1) is fulfilled and fix $x \in \Omega$. Given $\varepsilon > 0$ we can choose $(a_\varepsilon, b_\varepsilon) \in M$ so that $|a_\varepsilon x + b_\varepsilon|^2 < h(x) + \varepsilon$. If Ω contains the ball with radius δ and center at x, then $|a_\varepsilon x + b_\varepsilon| \ge a_\varepsilon\delta$ since $a_\varepsilon y + b_\varepsilon \ne 0$ when $y \in \Omega$, so we have a bound for a_ε, hence for b_ε. Letting $\varepsilon \to 0$ we can select a limit $(a, b) \in \mathsf{R}_+ \times \mathsf{R}^N$ and obtain $|ay + b|^2 \ge h(y)$, $y \in \Omega$, and $|ax + b|^2 = h(x)$. We have uniform bounds for a and b when x is in a compact subset of Ω, so the bound

$$\begin{aligned} h(y) - h(x) \le |ay+b|^2 - |ax+b|^2 &= a\langle y-x, a(y+x) + 2b\rangle \\ &= 2a\langle y-x, ax+b\rangle + a^2|y-x|^2 \end{aligned}$$

together with the analogue obtained by interchanging x and y proves that h is locally Lipschitz continuous. If h is differentiable at x, then

$$\langle y-x, h'(x)\rangle \le 2a\langle y-x, ax+b\rangle,$$
$$\text{hence } \ h'(x) = 2a(ax+b) \ \text{ and } \ h(x) = |ax+b|^2,$$

which gives $4a^2 = |h'(x)|^2/h(x)$ and proves (2.2). Taking the square root of (2.2) after multiplication by $h(x)$ proves the equivalence with (2.3), and (2.4), (2.5) are immediate consequences. It is also obvious that (2.2) implies (2.1) for some M, so what remains is to prove that (2.6) follows from (2.4), (2.5). Let $0 \le \chi \in C_0^\infty(\Omega)$ and note that when $t \in \mathsf{R}^N$

$$\int h(x)\langle \chi''(x)t, t\rangle\, dx = \lim_{\varepsilon\to 0} \int h(x) 2(\chi(x-\varepsilon t)-\chi(x)+\varepsilon\langle \chi'(x), t\rangle)/\varepsilon^2\, dx.$$

Since $\partial h/\partial x \in L^\infty_{\text{loc}}$ defines $\partial h/\partial x$ in the sense of distribution theory, we have

$$\int (h(x)\chi'(x) + h'(x)\chi(x))\, dx = 0, \quad \text{hence}$$

$$\int h(x)\langle \chi''(x)t, t\rangle\, dx = \lim_{\varepsilon\to 0} \int \chi(x) 2(h(x+\varepsilon t)-h(x)-\varepsilon\langle h'(x), t\rangle)/\varepsilon^2\, dx.$$

By (2.5) the upper limit as $\varepsilon\to 0$ of the integrand is $\le \chi(x)|t|^2|h'(x)|^2/(2h(x))$ almost everywhere, and (2.4) gives an upper bound $C|t|^2\chi(x)$ for some constant C. Hence it follows from Fatou's lemma that

$$\int h(x)\langle \chi''(x)t, t\rangle\, dx \le |t|^2 \int \chi(x)|h'(x)|^2/(2h(x))\, dx,$$

which proves (2.6). Taking t along a coordinate axis gives $\partial^2 h/\partial x_j^2 \le |h'|^2/2h$, $j = 1, \dots, N$, so $\partial^2 h/\partial x_j^2$ is a measure, and since $\partial^2 h/\partial x_j^2 + \partial^2 h/\partial x_k^2 + 2\partial^2 h/\partial x_j \partial x_k \le 2|h'|^2/h$, all second derivatives are measures and the proof is complete.

Remark. Note that (2.2) is valid with equality if $h(x) = |ax+b|^2$, the fundamental functions in (2.1), and for no other functions. The inequality (2.2) is inherited from this fact and (2.1). A non-constant quadratic polynomial $h(x) = a|x|^2 + 2\langle b, x\rangle + c$ in R^N is quadratically concave in $\Omega = \{x \in \mathsf{R}^N; h(x) > 0\}$ unless $\Omega = \mathsf{R}^N$, for $h(y) - h(x) - \langle y-x, h'(x)\rangle = a|y-x|^2$ so (2.2) means that $ah(x) \le |ax+b|^2$ when $x \in \Omega$. This is true unless $a > 0$ and $ac > |b|^2$, and then we have $h \ge c - |b|^2/a > 0$.

Our next goal is to prove that conversely (2.6) implies (2.2) when $0 < h \in C^1(\Omega)$ and Ω is the open unit ball B in R^N. At first we assume that $h \in C^2(\overline{B})$, that $h > 0$ in $\overline{B}$, and that

$$\langle x, h'(x)\rangle < h(x), \quad x \in \partial B. \tag{2.7}$$

This is a consequence of the inequality (2.3) which we wish to prove, for when $h > 0$ in $\overline{B}$ it follows from (2.3) that

$$\left|\frac{h'(x)}{|h'(x)|}h(x) - \tfrac{1}{2}x|h'(x)|\right| \ge \tfrac{1}{2}|h'(x)| + \inf_B h > \tfrac{1}{2}|h'(x)|,$$

and if we square it follows that $h(x)^2 - h(x)\langle x, h'(x)\rangle > 0$ when $|x| \le 1$. The following lemma is based on an idea in the proof of Proposition 1.2.

LEMMA 2.3. *Let* $0 < h \in C^2(\overline{B})$ *and assume that* h *satisfies* (2.7) *in* ∂B *and* (2.6) *with strict inequality for* $t \neq 0$ *at every point in* $\overline{B}$. *Then the open subset of* $\overline{B}$ *defined by*

$$\Omega_{a,b} = \{x \in \overline{B};\, h(x) > |ax + b|^2\} \tag{2.8}$$

is connected for arbitrary $(a, b) \in \mathsf{R}_+ \times \mathsf{R}^N$.

PROOF. Since $\Omega_{0,0} = \overline{B}$ we may assume that $(a, b) \neq (0, 0)$. Set $g(x) = h(x) - f(x)$ where $f(x) = |ax+b|^2$. If $x \in \overline{B}$ and $g(x) = 0$, $g'(x) = 0$, then

$$\begin{aligned} f(y) &= f(x) + \langle y - x, f'(x)\rangle + \tfrac{1}{4}|f'(x)|^2|y - x|^2/f(x) \\ &= h(x) + \langle y - x, h'(x)\rangle + \tfrac{1}{4}|h'(x)|^2|y - x|^2/h(x). \end{aligned} \tag{2.9}$$

However, since (2.6) is valid with strict inequality for $t \neq 0$ it follows from Taylor's formula that the right-hand side is $\ge h(y)$ in a neighborhood of x, hence $g(y) \le 0$ there, so x cannot belong to the closure of $\Omega_{a,b}$. Thus $B \cap \partial\Omega_{a,b}$ is a C^2 surface. Suppose now that $x \in \partial B$ and that $g(x) = 0$, but $g'(x) \neq 0$. If $g'(x)$ is not proportional to x then the zeros of g are a C^2 surface intersecting ∂B transversally, and $\Omega_{a,b}$ is in a neighborhood of x in $\overline{B}$ the connected subset on one side of this surface. The situation is more complicated if $g'(x)$ is proportional to x, thus

$$g'(x) = h'(x) - f'(x) = 2Cx, \quad \text{where } C \neq 0.$$

By (2.6) and Taylor's formula

$$g(x + y) \le \langle y, 2Cx\rangle + \tfrac{1}{4}|y|^2\big(|h'(x)|^2/h(x) - |f'(x)|^2/f(x)\big)$$

when $x + y \in \overline{B}$ and $|y|$ is small. When $x + y \in \overline{B}$ then $|y|^2 + 2\langle y, x\rangle \le 0$, and if $C > 0$ it follows that

$$\begin{aligned} g(x + y) &\le |y|^2\big(-C + \tfrac{1}{4}\langle h'(x) - f'(x), h'(x) + f'(x)\rangle/h(x)\big) \\ &= |y|^2\big(-C + \tfrac{1}{4}\langle 2Cx, 2h'(x) - 2Cx\rangle/h(x)\big) \\ &= C|y|^2\big(\langle x, h'(x)\rangle/h(x) - 1 - C/h(x)\big) \end{aligned}$$

for small $|y|$ when $x + y \in \overline{B}$. Hence, by (2.7), x is not in the closure of $\Omega_{a,b}$. Finally, if $C < 0$ then the derivative of g at x in the direction of the interior unit normal $-x$ is positive, and then it is obvious that $\Omega_{a,b}$ is locally connected at x; we have $tx \in \Omega_{a,b}$ if $1 - t$ is positive and small enough.

It is now easy to conclude that $\Omega_{a,b}$ is connected. Indeed, if $x, y \in \Omega_{a,b}$ it is clear that x and y are in the same component of $\Omega_{\varepsilon a,\varepsilon b}$ if ε is positive and small enough, for h has a positive lower bound on a connecting curve in $\overline{B}$. The set E of $\varepsilon \in [0, 1]$ such that x and y are in the same component of $\Omega_{\varepsilon a,\varepsilon b}$ is an open subset. If $1 \notin E$ then there is some $\varepsilon \in (0, 1]$ such that $[0, \varepsilon) \subset E$ but $\varepsilon \notin E$. Let Ω^x and Ω^y be the components of $\Omega_{\varepsilon a,\varepsilon b}$ containing x and y. The closures $\overline{\Omega}^x$, $\overline{\Omega}^y$ are disjoint for $\Omega_{\varepsilon a,\varepsilon b}$ is locally connected at each boundary point, and they have neighborhoods $\widetilde{\Omega}^x$, $\widetilde{\Omega}^y$ containing no other points z with $h(z) \geq |\varepsilon az + \varepsilon b|^2$. If we choose them compact and disjoint, it follows that $h(z) < |\delta az + \delta b|^2$ when $z \in \partial\widetilde{\Omega}^x \cup \partial\widetilde{\Omega}^y$ if $\delta < \varepsilon$ and $\varepsilon - \delta$ is sufficiently small. Hence $\widetilde{\Omega}^x$ (resp. $\widetilde{\Omega}^y$) contains the component of x (resp. y) in $\Omega_{\delta a,\delta b}$ then which contradicts that x and y are in the same component. Thus $E = [0, 1]$ and $\Omega_{a,b}$ is connected.

THEOREM 2.4. *Let h be a positive function in $C^1(B)$ where $B \subset \mathsf{R}^N$ is an open ball, and assume that* (2.6) *is valid for $\Omega = B$. Then* (2.2) *follows with $\Omega = B$, so h is quadratically concave in B.*

PROOF. We may assume that B is the open unit ball. The proof proceeds in three steps.

a) Assume at first that $0 < h \in C^2(\overline{B})$, that (2.6) is fulfilled with strict inequality when $t \neq 0$, and that (2.7) holds. Given $x \in \overline{B}$ we define a and b by

$$|ay + b|^2 = h(x) + \langle y - x, h'(x)\rangle + \tfrac{1}{4}|h'(x)|^2|y - x|^2/h(x),$$

that is, $a = \frac{1}{2}|h'(x)|/\sqrt{h(x)}$, $b = \sqrt{h(x)}h'(x)/|h'(x)| - \frac{1}{2}x|h'(x)|/\sqrt{h(x)}$ (cf. (2.3)). (If $h'(x) = 0$ we interpret $h'(x)/|h'(x)|$ as any unit vector.) By Lemma 2.3 the set $\Omega_{(1-\varepsilon)a,(1-\varepsilon)b} = \{y \in \overline{B}; h(y) > (1 - \varepsilon)^2|ay + b|^2\}$ is connected if $0 < \varepsilon < 1$. Since the inequality in (2.6) is assumed strict for $t \neq 0$, it follows from Taylor's formula that it has one component which shrinks to $\{x\}$ when $\varepsilon \to 0$. Hence it cannot have another component for small $\varepsilon > 0$ which proves that $\Omega_{a,b} = \emptyset$, that is, that (2.2) is valid.

b) Now we just assume that $0 < h \in C^2(\overline{B})$ and that there is strict inequality in (2.6) when $t \neq 0$, but we no longer assume that (2.7) is fulfilled. As in [4, p. 93] the proof is then obtained by a standard continuity argument sometimes referred to as "continuous induction". For $0 < r \leq 1$ let $h^r(x) = h(rx)$, $x \in \overline{B}$. For small r the condition (2.7) is satisfied if h is replaced by h^r, and so is (2.6) with strict inequality when $t \neq 0$. Hence, by a), (2.2) is valid with h replaced by h^r. The set M of all $r \in (0, 1]$ such that (2.2) is valid with h replaced by h^r is closed. To prove that it is open we note that if $r \in M$ then the condition (2.7) is fulfilled with h replaced by h^r, as we saw just after the statement of (2.7). For reasons of continuity it follows that (2.7) is also satisfied by h^ϱ if $|r-\varrho|$ is sufficiently small and $\varrho \le 1$, so M is open, hence equal to $(0, 1]$. Thus (2.2) is valid for $h = h^1$.

c) To prove that (2.6) implies (2.2) when h is just in $C^1(B)$ we first assume that $h \in C^1$ and that h is positive in a neighborhood of $\overline{B}$. Writing $h_\varepsilon(x) = h(x) - \varepsilon(|x|^2+1)$ and $g_\varepsilon(x) = \varepsilon(|x|^2+1)$ with a small $\varepsilon > 0$, we have

$$\begin{aligned}\langle h_\varepsilon'' t, t\rangle &\le \big(\tfrac12 |h'|^2/h - 2\varepsilon\big)|t|^2 \le \big(\tfrac12 |h_\varepsilon'|^2/h_\varepsilon + \tfrac12 |g_\varepsilon'|^2/g_\varepsilon - 2\varepsilon\big)|t|^2\\ &= \big(\tfrac12 |h_\varepsilon'|^2/h_\varepsilon - 2\varepsilon/(|x|^2+1)\big)|t|^2,\end{aligned}$$

for $(t, t_0) \mapsto |t|^2/t_0$ is a convex and homogeneous, hence subadditive, function of (t, t_0) when $t \in \mathsf{R}^N$ and $t_0 > 0$. If $0 \le \psi \in C_0^\infty(\mathsf{R}^N)$ and ψ has support sufficiently close to the origin, $\int \psi\, dx = 1$, it follows that $H = h_\varepsilon * \psi \in C^\infty(\overline{B})$, that $H > 0$ in $\overline{B}$ if ε is small, and that (2.6) is valid in $\overline{B}$ with strict inequality when $|t| = 1$ when h is replaced by H. Hence (2.2) is valid with h replaced by H, and (2.2) follows for h itself when $\operatorname{supp}\psi \to \{0\}$. If h just satisfies the hypotheses in the theorem we conclude that (2.2) is valid with h replaced by h^r when $0 < r < 1$, if h^r is defined as in part b) above. When $r \to 1$ the statement follows for h itself.

Remark. The proof of Theorem 2.4 for $N = 2$ given in [4] relied on approximating a Hartogs domains in C^2 defined by h with domains for which Proposition 1.3 can be applied. This approximation also depended on the condition (2.7) but in a more technical and less geometrically motivated way than in Lemma 2.3 here. (See also [1], Section 2.5.)

Kiselman [4, Section 8] proved when $N = 2$ that if the conclusion of Theorem 2.4 is valid for an open bounded bounded set $\Omega \subset \mathsf{R}^2$ then $\overline{\Omega}$ is a disk. We shall simplify his proof and generalize the result at the end of Section 3. However, the approximation in part c) of the proof can be applied quite generally:

Theorem 2.5. *If h is a positive quadratically concave function in $C^1(\Omega)$ where Ω is an open set in R^N, and if $\omega \Subset \Omega$ is another open set, then there exists a sequence $h_j \in C^\infty(\omega)$ of positive quadratically concave functions converging to h in $C^1(\omega)$.*

Proof. With the notation in part c) of the proof of Theorem 2.4 we have $h_\varepsilon > 0$ in a neighborhood of $\bar\omega$ for small $\varepsilon > 0$. If ε is sufficiently small and $\operatorname{supp}\psi$ is sufficiently close to the origin then $H = h_\varepsilon * \psi$ is in $C^\infty(\omega)$ and $\langle H'' t, t\rangle < \frac12 |t|^2 |H'|^2/H$ when $t \ne 0$, in a neighborhood of ω, as in the proof of Theorem 2.4. By Theorem 2.4 this implies that (2.2) is valid with h replaced by H if $x, y \in \bar{\omega}$ and $|x-y| < r$, say. Now

$$\begin{aligned} h_\varepsilon(y) - h_\varepsilon(x) - \langle y-x, h'_\varepsilon(x)\rangle - \tfrac{1}{4}|h'_\varepsilon(x)|^2|y-x|^2/h_\varepsilon(x) \\ = \varepsilon(|x|^2 - |y|^2) + h(y) - h(x) \\ - \langle y-x, h'(x) - 2\varepsilon x\rangle - \tfrac{1}{4}|h'_\varepsilon(x)|^2|y-x|^2/h_\varepsilon(x) \\ \le -\varepsilon|x-y|^2 + \tfrac{1}{4}(|h'(x)|^2/h(x) - |h'_\varepsilon(x)|^2/h_\varepsilon(x))|y-x|^2 \\ \le -\varepsilon|y-x|^2/(1+|x|^2), \end{aligned}$$

as in the proof of Theorem 2.4. Hence H satisfies (2.2) when $x, y \in \omega$ and $|x-y| \ge r$, provided that ε is sufficiently small and $|z| < \delta_\varepsilon$ when $z \in \operatorname{supp}\psi$. This proves the statement.

Using just Theorem 2.4 we can now prove Theorem 1.4.

PROOF OF THEOREM 1.4. Choose $\delta > 0$ so that $h \in C^1(\Omega_\delta)$ where $\Omega_\delta = \{z \in \Omega; h(z) < \delta^2\}$. If $\zeta \in \partial\Omega$ then Ω_δ contains the open ball B_ζ of radius $\delta/2$ with $\zeta \in \partial B_\zeta$ and the same interior normal as Ω at ζ, and $h \in C^1(B_\zeta)$. The restriction of h to the intersection of B_ζ and a complex line is quadratically concave, for it follows from Proposition 1.6 and the second part of Theorem 2.2 that the hypotheses of Theorem 2.4 are fulfilled. Thus

(2.10)
$$h(w) \le h(z) + 2\Re\langle w-z, h'_z(z)\rangle + |\langle w-z, h'_z(z)\rangle|^2/h(z), \quad w, z \in B_\zeta.$$

If we choose z as the center of B_ζ, it follows that $h(w)$ is at most equal to the squared distance from w to $T_{\mathbf{C}}(\zeta)$, by the interpretation made in the proof of Proposition 1.5. If $w' \in T_{\mathbf{C}}(\zeta) \cap \Omega$ and $|w'-\zeta|$ is small, then the direction of the normal of $\partial\Omega$ at the point $\zeta' \in \partial\Omega$ closest to w' is close to that at ζ, and $|\zeta'-\zeta| \le 2|w'-\zeta|$ is also small. Hence the normal at ζ' will contain points $w \in B_\zeta$. The boundary distance $|w-\zeta'| = \sqrt{h(w)}$ is $|w-w'| + |w'-\zeta'| > |w-w'|$, and since the distance from w to $T_{\mathbf{C}}(\zeta)$ is at most equal to $|w-w'|$, this contradicts (2.10). Hence $\omega \cap T_{\mathbf{C}}(\zeta) \cap \Omega = \emptyset$ if ω is a sufficiently small neighborhood of ζ, and it follows from Proposition 1.2 that Ω is weakly linearly convex.

REMARK. The preceding proof used the interior ball condition in two ways. Without it we could still use Proposition 1.6 and the second part of Theorem 2.2 to conclude that (2.6) is valid for the restriction of h to a complex line,

$$t \mapsto h(at+b), \quad t \in \mathbf{C}, \ at+b \in \Omega,$$

provided that h is differentiable at $at+b$ for almost all t. The differentiability of h in Ω_δ allowed us to apply Theorem 2.4 to obtain (2.10). Secondly it allowed us to conclude that ζ' was the boundary point closest to w, which was equally important.

3. Weakly quadratically concave functions

In Theorem 2.4 it is not possible to replace the hypothesis that $h \in C^1$ by Lipschitz continuity. This is shown by the following example, adapted from [4, Example 3.1]. Let B be the open unit ball and

$$h_0(x) = |x|^2 + 4 - 4|x'|, \quad x' = (x_1, \dots, x_{N-1}), \ \ x \in \overline{B},$$

which is quadratically concave in B since $h_0(x) = \min |x+b|^2$ when $b_N = 0$ and $|b| = 2$. The maximum of h_0 in $\overline{B}$, equal to 5, is achieved when $x' = 0$ and $x_N = \pm 1$. Also $h_t = \min(h_0, t)$, $t > 0$, is quadratically concave. If $4 < t < 5$ then

$$h(x) = \begin{cases} h_t(x), & \text{if } x_N \geq 0, \\ h_0(x), & \text{if } x_N \leq 0 \end{cases}$$

satisfies (2.6) for the two definitions agree when $|x_N|$ is sufficiently small since $t > 4$. However, (2.2) is not valid if $x' = y' = 0$ and x_N is close to 1 and y_N is close to -1 since $t < 5$.

If one tries to remove the interior ball condition from the hypotheses of Theorem 1.4 one will encounter functions h satisfying (2.6) which are just Lipschitz continuous. We shall therefore study their properties in this section although, as remarked at the end of Section 2, this is not very likely to lead to any improvement of Theorem 1.4.

If h is Lipschitz continuous and satisfies (2.6), then the right-hand side of (2.6) is bounded by $2C|t|^2$ for some constant C, and then it follows that $C|x|^2 - h(x)$ is a convex function. We exploit this in the proof of the following theorem.

THEOREM 3.1. *If h is a positive locally Lipschitz continuous function satisfying* (2.6) *in an open set $\Omega \subset \mathsf{R}^N$, then for every compact set $K \subset \Omega \times \Omega$ there is a constant C_K such that* (2.4) *and* (2.5) *are valid when h is differentiable at x. Thus* (2.6) *is equivalent to the conjunction of* (2.4) *and* (2.5). *If Ω^* is the set of points in Ω where h is differentiable, then $\Omega^* \ni x \mapsto h'(x)$ is continuous.*

PROOF. If $x \in \Omega$ and $B_r(x) = \{x + y; |y| < r\} \Subset \Omega$, then (2.6) implies that

$$f(y) = \tfrac{1}{2} M_r |y - x|^2 - h(y)$$

is a convex function in $B_r(x)$ if M_r is the essential supremum of $\frac{1}{2}|h'(y)|^2/h(y)$ in $B_r(x)$. If $x \in \Omega^*$, that is, h is differentiable at x, then f is differentiable at x and $f(y) \geq f(x) + \langle y - x, f'(x)\rangle$, that is,

$$h(y) \leq h(x) + \langle y - x, h'(x)\rangle + \tfrac{1}{2} M_r |x - y|^2, \quad |y - x| < r, \tag{3.1}$$

which proves (2.4). If $g(y) = f(y) - f(x) - \langle y - x, f'(x)\rangle$ and $G_r = \sup_{B_r(x)} g$, then $0 \le g(y) \le G_r|y - x|/r$ when $y \in B_r(x)$, since $g(x) = g'(x) = 0$. Hence

$$|g'(y)| \le (G_r - g(y))/(r - |y - x|) \to G_r/r \quad \text{when } \Omega^* \ni y \to x,$$

and since $G_r/r \to 0$ when $r \to 0$ it follows that $f'(y) \to f'(x)$, that is, $h'(y) \to h'(x)$ when $\Omega^* \ni y \to x$. The complement of Ω^* is a null set by Rademacher's theorem, so this implies that $M_r \to \frac{1}{2}|h'(x)|^2/h(x)$ when $r \to 0$, and (2.5) follows from (3.1).

It is convenient to have a name for the functions in Theorem 3.1:

Definition 3.2. Positive locally Lipschitz continuous functions satisfying (2.6) in an open set $\Omega \subset \mathsf{R}^N$ will be called weakly quadratically concave.

As in the proof of Theorem 3.1 we can transfer to weakly quadratically concave functions a number of basic properties of convex functions:

Theorem 3.3. *If h is a positive locally Lipschitz continuous weakly quadratically concave function in the open set $\Omega \subset \mathsf{R}^N$, then the Gateau differential*

$$h'(x; y) = \lim_{\varepsilon \to +0} (h(x + \varepsilon y) - h(x))/\varepsilon \tag{3.2}$$

exists for every $x \in \Omega$ and $y \in \mathsf{R}^N$, and it is a concave and positively homogeneous function of y, thus

$$h'(x; y) = \min_{\xi \in \tilde{h}'(x)} \langle y, \xi\rangle, \quad \text{where } \tilde{h}'(x) = \{\xi \in \mathsf{R}^N; \langle y, \xi\rangle \ge h'(x; y), y \in \mathsf{R}^N\} \tag{3.3}$$

is a convex compact set. If Ω^* is the set of points where h is differentiable then $\Omega \setminus \Omega^*$ is a null set and for arbitrary $x \in \Omega$ and $y \in \mathsf{R}^N$,

$$h'(x; y) = \varliminf_{\Omega^* \ni x^* \to x} \langle h'(x^*), y\rangle; \qquad \varlimsup_{\Omega^* \ni x^* \to x} |h'(x^*)| = \max_{\xi \in \tilde{h}'(x)} |\xi|.$$

The distance from $h'(x^*)$ to $\tilde{h}'(x)$ tends to 0 when $\Omega^* \ni x^* \to x$, and if ξ is an extreme point of $\tilde{h}'(x)$ then $\varliminf_{\Omega^* \ni x^* \to x} |h'(x^*) - \xi| = 0$. If h_j is a locally uniformly Lipschitz continuous sequence of positive weakly quadratically concave functions and $h_j \to h$ where h is also positive, then h is weakly quadratically concave and

$$\varliminf_{j \to \infty} h'_j(x; y) \ge h'(x; y), \quad x \in \Omega, \quad y \in \mathsf{R}^N.$$

If h is differentiable at x then $h'_j(x;y) \to h'(x;y) = \langle h'(x), y\rangle$ for every $y \in \mathsf{R}^N$.

Proof. If f is convex in a convex open set $\omega \subset \mathsf{R}^N$ and ω^* is the subset where f is differentiable, then

$$\varlimsup_{\omega^* \ni x^* \to x} \langle f'(x^*), y\rangle = f'(x;y), \quad y \in \mathsf{R}^N, \ x \in \omega.$$

In fact, since $\langle f'(x^*), y\rangle \le (f(x^*+\varepsilon y) - f(x^*))/\varepsilon \to (f(x+\varepsilon y) - f(x))/\varepsilon$ as $x^* \to x$, if $\varepsilon > 0$, it is clear that the left-hand side is bounded by the right-hand side. Now

$$F_\varepsilon(y) = \sup_{|x^*-x|<\varepsilon, x^* \in \omega^*} \langle f'(x^*), y\rangle$$

is a convex positively homogeneous function and $f(x+y) - f(x) \le F_\varepsilon(y)$ when $|y| < \varepsilon$. In fact, $f(x'+y) - f(x') \le F_\varepsilon(y)$ if $x' + ty \in \omega^*$ for almost all $t \in [0,1]$ and $|x'-x| + |y| < \varepsilon$; by Fubini's theorem there exist such points x' arbitrarily close to x. Hence $f'(x;y) \le F_\varepsilon(y)$ which proves the statement. If

$$K_x = \{\xi \in \mathsf{R}^N;\ f'(x;y) \ge \langle y, \xi\rangle,\ y \in \mathsf{R}^N\},$$

then $f'(x;\cdot)$ is the supporting function of the convex compact set K_x. For $\varepsilon > 0$ we have $\langle f'(x^*), y\rangle \le f'(x;y) + \varepsilon|y|$ when $x^* \in \omega^*$ and $|x^*-x|$ is small enough, hence $f'(x^*) \in K_x + \{y; |y| \le \varepsilon\}$ then. If $\widetilde{K}_x$ is the set of limit points of $f'(x^*)$ as $\omega^* \ni x^* \to x$, then $\widetilde{K}_x \subset K_x$ is compact, and since the convex hull is equal to K_x it follows that $\widetilde{K}_x$ contains the extreme points of K_x. If these results and those in [3, Theorem 2.1.22] are applied to $M|x|^2 - h(x)$ for a suitably large M, we obtain the statements in the theorem.

Remark. With the notation in the proof of Theorem 3.1 the limit of M_r when $r \to 0$ is $\frac{1}{2}\max_{\xi \in \tilde{h}'(x)} |\xi|^2/h(x)$. Since $f(y) \ge f(x) + f'(x; y-x) = f(x) - h'(x; y-x)$ we obtain

$$h(y) \le h(x) + h'(x; y-x) + \tfrac{1}{2}M_r|y-x|^2, \quad |y-x| < r, \tag{3.1$'$}$$

so (2.5) can be strengthened to

$$\varlimsup_{y\to x}\big(h(y) - h(x) - h'(x; y-x)\big)/|y-x|^2 \le \tfrac{1}{4}\max_{\xi \in \tilde{h}'(x)} |\xi|^2/h(x), \quad x \in \Omega. \tag{2.5$'$}$$

If h is quadratically concave in Ω, it follows from Theorem 3.3 and (2.2) that for arbitrary $x, y \in \Omega$

$$h(y) \le h(x) + \langle y-x, \xi\rangle + \tfrac{1}{4}|y-x|^2|\xi|^2/h(x), \quad \text{if } \xi \text{ is an extreme point of } \tilde{h}'(x), \tag{3.4}$$

hence

$$h(y) \le h(x) + h'(x; y-x) + \tfrac{1}{4}|y-x|^2 \max_{\xi \in \tilde{h}'(x)} |\xi|^2/h(x).$$

If h_j is a sequence of quadratically concave functions in Ω converging pointwise to a positive function h in Ω, then the sequence is locally uniformly Lipschitz continuous by Theorem 2.2 and by Theorem 3.3 the inequality (2.2) is valid for the limit h so it is quadratically concave too.

If h_1 and h_2 are quadratically concave functions it is obvious that $h = \min(h_1, h_2)$ is quadratically concave. To prove the analogue for weakly quadratically concave functions we need an elementary lemma on convex functions of one variable.

LEMMA 3.4. *If f_1 and f_2 are convex functions in $(-1, 1)$ and $f = \max(f_1, f_2)$, then $f'' \ge \min(f_1'', f_2'')$ in the sense of measure theory.*

PROOF. For $\chi \in C_0^\infty(-1, 1)$ we have by Taylor's formula

$$\begin{aligned}\langle f'', \chi\rangle = \langle f, \chi''\rangle &= \lim_{\varepsilon\to 0} \int f(x)(\chi(x+\varepsilon) + \chi(x-\varepsilon) - 2\chi(x))/\varepsilon^2\, dx \\ &= \lim_{\varepsilon\to 0} \int (f(x+\varepsilon) + f(x-\varepsilon) - 2f(x))\chi(x)/\varepsilon^2\, dx.\end{aligned}$$

Set $d\mu_j = f_j''$ and $d\nu = \min(d\mu_1, d\mu_2)$. If $f(x) = f_1(x)$ then

$$\begin{aligned}f(x+\varepsilon) + f(x-\varepsilon) - 2f(x) &\ge f_1(x+\varepsilon) + f_1(x-\varepsilon) - 2f_1(x) \\ &= \int_{|t-x|<\varepsilon} (\varepsilon - |t-x|)\, d\mu_1(t) \ge \int_{|t-x|<\varepsilon} (\varepsilon - |t-x|)\, d\nu(t),\end{aligned}$$

and similarly if $f(x) = f_2(x)$. Hence we obtain if $\chi \ge 0$

$$\langle f'', \chi\rangle \ge \int d\nu(t) \lim_{\varepsilon\to 0} \int_{|t-x|<\varepsilon} (\varepsilon - |t-x|)\chi(x)\, dx/\varepsilon^2 = \int \chi(t)\, d\nu(t),$$

which proves that $f'' \ge d\nu = \min(f_1'', f_2'')$.

THEOREM 3.5. *If h_1 and h_2 are weakly quadratically concave functions in $\Omega \subset \mathsf{R}^N$, then $h = \min(h_1, h_2)$ is also weakly quadratically concave.*

PROOF. By Lemma 3.4 applied to $M|x|^2 - h_j(x)$ for sufficiently large M we have

$$\langle h''t, t\rangle \le \tfrac{1}{2}|t|^2 \max(|h_1'(x)|^2/h_1(x), |h_2'(x)|^2/h_2(x)), \quad t \in \mathsf{R}^N. \tag{3.5}$$

Let χ_1, χ_2, χ_0 be the characteristic functions of the sets

$$\Omega_1 = \{x \in \Omega;\, h_1(x) < h_2(x)\},$$
$$\Omega_2 = \{x \in \Omega;\, h_2(x) < h_1(x)\},$$
$$\Omega_0 = \{x \in \Omega;\, h_1(x) = h_2(x)\}.$$

The sets Ω_j with $j = 1, 2$ are open and $h = h_j$ there, hence

$$\chi_j \langle h''t, t\rangle \le \tfrac{1}{2}\chi_j |t|^2 |h'|^2/h, \quad t \in \mathsf{R}^N, \quad j = 1, 2. \tag{3.6}$$

To prove this for $j = 0$ we first observe that by (3.5) the positive part of the left-hand side is absolutely continuous. If $h_1 - h_2$ is differentiable at $x \in \Omega_0$ and the differential is not equal to 0, then x is not a point of density in Ω_0. Hence it follows from Rademacher's theorem that $h_1'(x) = h_2'(x)$ for almost every $x \in \Omega_0$, so (3.6) follows from (3.5) when $j = 0$. This completes the proof. (We could also have used the equivalent condition (2.5), for if h is differentiable at x and $h_1(x) = h_2(x)$, then h_1 and h_2 are differentiable at x and $h_1'(x) = h_2'(x) = h'(x)$.)

Positive locally uniform limits of quadratically concave functions in Ω are quadratically concave, but we shall now prove that limits of smooth quadratically concave functions satisfy a stronger version of (3.4). For the proof we need a preliminary lemma on convex functions.

Lemma 3.6. *Let f be a positively homogeneous convex function in R^N, thus the supporting function of a convex compact set*

$$K = \{\xi \in \mathsf{R}^N;\, \langle x, \xi\rangle \le f(x),\, x \in \mathsf{R}^N\}; \quad f(x) = \sup_{\xi \in K}\langle x, \xi\rangle.$$

If g is a convex function in $C^2(\{x \in \mathsf{R}^N;\, |x| \le R\})$ then

$$K \subset \{g'(x);\, |x| \le R\} + 4\{\xi;\, |\xi| \le \sup_{|x|\le R} |f(x) - g(x)|/R\}. \tag{3.7}$$

Proof. If a linear form $\langle x, \theta\rangle$ is added to $f(x)$ and to $g(x)$ then K is replaced by $K + \{\theta\}$ and $g'(x)$ is replaced by $g'(x) + \theta$, so the statement does not change. It is therefore sufficient to prove that if $0 \in K$, that is, $f \ge 0$, then $|g'(x)| \le 4\varepsilon/R$ for some x with $|x| < R$ if $\varepsilon = \sup_{|x|\le R} |f(x) - g(x)|$. Since

$$g(x) + 2\varepsilon|x|^2/R^2 \ge f(x) - \varepsilon + 2\varepsilon|x|^2/R^2 \ge \varepsilon \quad \text{when } |x| = R,$$

and $g(0) \le \varepsilon$, it follows that $g(x) + 2\varepsilon|x|^2/R^2$ has a minimum point in the open ball, thus $g'(x) + 4\varepsilon x/R^2 = 0$ and $|g'(x)| \le 4\varepsilon/R$ for some x with $|x| < R$. This proves the lemma.

THEOREM 3.7. *If h is a positive quadratically concave function in an open set $\Omega \subset \mathsf{R}^N$ which in every relatively compact subset is a uniform limit of smooth quadratically concave functions, then*

$$h(y) \le h(x) + \langle y - x, \xi\rangle + \tfrac{1}{4}|y-x|^2|\xi|^2/h(x), \qquad x, y \in \Omega,\ \ \xi \in \tilde{h}'(x), \tag{3.8}$$

$$\tilde{h}'(x) = \{\xi \in \mathsf{R}^N;\ \langle z, \xi\rangle \ge h'(x; z), z \in \mathsf{R}^N\}. \tag{3.9}$$

Recall that by (3.4) the inequality (3.8) is valid for every quadratically concave function when ξ is an extreme point of $\tilde{h}'(x)$. For the example discussed at the beginning of this section,

$$h(x) = |x|^2 + 4 - 4|x'|, \quad x' = (x_1, \dots, x_{N-1}),\ \ |x| < 1,$$

we have $h'(x; z) = 2x_N z_N - 4|z'|$, hence $\tilde{h}'(x) = \{(\xi', 2x_N); |\xi'| \le 4\}$, when $x' = 0$, but (3.8) is not valid when $|\xi'| < 4$. In fact, if $y' = 0$ too then (3.8) requires that

$$y_N^2 \le x_N^2 + 2x_N(y_N - x_N) + \tfrac{1}{4}(y_N - x_N)^2(|\xi'|^2 + 4x_N^2)/(x_N^2 + 4),$$

which simplifies to $16 \le |\xi'|^2$ if $y_N \ne x_N$. – By Theorem 2.5 it does not matter in Theorem 3.7 if by smooth we mean C^1, C^2, or C^∞.

PROOF OF THEOREM 3.7. We may assume that $x = 0$. Let $0 \in \omega \Subset \Omega$ and choose a sequence $h_j \in C^2(\bar{\omega})$ of quadratically concave functions converging uniformly to h in $\bar{\omega}$. Let $\frac{1}{4}|h_j'|^2/h_j \le M$ in $\bar{\omega}$, which implies that $f_j(x) = M|x|^2 - h_j(x)$ is convex in $\bar{\omega}$; so is the limit $f(x) = M|x|^2 - h(x)$, and $f'(0; \cdot) = -h'(0; \cdot)$. If $\xi \in \tilde{h}'(0)$ then $\langle z, \xi\rangle \ge -f'(0; z)$, that is, $\langle z, -\xi\rangle \le f'(0; z)$. We have

$$f'(0; z) = \lim_{\delta \to +0}(f(\delta z) - f(0))/\delta$$

with uniform convergence when $|z| \le 1$. For arbitrary $\varepsilon > 0$ it follows then that

$$|f'(0; z) - (f_j(\delta z) - f_j(0))/\delta| < \varepsilon, \quad \text{if } |z| \le 1,\ 0 < \delta < \delta_\varepsilon \text{ and } j > J_{\varepsilon,\delta}.$$

Let $\xi \in \tilde{h}'(0)$ and fix $\delta < \delta_\varepsilon$ for a moment. By Lemma 3.6 we can then for large j find z_j with $|z_j| \le 1$ such that $|f_j'(\delta z_j) + \xi| \le 4\varepsilon$, hence $|h_j'(\delta z_j) - \xi| \le 4\varepsilon + 2M\delta$. If we apply (2.2) to h_j with x replaced by δz_j it follows when $j \to \infty$ that for some z with $|z| \le \delta$ and $\tilde{\xi}$ with $|\tilde{\xi} - \xi| \le 4\varepsilon + 2M\delta$ we have

$$h(y) \le h(z) + \langle y - z, \tilde{\xi}\rangle + \tfrac{1}{4}|y - z|^2|\tilde{\xi}|^2/h(z), \quad y \in \omega.$$

When ε and $\delta \to 0$ then $z \to 0$ and $\tilde{\xi} \to \xi$, so we obtain (3.8) with $x = 0$ and $y \in \omega$. The proof is complete.

The following corollary makes the condition in Theorem 3.7 concrete in a special case close to Example 3.1 in [4].

COROLLARY 3.8. *Let h_1 and h_2 be positive C^2 functions in a neighborhood of $0 \in \mathsf{R}^N$ such that $h_1(0) = h_2(0)$. Then $h = \min(h_1, h_2)$ is not in any ball $\overline{B}$ with center at 0 a uniform limit of C^2 quadratically concave functions in $\overline{B}$ unless*

$$\langle v, \partial\rangle^2(\lambda_1 h_1 + \lambda_2 h_2)(0) \le \tfrac{1}{2}|\partial(\lambda_1 h_1 + \lambda_2 h_2)(0)|^2/h(0), \tag{3.10}$$

when $\lambda_1 + \lambda_2 = 1$, $\lambda_1, \lambda_2 \ge 0$ and v is a unit vector such that $\langle v, \partial\rangle h_1(0) = \langle v, \partial\rangle h_2(0)$. This is equivalent to

$$\begin{gathered} \langle v, \partial\rangle^2 h_j(0) \le \tfrac{1}{2}|h_j'(0)|^2/h(0), \quad j = 1, 2, \\ \sum_1^2 \big(|h_j'(0)|^2 - 2h(0)\langle v, \partial\rangle^2 h_j(0)\big)^{\frac{1}{2}} \ge |h_1'(0) - h_2'(0)|. \end{gathered} \tag{3.10$'$}$$

PROOF. We can assume that $h_1'(0) \ne h_2'(0)$ for otherwise there is nothing to prove. Since

$$h'(0; x) = \min_{j=1,2} \langle h_j'(0), x\rangle,$$

the condition on ξ in Theorem 3.7 is that

$$\xi = \lambda_1 h_1'(0) + \lambda_2 h_2'(0), \quad \text{where } \lambda_1 \ge 0,\ \lambda_2 \ge 0,\ \lambda_1 + \lambda_2 = 1.$$

The equation $h_1(x) = h_2(x)$ defines a C^2 hypersurface Γ with tangent v at the origin, and on this surface we have $h(x) = h_j(x)$ for $j = 1, 2$, hence

$$h(x) = h(0) + \langle x, h_j'(0)\rangle + \tfrac{1}{2}\langle h_j''(0)x, x\rangle + o(|x|^2), \quad x \in \Gamma,\ j = 1, 2,$$

thus

$$h(x) = h(0) + \langle x, \xi\rangle + \tfrac{1}{2}\langle(\lambda_1 h_1''(0) + \lambda_2 h_2''(0))x, x\rangle + o(|x|^2).$$

When $x \to 0$ along a curve in Γ with tangent v at 0, it follows that h cannot be approximated by smooth quadratically concave functions in a neighborhood of the origin unless

$$\tfrac{1}{2}\langle(\lambda_1 h_1''(0)+\lambda_2 h_2''(0))v, v\rangle \le \tfrac{1}{4}|\xi|^2|v|^2/h(0) = \tfrac{1}{4}|\lambda_1 h_1'(0)+\lambda_2 h_2'(0)|^2/h(0).$$

This completes the proof of (3.10). To prove the equivalence of (3.10) and (3.10)′ we observe that a quadratic polynomial $p(\lambda)$ with leading term $c\lambda^2$, $c \geq 0$, is non-negative in $[0, 1]$ if and only if $p(0) \geq 0$, $p(1) \geq 0$, and $\sqrt{c} \leq \sqrt{p(0)} + \sqrt{p(1)}$. In fact, for $0 < \lambda < 1$,

$$\begin{aligned} p(\lambda) &= \lambda p(1) + (1-\lambda)p(0) + c\lambda(\lambda - 1) \\ &= \big(\lambda\sqrt{p(1)} + (\lambda-1)\sqrt{p(0)}\,\big)^2 + \big((\sqrt{p(0)} + \sqrt{p(1)})^2 - c\big)\lambda(1-\lambda). \end{aligned}$$

The first term on the right vanishes for some $\lambda \in (0, 1)$ unless $p(1) = 0$ or $p(0) = 0$. If $p(0) = 0 < p(1)$ then the right-hand side is $(p(1)-c)\lambda + O(\lambda^2)$ as $\lambda \to 0$, so $c \leq p(1)$ if $p \geq 0$ in $(0, 1)$. Similarly $c \leq p(0)$ if $p(1) = 0 < p(0)$, and $c = 0$ if $p(0) = p(1) = 0$. This proves the necessity, and the sufficiency follows at once from the formula.

The condition (3.10) is also sufficient for approximation by smooth quadratically concave functions to be possible: Let $0 \leq \psi \in C_0^\infty(\mathsf{R}^N)$, $\int \psi(x)\,dx = 1$, and set $\psi_\delta(x) = \delta^{-N}\psi(x/\delta)$. If h_1 and h_2 are positive C^2 functions in a neighborhood Ω of a compact set $K \subset \mathsf{R}^N$ satisfying (2.6), $h = \min(h_1, h_2)$, and if the analogue of (3.10) is fulfilled where $h_1 = h_2$, then the regularisation of $h_\varepsilon(x) = h(x) - \varepsilon(|x|^2 + 1)$ by convolution with ψ_δ satisfies (2.6) in a neighborhood of K if $0 < \delta < \delta_\varepsilon$, $0 < \varepsilon < \varepsilon_0$, and converges to h uniformly and with a uniform Lipschitz bound when first δ and then ε tend to 0.

The proof is straightforward but rather long and will be omitted. We have not been able to decide if there is a general converse of Theorem 3.7, that is, whether all quadratically concave functions satisfying (3.8) are locally uniform limits of smooth quadratically concave functions. However, this is true for the simplest functions suggested by (3.8):

Proposition 3.9. *Let K be a convex compact subset of R^N and set with $\gamma > 0$*

$$h(x) = \min_{\xi \in K}\big(\gamma + \langle x, \xi\rangle + \tfrac14 |x|^2|\xi|^2/\gamma\big). \tag{3.11}$$

Then h is positive and quadratically concave in the complement of $\{-2\gamma\xi/|\xi|^2; 0 \neq \xi \in K\}$, and h is on compact subsets a uniform limit of smooth quadratically concave functions. Moreover, $h \in C^{1,1}$ in $\complement\{0\}$, and

$$h''(x) = O(|x|^{-2}), \quad h(x) = \gamma + h'(0; x) + O(|x|^2), \quad \textit{when } x \to 0.$$

Here $h'(0; x) = \min_{\xi\in K}\langle x, \xi\rangle$, thus $x \mapsto -h'(0; -x)$ is the supporting function of K.

Proof. If $\xi \neq 0$ then $\gamma + \langle x, \xi\rangle + \frac14|x|^2|\xi|^2/\gamma = \big|\gamma\xi/|\xi| + \frac12 x|\xi|\big|^2/\gamma$, so $h(x)$ is positive in the complement Ω of $\{-2\gamma\xi/|\xi|^2; 0 \neq \xi \in K\}$, which is essentially the inversion of K. Thus h is a positive quadratically concave function in Ω, and when $x \to 0$ we have $h(x) = \gamma + h'(0; x) + O(|x|^2)$. When $x \neq 0$ the minimum in (3.11) is atttained at a unique point $\xi = \xi(x) \in K$ because the minimized function is strictly convex and K is convex. It is clear that $\xi(x)$ must be a continuous function of x when $x \neq 0$, thus $h \in C^1(\Omega \setminus \{0\})$. To prove that $h \in C^{1,1}$ there we observe that if $p(\xi)$ is a quadratic polynomial with principal part $c|\xi|^2$ where $c > 0$ and $\min_{\xi \in K} p(\xi)$ is attained at $\xi \in K$, then $p(\eta) \geq p(\xi) + c|\eta - \xi|^2$ when $\eta \in K$ by Taylor's formula, for $\langle p'(\xi), \eta - \xi \rangle = \lim_{\varepsilon \to +0}(p(\xi + \varepsilon(\eta - \xi)) - p(\xi))/\varepsilon \geq 0$. Now let the minimum in the definition of $h(x)$ and of $h(y)$ be attained at $\xi \in K$ and $\eta \in K$ respectively. To estimate $\xi - \eta$ in terms of $x - y$ we first note that as just observed

$$\begin{aligned} \gamma + \langle x, \eta \rangle + \tfrac{1}{4}|x|^2|\eta|^2/\gamma &\geq h(x) + \tfrac{1}{4}|x|^2|\xi - \eta|^2/\gamma \\ &= \gamma + \langle x, \xi \rangle + \tfrac{1}{4}|x|^2(|\xi|^2 + |\xi - \eta|^2)/\gamma. \end{aligned}$$

Interchanging x, ξ and y, η we get

$$\gamma + \langle y, \eta \rangle + \tfrac{1}{4}|y|^2(|\eta|^2 + |\xi - \eta|^2)/\gamma \leq \gamma + \langle y, \xi \rangle + \tfrac{1}{4}|y|^2|\xi|^2/\gamma,$$

and subtraction gives after multiplication by 4γ

$$(|x|^2 + |y|^2)|\xi - \eta|^2 \leq 4\gamma\langle y - x, \xi - \eta \rangle + (|y|^2 - |x|^2)(|\xi|^2 - |\eta|^2).$$

Hence

$$|\xi - \eta| \leq |x - y|(4\gamma + |x + y||\xi + \eta|)/(|x|^2 + |y|^2),$$

and since $h'(x) = \xi(x) + \frac{1}{2}x|\xi(x)|^2/\gamma$, it follows that h' is locally Lipschitz continuous in $\complement\{0\}$, with $|h''(x)| \leq 2(1 + |x||\xi(x)|/\gamma)(\gamma + |x||\xi(x)|)/|x|^2 + \frac{1}{2}|\xi(x)|^2/\gamma$ almost everywhere.

Let $|\xi|^2 \leq A$ when $\xi \in K$, and set for $\delta > 0$

$$h_\delta(x) = \min_{\xi \in K}\bigl(\gamma + \delta(|\xi|^2 - A) + \langle x, \xi \rangle + \tfrac{1}{4}|x|^2|\xi|^2/\gamma\bigr).$$

It is clear that $h_\delta \uparrow h$ locally uniformly as $\delta \downarrow 0$, and h_δ is quadratically concave for the minimized functions are quadratically concave. Since they are strictly convex with respect to ξ, also when $x = 0$, the minimum is for every x attained at a unique point $\xi = \xi_\delta(x) \in K$, so h_δ is differentiable with $h'_\delta(x) = \xi_\delta(x) + \frac{1}{2}x|\xi_\delta(x)|^2/\gamma$, which proves that $h_\delta \in C^1$. The proof is complete.

EXAMPLE. If $K = \{\xi \in \mathsf{R}^N; |\xi| \le R, \xi'' = 0\}$, where $\xi' = (\xi_1, \dots, \xi_k)$, $\xi'' = (\xi_{k+1}, \dots, \xi_N)$ for some k with $0 < k < N$, then (3.11) gives

$$h(x) = \begin{cases} \gamma - \gamma|x'|^2/|x|^2, & \text{if } 2\gamma|x'| \le R|x|^2 \\ \gamma - R|x'| + \frac{1}{4}|x|^2R^2/\gamma, & \text{if } 2\gamma|x'| \ge R|x|^2. \end{cases}$$

Let $0 \le \chi \in C_0^\infty(\mathsf{R}^N)$ be an even function with $\int \chi(x)\,dx = 1$, and set $h_\varepsilon = h * \chi_\varepsilon$ where $\chi_\varepsilon(x) = \chi(x/\varepsilon)/\varepsilon^N$. Then the regularization h_ε is an even function so $h'_\varepsilon(0) = 0$. If $t \in \mathsf{R}^N$, $|t| = 1$ and $t' = 0$, then $\langle \partial, t\rangle^2 h(x)$ is equal to $R^2/2\gamma$ when $2\gamma|x'| > R|x|^2$ and equal to $-\gamma|x'|^2\langle\partial, t\rangle^2|x|^{-2}$ when $2\gamma|x'| < R|x|^2$, hence bounded then. Thus

$$\langle\partial, t\rangle^2 h_\varepsilon(0) - R^2/(2\gamma) = \int_{2\gamma|x'|<R|x|^2} \big(\langle\partial, t\rangle^2 h(x) - R^2/(2\gamma)\big)\chi_\varepsilon(x)\,dx$$

can be estimated by $C\int_{2\gamma|x'|\le\varepsilon R|x|^2}\chi(x)\,dx = O(\varepsilon^k)$ when $\varepsilon \to 0$. This means that h_ε is very far from satisfying (2.6) at the origin, for the left-hand side is $R^2/(2\gamma) + O(\varepsilon^k)$ while the right-hand side vanishes. It is therefore impossible to prove a converse of Theorem 3.7 by a standard regularization using a convolution, even if it is combined with some minor modification as in part c) of the proof of Theorem 2.4.

We shall now extend Theorem 2.4 by proving that functions in a ball satisfying an infinitesimal version of (3.8) in a very weak sense are quadratically concave there. The main step is to prove an analogue of Lemma 2.3.

LEMMA 3.10. *Let h be positive and weakly quadratically concave in a neighborhood of the closure of the unit ball B. Assume that*

$$(3.8)' \quad \varlimsup_{y\to x}(h(y) - h(x) - \langle y - x, \xi\rangle)/|y - x|^2 < \tfrac{1}{4}|\xi|^2/h(x), \qquad x \in \overline{B},\ \xi \in \tilde{h}'(x),$$

$$(3.12) \quad \langle x, \xi\rangle < h(x), \quad x \in \partial B,\ \xi \in \tilde{h}'(x).$$

Then the open subset $\Omega_{a,b} = \{x \in \overline{B}; h(x) > |ax + b|^2\}$ of $\overline{B}$ is connected for arbitrary $(a, b) \in \mathsf{R}_+ \times \mathsf{R}^N$.

PROOF. We shall follow the proof of Lemma 2.3 closely. As there we may assume that $(a, b) \ne (0, 0)$, and we set $g(x) = h(x) - f(x)$ where $f(x) = |ax + b|^2$. If $x \in \overline{B}$ and $g(x) = h(x) - f(x) = 0$, $f'(x) = \xi \in \tilde{h}'(x)$, then it follows from (3.8)′ that

$$\begin{aligned} f(y) &= f(x) + \langle y - x, f'(x)\rangle + \tfrac{1}{4}|f'(x)|^2|y - x|^2/f(x) \\ &= h(x) + \langle y - x, \xi\rangle + \tfrac{1}{4}|\xi|^2|y - x|^2/h(x) \ge h(y), \end{aligned}$$

when y is in a neighborhood of x. Thus $g(y) \le 0$, so x is not in the closure of $\Omega_{a,b}$. If $x \in B \cap \partial\Omega_{a,b}$ it follows that $f'(x) \notin \tilde{h}'(x)$, which means that $\langle z, f'(x)\rangle < h'(x; z)$ for some $z \in \mathsf{R}^N$. By the continuity of f' and the semi-continuity of h' this remains true with x replaced by any y in a neighborhood of x, and for all z in an open set. Hence g is strictly increasing in directions near z, in a neighborhood of x, so the equation $g = 0$ defines a Lipschitz surface satisfying a cone condition there. Since $\Omega_{a,b}$ consists of the points on one side, it is locally connected.

Suppose now that $x \in \partial B \cap \partial\Omega_{a,b}$, hence $g(x) = h(x) - f(x) = 0$, and that $f'(x) \notin \tilde{h}'(x)$. If $\langle z, f'(x)\rangle < h'(x; z)$ for some z with $\langle z, x\rangle = 0$, that is, tangent to ∂B, then it follows as when $x \in B$ that $\Omega_{a,b}$ is locally connected at x. On the other hand, if $\langle z, f'(x)\rangle \ge h'(x; z)$ when $\langle z, x\rangle = 0$ then the restriction of the linear form $z \to \langle z, f'(x)\rangle$ to the orthogonal space of x can be extended to a form satisfying such an inequality for all z, that is, $f'(x) + 2Cx = \xi \in \tilde{h}'(x)$ for some $C \ne 0$. (This may be true for all C in some interval not containing 0, but the sign of C is unique in any case since $f'(x) \notin \tilde{h}'(x)$.) By (3.8)$'$

$$
\begin{aligned}
g(x+y) &= h(x+y) - f(x+y) \\
&\le h(x) + \langle y, \xi\rangle + \tfrac{1}{4}|y|^2|\xi|^2/h(x) \\
&\qquad - f(x) - \langle y, f'(x)\rangle - \tfrac{1}{4}|f'(x)|^2|y|^2/f(x) \\
&= \langle y, 2Cx\rangle + \tfrac{1}{4}|y|^2(|\xi|^2 - |f'(x)|^2)/h(x) \\
&= \langle y, 2Cx\rangle + \tfrac{1}{4}|y|^2\langle 2Cx, 2\xi - 2Cx\rangle/h(x) \\
&= C\big(2\langle y, x\rangle + |y|^2(\langle x, \xi\rangle - C)/h(x)\big)
\end{aligned}
$$

when $|y|$ is sufficiently small. Since $2\langle y, x\rangle \le -|y|^2$ if $x + y \in \overline{B}$, it follows if $C > 0$ that

$$
g(x+y) \le C|y|^2(\langle x, \xi\rangle/h(x) - 1 - C/h(x)) \le 0
$$

by (3.12). Hence x is not in the closure of $\Omega_{a,b}$.

Finally, if $C < 0$, then $\tilde{h}'(x) - f'(x)$ contains $2Cx$ which is directed along the interior normal of B at x. Thus $g(x)$ is strictly increasing in a neighborhood of x in all directions close to the interior normal. Hence $\Omega_{a,b}$ is locally connected at x and contains the interior normal in a deleted neighborhood of x.

The proof of the lemma can now be completed by repeating the end of the proof of Lemma 2.3, so we leave this for the reader.

THEOREM 3.11. *Let h be a weakly quadratically concave function in the open unit ball B, and assume that*

$$(3.8)'' \quad \overline{\lim_{y\to x}}(h(y)-h(x)-\langle y-x,\xi\rangle)/|y-x|^2 \le \tfrac14|\xi|^2/h(x), \qquad x\in B,\ \xi\in\tilde h'(x).$$

Then it follows that (3.8) *is valid in B; in particular h is quadratically concave in B.*

PROOF. We shall proceed in three steps as in the proof of Theorem 2.4.

a) Assume at first that h is positive and satisfies (3.8)$'$ in a neighborhood of $\overline{B}$, and that (3.12) is fulfilled. Given $x\in B$ and $\xi\in\tilde h'(x)$ we set

$$|ay+b|^2 = h(x)+\langle y-x,\xi\rangle+\tfrac14|y-x|^2|\xi|^2/h(x), \quad \text{that is,}$$
$$a=\tfrac12|\xi|/\sqrt{h(x)}, \quad b=\sqrt{h(x)}\xi/|\xi|-\tfrac12 x|\xi|/\sqrt{h(x)}.$$

(If $\xi=0$ then $a=0$ and $b/\sqrt{h(x)}$ is any unit vector.) For some $\delta>0$ we have

$$\begin{aligned}
h(y)&-|ay+b|^2\\
&\le h(x)+\langle y-x,\xi\rangle+|y-x|^2\big(\tfrac14|\xi|^2/h(x)-\delta\big)-|ay+b|^2\\
&=-|y-x|^2\delta,
\end{aligned}$$

when $|y-x|$ is sufficiently small. For small $\varepsilon>0$ the set

$$\Omega_{(1-\varepsilon)a,(1-\varepsilon)b}=\{y\in\overline{B};\, h(y)>(1-\varepsilon)^2|ay+b|^2\}$$

has one component which is a neighborhood of x shrinking to $\{x\}$ when $\varepsilon\to0$. By Lemma 3.10 there can be no other component which proves that $\Omega_{a,b}=\emptyset$, hence that (3.8) is valid.

b) Now we just assume that h is positive and satisfies (3.8)$'$ in a neighborhood of $\overline{B}$ but we no longer assume (3.12). If (3.8) is valid in $\overline{B}$ and $x\in\partial B$, $0\ne\xi\in\tilde h'(x)$, then for $|y|\le1$

$$\big|h(x)\xi/|\xi|+\tfrac12(y-x)|\xi|\big|\ge\sqrt{h(x)h(y)}, \quad \text{hence } \big|h(x)\xi/|\xi|-\tfrac12x|\xi|\big|>\tfrac12|\xi|.$$

If we square it follows that $h(x)(h(x)-\langle x,\xi\rangle)>0$ which proves (3.12). (If $\xi=0$ then (3.12) is obvious.)

The proof of the theorem is now completed by “continuous induction”: If $h^r(x)=h(rx)$, then (3.12) holds and the statement is true when h is replaced by h^r and $r>0$ is sufficiently small. The set of such $r\le1$ is closed, and it follows from part a) of the proof that it is open, since (3.12) is valid for h^r when r is in this set. (See also the proof of Theorem 2.4.)

c) Assume now just that h is positive and that (3.8)$''$ is valid in a neighborhood of $\overline{B}$. Set $h_\varepsilon(x) = h(x) - g_\varepsilon(x)$ where $g_\varepsilon(x) = \varepsilon(|x|^2 + 1)$ with $\varepsilon > 0$ so small that $h_\varepsilon > 0$ in $\overline{B}$. Then $\tilde{h}'_\varepsilon(x) = \tilde{h}'(x) - g'_\varepsilon(x)$, and if $\xi \in \tilde{h}'(x)$ then

$$\begin{aligned}
&h_\varepsilon(y) - h_\varepsilon(x) - \langle y - x, \xi - g'_\varepsilon(x)\rangle - \tfrac{1}{4}|y - x|^2|\xi - g'_\varepsilon(x)|^2/h_\varepsilon(x)\\
&= h(y) - h(x) - \langle y - x, \xi\rangle - \tfrac{1}{4}|y - x|^2|\xi|^2/h(x)\\
&- \big(g_\varepsilon(y) - g_\varepsilon(x) - \langle y - x, g'_\varepsilon(x)\rangle - \tfrac{1}{4}|y - x|^2|g'_\varepsilon(x)|^2/g_\varepsilon(x)\big)\\
&+ \tfrac{1}{4}|y - x|^2\big(|\xi|^2/(h_\varepsilon(x) + g_\varepsilon(x)) - |\xi - g'_\varepsilon(x)|^2/h_\varepsilon(x) - |g'_\varepsilon(x)|^2/g_\varepsilon(x)\big).
\end{aligned}$$

The last parenthesis is ≤ 0, and the preceding one is equal to

$$\varepsilon|y - x|^2(1 - |x|^2/(|x|^2 + 1)) = \varepsilon|y - x|^2/(|x|^2 + 1).$$

Hence the upper limit in (3.8)$'$ is $\leq \frac{1}{4}|\xi - g'_\varepsilon(x)|^2/h_\varepsilon(x) - \varepsilon/(|x|^2 + 1)$ when h is replaced by h_ε, which proves that (3.8) is fulfilled in B when h is replaced by h_ε. When $\varepsilon \to 0$ we conclude that (3.8) is valid for h, under the additional assumption above that (3.8)$'$ is valid in a neighborhood of $\overline{B}$. If we drop that hypothesis too we just have to apply this conclusion to h^r and let $r \uparrow 1$ afterwards to complete the proof.

The relevance of Lemma 3.10 for the proof of Theorem 3.11 is underlined by the following general construction of examples of the kind given at the beginning of this section.

Proposition 3.12. *Assume that h is a positive (weakly) quadratically concave function in an open set $\Omega \subset \mathsf{R}^N$ and that for some $(a, b) \in \mathsf{R}_+ \times \mathsf{R}^N$ the open set*

$$\Omega_{a,b} = \{x \in \Omega;\, h(x) > |ax + b|^2\}$$

is not connected. Let ω be a component of $\Omega_{a,b}$ where $ax + b \neq 0$. Then

$$H(x) = \begin{cases} |ax + b|^2, & \text{when } x \in \omega,\\ h(x), & \text{when } x \in \Omega \setminus \omega,\end{cases}$$

is weakly quadratically concave but not *quadratically concave in Ω.*

Proof. Let ω' be the union of the components of $\Omega_{a,b}$ other than ω, thus $\omega \cap \omega' = \emptyset$ and $\omega \cup \omega' = \Omega_{a,b}$. If $t > 1$ and $t - 1$ is small enough then the open sets

$$\omega_t = \{x \in \omega;\, h(x) > t|ax + b|^2\}, \quad \omega'_t = \{x \in \omega';\, h(x) > t|ax + b|^2\}$$

are not empty, and $h(x) \leq t|ax + b|^2$ in $\Omega \setminus (\omega_t \cup \omega'_t)$. If we define

$$H_t(x) = \begin{cases} t|ax + b|^2, & \text{when } x \in \omega_t,\\ h(x), & \text{when } x \in \Omega \setminus \omega_t,\end{cases}$$

then $H_t(x) = \min(h(x), t|ax + b|^2)$ in $\Omega \setminus \overline{\omega'_t}$, for the minimum is equal to $t|ax + b|^2$ in ω_t and equal to $h(x)$ in $\Omega \setminus (\omega_t \cup \omega'_t)$. Thus H_t is weakly quadratically concave in each of the open sets $\Omega \setminus \overline{\omega_t}$ and $\Omega \setminus \overline{\omega'_t}$ which cover Ω since $\overline{\omega_t} \cap \overline{\omega'_t} \subset \omega \cap \omega' = \emptyset$. When $t \downarrow 1$ then $H_t \downarrow H_1 = H$, so H is weakly quadratically concave. However, $H(x) = |ax + b|^2$ when $x \in \omega$ and $H(x) > |ax + b|^2$ when $x \in \omega'$ so H is not quadratically concave.

The following proposition combined with Proposition 3.12 yields a general version of the example given at the beginning of the section.

PROPOSITION 3.13. *Let K be a compact set in R^N such that the convex hull* $\mathrm{ch}(K)$ *is of dimension $N - 1$. If K is not convex and ξ^0 is in the interior of* $\mathrm{ch}(K) \setminus K$ *in the affine hyperplane Π spanned by K, then the hypotheses of Proposition 3.12 are fulfilled if h is defined by* (3.11) *and*

$$|ax + b|^2 = \gamma + \langle x, \xi^0 \rangle + \tfrac{1}{4}|x|^2|\xi^0|^2/\gamma,$$

provided that Ω is a sufficiently small neighborhood of the origin.

PROOF. Since

$$\begin{aligned}\gamma + \langle x, \xi \rangle + \tfrac{1}{4}|x|^2|\xi|^2/\gamma &= \gamma\left|\tfrac{1}{2}x|\xi|/\gamma + \xi/|\xi|\right|^2 \\ &= \gamma\left|\tfrac{1}{2}|x|\xi/\gamma + x/|x|\right|^2\end{aligned}$$

we have for $x \neq 0$ with the notation $y = x/|x|^2$

$$h(x) = \gamma|x|^2 \min_{\xi \in K} |\xi/(2\gamma) + y|^2, \quad |ax + b|^2 = \gamma|x|^2|\xi^0/(2\gamma) + y|^2.$$

If we write

$$K_\gamma = \{-\xi/(2\gamma); \xi \in K\}, \quad \xi^0_\gamma = -\xi^0/(2\gamma),$$

then $h(x)$ is equal to $\gamma|x|^2$ times the distance from y to K_γ and $|ax + b|^2$ is equal to $\gamma|x|^2$ times the distance from y to ξ^0_γ. Hence $|ax + b|^2 > h(x)$ if $|y| > R = \sup_{\xi \in K_\gamma} |\xi|$ and $y \in -\Pi/(2\gamma)$, that is, if $|x| < 1/R$ and $0 \neq x \in S$ where S is the sphere (or hyperplane) through the origin obtained by inversion of $-\Pi/(2\gamma)$. Note that a unit normal ν of Π is a unit normal of S at 0. If instead $y = \xi^0_\gamma + \tau\nu$ for some $\tau \neq 0$ then ξ^0_γ is the only point in $-\Pi/(2\gamma)$ at distance τ from y, so $|\tau| < \min_{\xi \in K_\gamma} |y - \xi|$, hence $|ax + b|^2 < h(x)$ if $x = y/|y|^2$, which means that x is on a circle (or line) in the plane spanned by ν and ξ^0, with tangent ν at the origin. If Ω is a neighborhood of the origin and $|x| < 1/R$ when $x \in \Omega$ it follows that $\Omega_{a,b} \cap S = \emptyset$ but that $\Omega_{a,b}$ contains circular arcs (line segments) approaching the origin in the directions

$\pm\nu$. Hence the intersection of $\Omega_{a,b}$ with the interior and the exterior of S are non-empty, which proves the proposition.

Propositions 3.12 and 3.13 show that when $\tilde{h}'(x)$ has dimension $N-1$ it is unavoidable to assume in Theorem 3.11 that (3.8)$''$ is valid for all ξ in the convex set $\tilde{h}'(x)$ and not only for the extreme points. However, this is not at all clear when the dimension is N or $\le N-2$. Indeed, if K is a compact convex set with interior points then the proof of Proposition 3.13 shows that (3.11) does not change if the convex set K is replaced by its boundary. If the dimension is $\le N-2$ then the proof of Proposition 3.13 breaks down because the complement of the sphere $S^{N-k} \subset \mathsf{R}^N$ is connected when $k \ge 2$.

We shall finally prove that the hypothesis in Theorems 2.4 and 3.11 that B is a ball is essential. We begin with an observation on the analogous situation for convex (or concave) functions.

PROPOSITION 3.14. *If $\Omega \subset \mathsf{R}^N$ is an open set and $\overline{\Omega}$ is not convex, one can find $x, y \in \Omega$ and $\psi \in C^\infty(\Omega)$ such that $\psi'' \ge 0$ in Ω but $\psi(y) < \psi(x) + \langle y-x, \psi'(x)\rangle$.*

Thus ψ satisfies a local convexity condition but not a global one.

PROOF. If $\overline{\Omega}$ is not convex we can choose x^0, $y^0 \in \overline{\Omega}$ so that some point z^0 in the interval $[x^0, y^0]$ is not in $\overline{\Omega}$. If x and y are in Ω and $|x-x^0|+|y-y^0|$ is sufficiently small, then $[x, y]$ also contains a point z^1 so close to z^0 that $z^1 \notin \overline{\Omega}$. We can choose the coordinates so that $z^1 = 0$, thus $|z| < \delta$ implies $z \notin \overline{\Omega}$ for some $\delta > 0$, and x, y are on the x_1 axis, $x = (a, 0, \dots, 0)$, $y = (b, 0, \dots, 0)$ where $a < 0 < b$. Choose convex C^∞ functions ψ_+ and ψ_- of $z' = (z_2, \dots, z_N)$ such that $\psi_+(0) < \psi_-(0)$ but $\psi_+ = \psi_-$ when $|z'| > \delta/2$. We can for example choose ψ_+ strictly convex first and define ψ_- by adding a small non-negative function which is positive at the origin. If we define $\psi(z) = \psi_\pm(z')$ when $\pm z_1 \ge 0$ then $\psi \in C^\infty(\Omega)$ has the desired property.

We could have made ψ strictly convex by choosing $\psi_\pm$ strictly convex and adding εz_1^2 for some small positive ε. An analogue for quadratically concave functions follows at once:

LEMMA 3.15. *If $\Omega \subset \mathsf{R}^N$ is an open bounded set and $\overline{\Omega}$ is not convex, then one can choose $x, y \in \Omega$ and a positive function $h \in C^\infty(\Omega)$ such that h satisfies* (2.6) *but*

$$h(y) > h(x) + \langle y-x, h'(x)\rangle + \tfrac{1}{4}|y-x|^2|h'(x)|^2/h(x).$$

PROOF. Since Ω is bounded the function ψ constructed in Proposition 3.14 is bounded in Ω. Hence $h = 1 - \varepsilon\psi$ is positive in Ω if ε is a small positive number, and (2.6) is fulfilled since $h'' = -\varepsilon\psi'' \le 0$. We have

$$h(y) - h(x) - \langle y - x, h'(x)\rangle - \tfrac{1}{4}|y - x|^2|h'(x)|^2/h(x)$$
$$= -\varepsilon\big(\psi(y) - \psi(x) - \langle y - x, \psi'(x)\rangle + \tfrac{1}{4}|y - x|^2\varepsilon|\psi'(x)|^2/(1 - \varepsilon\psi(x))\big) > 0$$

if ε is sufficiently small.

Quadratic concavity and weak quadratic concavity are obviously invariant under Euclidean motions. To strengthen the conclusion of Lemma 3.15 we must use the invariance under inversions also, as in [4, Section 8]. We recall that the inversion in R^N with respect to the origin is defined by $x^* = x/|x|^2$ when $x \neq 0$; then $(x^*)^* = x$. If h is a function defined in an open set $\Omega \subset \mathsf{R}^N$ then composition with inversion gives a function in $\Omega^* = \{x^*; x \in \Omega\}$. When $h(x) = |ax + b|^2$ as in (2.1) then

$$h(x) = |ax^*|x|^2 + b|^2 = a^2|x^*|^2|x|^4 + 2a|x|^2\langle x^*, b\rangle + |b|^2$$
$$= |x|^2\big||b|x^* + ab/|b|\big|^2$$

where $b/|b|$ should be read as a unit vector if $b = 0$. If we define quite generally for functions h in Ω

$$h^*(x^*) = h(x)/|x|^2 = |x^*|^2 h(x^*/|x^*|^2), \quad x^* \in \Omega^*$$

then it follows from the definition (2.1) that quadratically concave functions are mapped to quadratically concave functions. If $h \in C^2(\Omega)$ satisfies (2.6) then $h^* \in C^2(\Omega^*)$ satisfies (2.6), for (2.6) means that $h'' \le h_0''$ at $x \in \Omega$ if $h_0(y) = |ay + b|^2$ and $h = h_0$, $h' = h_0'$ at x, which implies that $h^* = h_0^*$, $h^{*\prime} = h_0^{*\prime}$ and $h^{*\prime\prime} \le h_0^{*\prime\prime}$ at x^*. Since h_0^* is of the same form as h_0, the assertion follows. It can of course also be proved by a direct computation.

Theorem 3.16. *If Ω is an open bounded set in R^N such that* (2.6) *implies* (2.2) *when $h \in C^\infty(\Omega)$, then $\overline{\Omega}$ is a ball.*

Proof. By Lemma 3.15 $K = \overline{\Omega}$ must be convex, and by the precding observations on inversions K must remain convex after inversion in any point $x^0 \in \complement K$, defined as the inversion of $K - x^0$. Now let B be an open ball with minimal radius such that $\overline{B} \supset K$. Then $\partial B \cap \partial K$ must contain at least two points, for if there is just one such point a translation of B in the radial direction there would give a congruent open ball containing K. If $K \neq \overline{B}$ we can choose a point $x^0 \in B \setminus K$ and consider the inversion K^* of K with respect to x^0. Since $B \setminus \{x^0\}$ is mapped to the exterior of the sphere S obtained by inversion of ∂B, it follows that K^* contains no points in the interior of S but at least two points on S. This contradicts the convexity of K^*, for the open interval joining these points on S is in the interior of S, and the theorem is proved.

REFERENCES

1. Andersson, M., Passare, M., and Sigurdsson, R., *Complex Convexity and Analytic Functionals*, Birkhäuser, Basel-Boston-Berlin, 2004.
2. Behnke, H., and Peschl E., *Zur Theorie der Funktionen mehrerer komplexer Veränderlichen. Konvexität in bezug auf analytische Ebenen im kleinen und großen*, Math. Ann. 111 (1935), 158–177.
3. Hörmander, L., *Notions of Convexity*, Birkhäuser, Basel-Boston-Berlin, 1994.
4. Kiselman, C. O., *Lineally convex Hartogs domains*, Acta Math. Vietnam. 21 (1996), 69–94.
5. Kiselman, C. O., *A differential inequality characterizing weak lineal convexity*, Math. Ann. 311 (1998), 1–10.

DEPARTMENT OF MATHEMATICS
LUND UNIVERSITY
BOX 118
SE-221 00 LUND
SWEDEN
E-mail: lvh@maths.lth.se