MATH. SCAND. 102 (2008), 131–148

GRAND ORLICZ SPACES AND GLOBAL INTEGRABILITY OF THE JACOBIAN

C. CAPONE, A. FIORENZA and G. E. KARADZHOV*

Abstract

We introduce and investigate the grand Orlicz spaces and the grand Lorentz-Orlicz spaces. An application to the problem of global integrability of the Jacobian of orientation preserving mappings is given.

1. Introduction

The goal of this paper is to introduce and investigate the grand Orlicz spaces and the grand Lorentz-Orlicz spaces. The grand Lebesgue spaces were introduced by Iwaniec and Sbordone [12] and they found many applications in Analysis, see [12], [10]. The small Lebesgue spaces were introduced by A. Fiorenza [5] as associate to grand spaces. They have applications to some boundary value problems, see [17], [6]. Our investigation is closely related to [7], [4], where the second and the third authors studied the grand and small Lebesgue spaces and their analogs. The main difference with [7] is that now we do not use the general interpolation-extrapolation theory, although the technique from [13] is applied. The reason for this choice is that the real interpolation of the Orlicz spaces requires too strong conditions on the Orlicz functions. On the other hand, we use Lorentz-Orlicz classes, the quasinorm in which has the same structure as that given by the K-functional in the real interpolation. Therefore a direct approach is possible, which enable us to give characterizations of the grand Lorentz-Orlicz spaces and the grand Orlicz spaces (Theorem 2.4 and Proposition 2.6), similar to those for the grand Lebesgue spaces in [7].

In Section 3 we give some applications to the problem of global integrability of the Jacobian of orientation preserving mappings in R^n. Our results (Theorem 3.1 and Theorem 3.5) are analogs to the corresponding local estimates by Müller [15], Iwaniec and Sbordone [12], Greco [8], Iwaniec and Martin [11], Koskela and Zhong [14]. Note also that our technique could be applied

* The third author was partially supported by C.N.R. - Short Term Mobility Program 2006.
The authors are grateful to the referee whose suggestions improved the paper.
Received October 5, 2006.

to the problem of local integrability, but the results obtained will not be more general than those in Koskela and Zhong [14].

2. Grand Orlicz spaces

Let (X, μ) be σ-finite measure space. First we consider Orlicz classes $L_\Phi(X, \mu)$, denoted simply by L_Φ, where Φ is an Orlicz function, i.e. a positive, continuous, strictly increasing function on $(0, \infty)$, such that $\Phi(0) = 0$, $\Phi(1) = 1$, $\Phi(\infty) = \infty$.

DEFINITION 2.1 (Orlicz classes L_Φ).

$$L_\Phi := \left\{ f : \rho(f; \Phi) := \int_X \Phi(|f(x)|)\, d\mu = \int_0^\infty \Phi(f^*(t))\, dt < \infty \right\}. \tag{2.1}$$

where f^* denotes the decreasing rearrangement of f (see e.g. [3]).

The Orlicz class L_Φ is not a linear set in general. The Orlicz space, denoted by L^Φ, is defined as the set of all f such that

$$\|f\|_{L^\Phi} := \inf\{\lambda > 0 : \rho(f/\lambda; \Phi) \le 1\} < \infty.$$

This functional is a quasinorm if Φ satisfies the following conditions:

$$\Phi((1-\alpha)s + \alpha t) \le c[\Phi(s) + \Phi(t)], \qquad 0 < s, t < \infty, 0 < \alpha < 1, \tag{2.2}$$

for some constant $c \ge 1$, and

$$\Phi(\alpha t) \le c(\alpha)\Phi(t), \qquad 0 < \alpha < 1, \tag{2.3}$$

where $c(\alpha) \to 0$ as $\alpha \to 0$.

For instance, if Φ is convex, or equivalent to a convex function, then the conditions (2.2), (2.3) are satisfied. Other examples are the functions $\Phi(t) = t^p(1 + \log(1+t))^q$, $p > 0, q > 0$.

It will be convenient to introduce more general classes, the Lorentz-Orlicz classes $L_{h,\Phi}$, where h is a positive continuous weight on $(0, \infty)$.

DEFINITION 2.2 (Lorentz-Orlicz classes $L_{h,\Phi}$).

$$L_{h,\Phi} := \left\{ f : \rho(f; h, \Phi) := \int_0^\infty h(t)\Phi(f^*(t))\, dt < \infty \right\}. \tag{2.4}$$

The Lorentz-Orlicz space, denoted by $L^{h,\Phi}$, is defined as the set of all f such that

$$\|f\|_{L^{h,\Phi}} := \inf\{\lambda > 0 : \rho(f/\lambda; h, \Phi) \le 1\} < \infty$$

and this functional is a quasinorm if Φ satisfies (2.2) and (2.3).

For example, if $\Phi(t) = t^p$, $h(t) = t^{p/q-1}b^p(t)$, $0 < p, q < \infty$, and $b(t)$ is slowly varying on $(0, \infty)$, then $L^{h,\Phi}$ is the Lorentz-Karamata space $L_b^{q,p}$ with a quasinorm

$$\|f\|_{L_b^{q,p}} = \left(\int_0^\infty t^{p/q-1}[b(t)f^*(t)]^p\,dt\right)^{1/p}.$$

If $q = p$ we get the Lebesgue-Karamata space denoted by L_b^p. Recall the definition of the slowly varying function (see [16], [7]): if $s(t)$ is a positive continuous function on the interval $[1, \infty)$, we say that s is slowly varying on $[1, \infty)$ (in the sense of Karamata) if for all $\varepsilon > 0$ the function $t^\varepsilon s(t)$ is equivalent to a non-decreasing function and the function $t^{-\varepsilon}s(t)$ is equivalent to a non-increasing function. By symmetry, we say that a positive continuous function s on the interval $(0, 1]$ is slowly varying on $(0, 1]$ if the function $t \to s(1/t)$ is slowly varying on $[1, \infty)$. Finally, a positive continuous function s on $(0, \infty)$ is said to be slowly varying on $(0, \infty)$ if it is slowly varying on both $(0, 1]$ and $[1, \infty)$.

Note that the space $L_b^{p/(1+\sigma),p}$ has the quasinorm $\left(\int_0^\infty t^\sigma[b(t)f^*(t)]^p dt\right)^{1/p}$, $0 < \sigma < 1$. Comparing with [7], this suggests the following definition.

DEFINITION 2.3 (N-grand Lorentz-Orlicz classes $L_{h),\Phi),N}$).

$$L_{h),\Phi),N} := \Big\{f : \rho(f; h, \Phi, N) := \sup_{0<\sigma<\sigma_0} N(\sigma)\rho(f; h_\sigma, \Phi) < \infty\Big\}, \tag{2.5}$$

where $h_\sigma(t) := t^\sigma h(t)$, $0 < \sigma_0 < 1/2$, and $N(\sigma)$ is a positive continuous and increasing weight on $(0, 1)$, tempered in the sense that $N(2\sigma) \approx N(\sigma)$ (see [9]). For simplicity we write $L_{\Phi),N}$ if $h = 1$. The corresponding N-grand Lorentz-Orlicz space $L^{h),\Phi),N}$ is defined as the set of all f such that

$$\|f\|_{L^{h),\Phi),N}} := \inf\big\{\lambda > 0 : \rho(f/\lambda; h, \Phi, N) \le 1\big\} < \infty$$

and this functional is a quasinorm if Φ satisfies (2.2) and (2.3).

For example, if $\Phi(t) = t^p$, $h(t) = 1$, $N(\sigma) = \sigma$, and $\mu(X) = 1$, then we get the grand Lebesgue space $L^{p)}$ ([12], [7]). In this case (see Proposition 2.6 below)

$$f \in L^{p)} \qquad \text{iff} \qquad \sup_{0<\sigma<\sigma_0} \sigma \int_X |f(x)|^{p/(1+\sigma)}\,d\mu < \infty.$$

Now we introduce the grand Orlicz space. Its definition is suggested by the applications below where we consider a weight N_Φ, generated by the function

Φ:

$$\frac{1}{N_\Phi(\sigma)} := \int_2^\infty [\Phi(t)]^{\frac{1}{1+\sigma}} t^{-p-1}\,dt, \qquad \int_2^\infty \Phi(t)t^{-p-1}\,dt = \infty. \tag{2.6}$$

Since $\Phi(t) > 1$ for $t > 1$, we see that $N_\Phi(\sigma) \to 0$ as $\sigma \to 0$, decreasingly. Suppose that N_Φ is tempered on $(0, \sigma_0)$. Then the space $L^{\Phi),N_\Phi}$ is denoted by $L^{\Phi)}$ and called grand Orlicz space. The corresponding class $L_{\Phi)}$ is called grand Orlicz class. For example, if $\Phi(t) = t^p$, then $N_\Phi(\sigma) \approx \sigma$ and we get the grand Lebesgue space $L^{p)}$.

Our main result is the following characterization of the grand classes $L_{h),\Phi),N}$.

THEOREM 2.4.

$$L_{h),\Phi),N} = \Big\{ f : q(f; h, \Phi, w_N) := \sup_{0<t<1} w_N(t) \int_t^1 h(s)\Phi(f^*(s))\,ds + \int_1^\infty s^{\sigma_0} h(s)\Phi(f^*(s))\,ds < \infty \Big\},$$

where

$$w_N(t) := \sup_{0<\sigma<\sigma_0} N(\sigma)t^\sigma. \tag{2.7}$$

Moreover, if the condition (2.3) *is satisfied, then* $\rho(f; h, \Phi, N) \approx q(f; h, \Phi, w_N)$ *hence*

$$\|f\|_{L^{h),\Phi),N}} \approx \|f\| := \inf\{\lambda > 0 : q(f/\lambda; h, \Phi, w_N) \le 1\}. \tag{2.8}$$

Note that w_N is a slowly varying function on (0, 1) (cf. [7]).

PROOF. Using the identity

$$\alpha \int_0^1 t^{\alpha-1} \int_t^1 h(s)\Phi(f^*(s))\,ds\,dt = \int_0^1 s^\alpha h(s)\Phi(f^*(s))\,ds, \qquad \alpha > 0,$$

we can write

$$\rho(f; h, \Phi, N) \approx \sup_{0<\sigma<\sigma_0} N(\sigma)\sigma \int_0^1 t^{\sigma-1} g(t)dt + \sup_{0<\sigma<\sigma_0} N(\sigma) \int_1^\infty t^\sigma h(t)\Phi(f^*(t))\,dt = I + II, \tag{2.9}$$

where $g(t) := \int_t^1 h(s)\Phi(f^*(s))\,ds$.

To estimate I, we change the variables $\sigma \to 2\sigma$:

$$I \prec \sup_{0<\sigma<\sigma_0} N(\sigma)\sigma \int_0^1 t^{2\sigma-1} g(t)\,dt \prec \sup_{0<\sigma<\sigma_0} N(\sigma)\sigma \sup_{0<t<1} t^\sigma g(t) \int_0^1 t^{\sigma-1}\,dt,$$

whence

$$I \prec q(f; h, \Phi, w_N). \tag{2.10}$$

The estimate of II is as follows:

$$II \prec \int_1^\infty \sup_{0<\sigma<\sigma_0} N(\sigma) t^\sigma h(t)\Phi(f^*(t))\,dt \prec \int_1^\infty t^{\sigma_0} h(t)\Phi(f^*(t))\,dt. \tag{2.11}$$

The estimates (2.10), (2.11) and (2.9) give

$$\rho(f; h, \Phi, N) \prec q(f; h, \Phi, w_N). \tag{2.12}$$

To prove the reverse, we split the integral

$$\rho(f; h_\sigma, \Phi) := \int_0^\infty s^\sigma h(s)\Phi(f^*(s))\,ds = I(\sigma) + II(\sigma), \tag{2.13}$$

$$I(\sigma) = \int_0^1 s^\sigma h(s)\Phi(f^*(s))\,ds,$$

$$II(\sigma) = \int_1^\infty s^\sigma h(s)\Phi(f^*(s))\,ds.$$

Then

$$I(\sigma) \geq t^\sigma \int_t^1 h(s)\Phi(f^*(s))\,ds,\ 0 < t < 1,$$

whence

$$\sup_{0<\sigma<\sigma_0} N(\sigma) I(\sigma) \geq \sup_{0<t<1} w_N(t) \int_t^1 h(s)\Phi(f^*(s))\,ds. \tag{2.14}$$

On the other hand,

$$II(\sigma) \geq t^{\sigma-\sigma_0} \int_1^t s^{\sigma_0} h(s)\Phi(f^*(s))\,ds, \qquad 1 < t < \infty, 0 < \sigma < \sigma_0,$$

hence

$$\sup_{0<\sigma<\sigma_0} N(\sigma) II(\sigma) \geq N(\sigma_0) \int_1^t s^{\sigma_0} h(s)\Phi(f^*(s))\,ds, \qquad t > 1. \tag{2.15}$$

Therefore (2.13), (2.14), (2.15) give

$$\rho(f; h, \Phi, N) \succ q(f; h, \Phi, w_N). \tag{2.16}$$

Finally, to prove (2.8), first we use (2.16) and get

$$\|f\|_{L^{h),\Phi),N}} \geq \|f\|. \tag{2.17}$$

For the reverse, let $q(f/\lambda; h, \Phi, w_N) \leq 1$. Then (2.12) gives $\rho(f/\lambda; \Phi, h, N) \leq c$ for some constant $c > 1$. Using the property (2.3), we obtain

$$\rho(\alpha f/\lambda; h, \Phi, N) \leq c(\alpha)\rho(f/\lambda; h, \Phi, N) \leq cc(\alpha) \leq 1,$$

choosing $\alpha > 0$ so that $cc(\alpha) \leq 1$. Therefore,

$$\|f\|_{L^{h),\Phi),N}} \leq \|f\|/\alpha.$$

The theorem is proved.

REMARK 2.5. If the condition (2.3) is not satisfied, we still have the inequalities (2.16), (2.17).

As a particular case, let us consider the N-grand Lorentz-Karamata spaces $L^{q),p}_{b,N}$ $0 < q, p < \infty$, and b - slowly varying on $(0, \infty)$. By definition, $L^{q),p}_{b,N} = L^{h),\Phi),N}$, where $\Phi(t) = t^p$, $h(t) = t^{p/q-1}b^p(t)$. Hence

$$\|f\|_{L^{q),p}_{b,N}} = \sup_{0<\sigma<\sigma_0} [N(\sigma)]^{1/p} \|f\|_{L_b^{q/(1+\sigma),p}} \tag{2.18}$$

and Theorem 2.4 gives

$$\|f\|^p_{L^{q),p}_{b,N}} \approx \sup_{0<t<1} w_N(t) \int_t^1 t^{p/q-1}[b(t)f^*(t)]^p\, dt + \int_1^\infty t^{\sigma_0+p/q-1}[b(t)f^*(t)]^p\, dt.$$

In particular, if $q = p$ we get the N-grand Lebesgue-Karamata space $L^{p)}_{b,N}$:

$$\|f\|^p_{L^{p)}_{b,N}} \approx \sup_{0<t<1} w_N(t) \int_t^1 [b(t)f^*(t)]^p\, dt + \int_1^\infty t^{\sigma_0}[b(t)f^*(t)]^p\, dt.$$

In the case $\mu(X) = 1$ these formulae simplify.

PROPOSITION 2.6. *If $\mu(X) = 1$ then*

$$\|f\|_{L^{q),p}_{b,N}} \approx \sup_{0<t<1} \left(w_N(t) \int_t^1 s^{p/q-1}[b(s)f^*(s)]^p\, ds \right)^{1/p}, \tag{2.19}$$

where

$$w_N(t) = \sup_{0<\sigma<\sigma_0} N(\sigma)t^{\sigma}, \qquad 0 < t < 1.$$

Moreover, if $q = p$ *then we get the* N*-grand Lebesgue-Karamata space* $L^{p)}_{b,N}$ *with a quasinorm*

$$\begin{aligned} \|f\|_{L^{p)}_{b,N}} &\approx \sup_{0<\sigma<\sigma_0} [N(\sigma)]^{1/p} \|f\|_{L_b^{p/(1+\sigma)}} \\ &\approx \sup_{0<t<1} \left(w_N(t) \int_t^1 [b(s) f^*(s)]^p \, ds \right)^{1/p}. \end{aligned} \tag{2.20}$$

Proof. It is sufficient to prove the first part of (2.20). To this end we start with the embedding

$$L_b^{p/(1+\sigma)} \subset L_b^{p/(1+\sigma),p} \text{ uniformly with respect to } 0 < \sigma < \sigma_0. \tag{2.21}$$

To see this let $q = p/(1+\sigma)$. Then

$$\int_0^1 [b(s) f^*(s)]^q \, ds \geq [f^*(t)]^q \int_0^t b(s)^q \, ds.$$

Since b is slowly varying, $s^{-\delta} b(s) > c_\delta t^{-\delta} b(t)$ for some small $\delta > 0$, therefore

$$\|f\|_{L_b^q} \succ t^{1/q} b(t) f^*(t).$$

Using this estimate, we get

$$\begin{aligned} \int_0^1 t^{p/q-1} [b(t) f^*(t)]^p \, dt &= \int_0^1 t^{p/q-1} [b(t) f^*(t)]^q [b(t) f^*(t)]^{p-q} \, dt \\ &\prec \|f\|_{L_b^q}^{p-q} \int_0^1 [b(t) f^*(t)]^q \, dt, \end{aligned}$$

hence (2.21) follows. Thus (2.20) is proved in one direction. To prove the reverse let

$$I := \sup_{0<\sigma<\sigma_0} [N(\sigma)]^{1/p} \left(\int_0^1 t^{\sigma} [b(t) f^*(t)]^{p/(1+\sigma)} \, dt \right)^{(1+\sigma)}.$$

Changing the variables $\sigma \to 2\sigma$ and applying Hölder's inequality we get

$$\begin{aligned} I &:= \sup_{0<\sigma<\sigma_0} [N(\sigma)]^{1/p} \left(\int_0^1 t^{\frac{\sigma}{1+2\sigma}} [b(t) f^*(t)]^{p/(1+2\sigma)} t^{-\frac{\sigma}{1+2\sigma}} \, dt \right)^{(1+2\sigma)/p} \\ &\prec \sup_{0<\sigma<\sigma_0} [N(\sigma)]^{1/p} \left(\int_0^1 t^{\sigma} [b(t) f^*(t)]^p \, dt \right)^{1/p} \left(\int_0^1 t^{-1/2} \, dt \right)^{2\sigma/p}. \end{aligned}$$

Hence

$$I \prec \sup_{0<\sigma<\sigma_0} [N(\sigma)]^{1/p} \|f\|_{L_b^{p/(1+\sigma),p}}. \tag{2.22}$$

Now the other direction of (2.20) follows from (2.22).

The proposition is proved.

In general, we have the following theorem of equivalence.

THEOREM 2.7 (Equivalence). *If* $\mu(X) = 1$ *then*

$$f \in L_{\Phi),N} \quad \textit{iff} \quad \sup_{0<\sigma<\sigma_0} N(\sigma)\rho(f; \Phi^{\frac{1}{1+\sigma}}) < \infty. \tag{2.23}$$

PROOF. Step 1. First we prove

$$\left[\frac{1}{2}\rho\big(f; \Phi^{\frac{1}{1+2\sigma}}\big)\right]^{1+2\sigma} \le \rho(f; h_\sigma, \Phi) \le \left[\rho\big(f; \Phi^{\frac{1}{1+\sigma}}\big)\right]^{1+\sigma}, \tag{2.24}$$

where $h_\sigma(t) = t^\sigma$, $0 < \sigma < \sigma_0 < 1/2$.

Indeed, we have

$$\begin{aligned}
\rho(f; \Phi^{\frac{1}{1+2\sigma}}) &= \int_0^1 \big[\Phi(f^*(t))\big]^{\frac{1}{1+2\sigma}}\, dt = \int_0^1 \big[t^\sigma \Phi(f^*(t))\big]^{\frac{1}{1+2\sigma}} t^{-\frac{\sigma}{1+2\sigma}}\, dt \\
&\le \left(\int_0^1 t^\sigma \Phi(f^*(t))\, dt\right)^{\frac{1}{1+2\sigma}} \left(\int_0^1 t^{-1/2}\, dt\right)^{\frac{2\sigma}{1+2\sigma}} \\
&\prec 2\big[\rho(f; h_\sigma, \Phi)\big]^{\frac{1}{1+2\sigma}},
\end{aligned}$$

whence the left part of (2.24) follows. To see the right part, we write

$$\rho\big(f; \Phi^{\frac{1}{1+\sigma}}\big) \ge \int_0^t \big[\Phi(f^*(s))\big]^{\frac{1}{1+\sigma}}\, ds \ge t\big[\Phi(f^*(t))\big]^{\frac{1}{1+\sigma}},$$

thus

$$\Phi(f^*(t)) \le t^{-1-\sigma}\big[\rho(f; \Phi^{\frac{1}{1+\sigma}})\big]^{1+\sigma}.$$

Therefore,

$$\begin{aligned}
\int_0^1 t^\sigma \Phi(f^*(t))\, dt &= \int_0^1 t^\sigma \big[\Phi(f^*(t))\big]^{\frac{\sigma}{1+\sigma}} \big[\Phi(f^*(t))\big]^{\frac{1}{1+\sigma}}\, dt \\
&\le \int_0^1 \big[\Phi(f^*(t))\big]^{\frac{1}{1+\sigma}}\, dt \big[\rho\big(f; \Phi^{\frac{1}{1+\sigma}}\big)\big]^\sigma \\
&= \big[\rho\big(f; \Phi^{\frac{1}{1+\sigma}}\big)\big]^{1+\sigma}.
\end{aligned}$$

Step 2. Let

$$I = \sup_{0<\sigma<\sigma_0} I(\sigma), \qquad I(\sigma) = N(\sigma)\rho(f; h_\sigma, \Phi),$$

$$J = \sup_{0<\sigma<\sigma_0} J(\sigma), \qquad J(\sigma) = N(\sigma)\rho\big(f; \Phi^{\frac{1}{1+\sigma}}\big).$$

If $J < \infty$, then

$$\sup_{0<\sigma<\sigma_0} [J(\sigma)]^{1+\sigma} = \max\left(\sup_{J(\sigma)\le 1} J(\sigma), \sup_{J(\sigma)>1} [J(\sigma)]^{1+\sigma_0}\right) \le \max(J, J^{1+\sigma_0}),$$

hence (2.24) implies $I \le \max(J, J^{1+\sigma_0}) < \infty$. Thus (2.23) is proved in one direction. The proof of the reverse is analogous.

3. Global integrability of the Jacobian

Let Ω be an open subset in R^n, $n \ge 2$, and let $f = (f_1, \ldots, f_n) : \Omega \to \mathsf{R}^n$, be in the Sobolev class $W^{1,1}_{\mathrm{loc}}$. Then the differential matrix $Df(x)$ is defined a.e. in Ω. Let $J = J(f; x) = \det Df(x)$ be the Jacobian determinant and suppose that $J(f, x) \ge 0$. Let $g(x) := |Df(x)|$ be the operator norm of the differential matrix $Df(x)$. In the theory of mappings of finite distortion, which has various applications in Analysis, in PDE, the problem of finding optimal conditions on the function g such that $J \in L^1_{\mathrm{loc}}$, is important. This problem was studied in various papers. Let us recall some results. First, by Hadamard's inequality, $|J(f; x)| \le g(x)^n$, hence we have a condition for global integrability:

$$\int_{\mathsf{R}^n} |J(f; x)|\, dx \le \|g\|^n_{L^n}. \tag{3.1}$$

Further, we have the following results for local integrability. S. Müller [15] proved that

$$g \in L^n \Rightarrow J \in L_{\mathrm{loc}} \log L.$$

T. Iwaniec and C. Sbordone [12] sharpened (3.1) (in the case of local integrability) as follows

$$g \in L^{n)} \Rightarrow J \in L^1_{\mathrm{loc}}.$$

L. Greco [8] showed that

$$g \in L^\Phi \Rightarrow J \in L^\Psi_{\mathrm{loc}},$$

for some Orlicz functions Φ and Ψ, generalizing the result of S. Müller.

P. Koskela and X. Zhong [14] proved that

$$g \in GL^\Phi \Rightarrow J \in L^1_{\mathrm{loc}},$$

where GL^{Φ} is some kind of grand Orlicz class, defined as follows

$$\limsup_{N\to\infty} \frac{1}{\widetilde{\Phi}(N)} \int_{g(x)\le N} \Phi(g(x))\,dx < \infty.$$

Here the Orlicz function Φ satisfies the conditions:

$$\frac{d}{dt}\left(\frac{\Phi(t)}{t^{n-1+\delta}}\right) \ge 0, \quad \text{for some } \ 0<\delta<1, \delta=0 \ \text{ if } \ n>2, \tag{3.2}$$

and

$$\widetilde{\Phi}(N) := \int_1^N \frac{\Phi(t)}{t^{n+1}}\,dt \to \infty \qquad \text{as} \quad N\to\infty.$$

In this section we give global estimates, when $\Omega = \mathsf{R}^n$ and $g \in L^{\Phi)}$. We shall use the conclusion in Remark 2.5 and define the grand space $L^{\Phi)}$ without the conditions (2.2) and (2.3). Moreover, the conditions on $\Phi(t)$ below are stated explicitly only for $t>1$, supposing that on the interval (0, 1) the function Φ is defined in such a way that it is an Orlicz function.

THEOREM 3.1. *Let Φ be an Orlicz function such that $\int_1^\infty \Phi(t)t^{-n-1}\,dt = \infty$. Moreover, let*

$$\frac{1}{N_\Phi(\sigma)} := \int_2^\infty [\Phi(t)]^{\frac{1}{1+\sigma}} t^{-n-1}\,dt \ \textit{be tempered on } (0,\sigma_0), \ \ \sigma_0 < 1/2, \tag{3.3}$$

and let the function $H_\sigma(t) := \Phi^{\frac{1}{1+\sigma}}(t)t^{1-\delta-n}$ be increasing in $(1,\infty)$ for some small $\delta>0$ and for $0\le\sigma\le\sigma_1<\sigma_0$. (We can take $\delta=0$ if $n>2$.) Let $\lim_{t\to\infty} t^{-n}\Phi^{\frac{1}{1+\sigma}}(t) < \infty$ if $0<\sigma<\sigma_0$. Finally, if $g\in L^{n-\varepsilon}$ for some small $\varepsilon\ge 0$, then

$$\int_{\mathsf{R}^n} J(f;x)\,dx \prec \|g\|^n_{L^{\Phi)}}. \tag{3.4}$$

Moreover, the following estimates are also valid

$$\int_{\mathsf{R}^n} J(f;x)\,dx \prec \limsup_{\sigma\to 0} N_\Phi(\sigma) \int_{\{g(x)>2\}} \Phi^{\frac{1}{1+\sigma}}(g(x))\,dx, \tag{3.5}$$

$$\int_{\mathsf{R}^n} J(f;x)\,dx \prec \sup_{0<t<\mu_g(2)} w_\Phi(t) \int_t^{\mu_g(2)} \Phi(g^*(s))\,ds, \tag{3.6}$$

where

$$w_\Phi(t) := \sup_{0<\sigma<\sigma_0} N_\Phi(\sigma)t^\sigma \tag{3.7}$$

and $\mu_g(t) := \mu\{x : g(x)>t\}$ is the distribution function of g.

PROOF. Since $g \in L^{n-\varepsilon}$ we can apply the argument from [18] (see formula (2.3)) and derive the estimate

$$\int_{\{M(x)<t\}} J(f;x)\,dx \prec t \int_{\{M(x)>t\}} |g(x)|^{n-1}\,dx, \qquad t>0, \tag{3.8}$$

where $M(x)$ is the Hardy-Littlwood maximal function of g. Using $M(x) \geq g(x)$ and an argument from [14], we can replace the set $\{M(x) > t\}$ in (3.8) by $\{2g(x) > t\}$ and get the estimate

$$\int_{\{M(x)<2t\}} J(f;x)\,dx \prec t^{1-\delta} \int_{\{g(x)>t\}} |g(x)|^{n-1+\delta}\,dx. \tag{3.9}$$

Here $\delta > 0$ is needed only if $n = 2$. Introduce the function

$$\psi_\sigma(t) := \int_2^t s^{-1+\delta}\,dH_\sigma(s).$$

Multiplying (3.9) by $t^{-1+\delta}\frac{d}{dt}H_\sigma(t)$, integrating and using Fubini, we get

$$\int_{\mathsf{R}^n} J(f;x) \int_{P(x)}^\infty d\psi_\sigma(t)\,dx \prec \int_2^\infty \int_{\{g(x)>t\}} |g(x)|^{n-1+\delta}\,dx\,dH_\sigma(t),$$

where $P(x) := \max(2, M(x)/2)$. Using again Fubini, we rewrite this as

$$\int_{\mathsf{R}^n} J(f;x)[\psi_\sigma(\infty) - \psi_\sigma(P(x))]\,dx \prec \int_{\{g(x)>2\}} \Phi^{\frac{1}{1+\sigma}}(g(x))\,dx.$$

Further, integrating by parts, we see that (since $\lim_{t\to\infty} t^{-n}\Phi^{\frac{1}{1+\sigma}}(t) < \infty$)

$$\psi_\sigma(\infty) \succ \frac{1-\delta}{N_\Phi(\sigma)} - 1,$$

hence

$$J_\sigma := \int_{\mathsf{R}^n} J(f;x)\left[1 - \frac{\psi_\sigma(P(x))}{\psi_\sigma(\infty)}\right] dx \prec N_\Phi(\sigma) \int_{\{g(x)>2\}} \Phi^{\frac{1}{1+\sigma}}(g(x))\,dx, \tag{3.10}$$

for $0 < \sigma < \sigma_1$ and small σ_1. From here we derive (3.5) taking the limit as $\sigma \to 0$. To prove (3.6), we are going to estimate appropriately the right-hand side of (3.10), using the same technique as in the proof of Theorem 2.4. Let

$$I(\sigma) := \int_{\{g(x)>2\}} \Phi^{\frac{1}{1+\sigma}}(g(x))\,dx.$$

Then

$$I(\sigma) = \int_0^{\mu_g(2)} \Phi^{\frac{1}{1+\sigma}}(g^*(t))\,dt$$

and by Hölder's inequality,

$$I(2\sigma) \prec \left(\int_0^{\mu_g(2)} t^\sigma \Phi(g^*(t))\,dt\right)^{\frac{1}{1+2\sigma}} [\mu_g(2)]^{\frac{2\sigma}{1+2\sigma}}.$$

If the integral $I(0)$ is divergent, we get the estimate

$$J_{2\sigma}[\mu_g(2)]^{-\frac{2\sigma}{1+2\sigma}} \prec N_\Phi(\sigma) \int_0^{\mu_g(2)} t^\sigma \Phi(g^*(t))\,dt, \qquad 0 < \sigma < \sigma_g < \sigma_0/2. \tag{3.11}$$

On the other hand, if the integral $I(0)$ is convergent, then

$$\begin{aligned} J_{2\sigma}[\mu_g(2)]^{-\frac{2\sigma}{1+2\sigma}} &\left(\int_0^{\mu_g(2)} t^\sigma \Phi(g^*(t))\,dt\right)^{\frac{2\sigma}{1+2\sigma}} \\ &\prec N_\Phi(\sigma) \int_0^{\mu_g(2)} t^\sigma \Phi(g^*(t))\,dt, \qquad 0 < \sigma < \sigma_0/2. \end{aligned} \tag{3.12}$$

In (3.11), (3.12) we replace again σ by 2σ and use the estimate

$$\begin{aligned} \int_0^{\mu_g(2)} t^{2\sigma} \Phi(g^*(t))\,dt &= 2\sigma \int_0^{\mu_g(2)} t^{2\sigma-1} \int_t^{\mu_g(2)} \Phi(g^*(s))\,ds\,dt \\ &\prec \sup_{0<t<\mu_g(2)} t^\sigma \int_t^{\mu_g(2)} \Phi(g^*(s))\,ds[\mu_g(2)]^\sigma. \end{aligned}$$

Thus

$$J_{4\varepsilon} A_g^\varepsilon \prec \sup_{0<\sigma<\sigma_0} N_\Phi(\sigma) \sup_{0<t<\mu_g(2)} t^\sigma \int_t^{\mu_g(2)} \Phi(g^*(s))\,ds \tag{3.13}$$

for all $0 < \varepsilon < \varepsilon_g$, where A_g is a positive functional, depending on g. Taking the limit as $\varepsilon \to 0$, we obtain (3.6), (3.7). In order to prove (3.4), we start with

$$\int_{\mathsf{R}^n} J(f;x)\,dx \prec \sup_{0<\sigma<\sigma_0} N_\Phi(\sigma) \sup_{0<t<\mu_g(2)} t^\sigma \int_t^{\mu_g(2)} \Phi(g^*(s))\,ds.$$

If $\mu_g(2) \le 1$ then

$$\int_{\mathsf{R}^n} J(f;x)\,dx \prec \sup_{0<t<1} w_N(t) \int_t^1 \Phi(g^*(s))\,ds. \tag{3.14}$$

If $\mu_g(2) > 1$, then

$$\begin{aligned}\int_{\mathsf{R}^n} & J(f;x)\,dx \\ &\prec \sup_{0<\sigma<\sigma_0} N_\Phi(\sigma) \sup_{0<t<1} \left(t^\sigma \int_t^1 \Phi(g^*(s))\,ds + \int_1^\infty s^\sigma \Phi(g^*(s))\,ds \right) \\ &\qquad + \sup_{t>1} \sup_{0<\sigma<\sigma_0} N_\Phi(\sigma) t^{\sigma-\sigma_0} \int_t^\infty s^{\sigma_0} \Phi(g^*(s))\,ds \\ &\prec \sup_{0<t<1} w_N(t) \int_t^1 \Phi(g^*(s))\,ds + \int_1^\infty s^{\sigma_0} \Phi(g^*(s))\,ds.\end{aligned}$$

From here and (3.14) we get

(3.15)
$$\int_{\mathsf{R}^n} J(f;x)\,dx \prec \sup_{0<t<1} w_N(t) \int_t^1 \Phi(g^*(s))\,ds + \int_1^\infty s^{\sigma_0} \Phi(g^*(s))\,ds.$$

Using Theorem 2.4 and Remark 2.5, we see that (3.15) means

$$\int_{\mathsf{R}^n} J(f;x)\,dx \prec \rho(g; 1, \Phi, N_\Phi). \tag{3.16}$$

Finally, using homogeneity we get (3.4).

The theorem is proved.

REMARK 3.2. Starting with the estimate

$$\int_{\mathsf{R}^n} J(f;x)\,dx \prec \sup_{0<\sigma<\sigma_0} N_\Phi(\sigma) \int_{\{g(x)>2\}} \Phi^{\frac{1}{1+\sigma}}(g(x))\,dx,$$

and taking the supremum under the sign of the integral, we get the estimate

$$\int_{\mathsf{R}^n} J(f;x)\,dx \prec \int_{\{g(x)>2\}} \Phi(g(x)) v_\Phi(g(x))\,dx$$

if $g \in L^{n-\varepsilon}$ for some small $\varepsilon \geq 0$, where

$$v_\Phi(t) = \sup_{0<\sigma<\sigma_0} N_\Phi(\sigma)[\Phi(t)]^{-\frac{\sigma}{1+\sigma}}.$$

In particular, if $\Phi(t) = t^n$, then $N_\Phi(\sigma) \approx \sigma$ and $v_\Phi(t) \approx (\log t)^{-1}$ for $t > 2$. Hence if $g \in L^{n-\varepsilon}$ for some small $\varepsilon \geq 0$, then

$$\int_{\mathsf{R}^n} J(f;x)\,dx \prec \int_{\{g(x)>2\}} (g(x))^n (\log g(x))^{-1}\,dx.$$

Another corollary is the estimate

$$\int_{\mathbb{R}^n} J(f;x)\,dx \prec \int_{\mathbb{R}^n} \frac{(g(x))^n}{\log(2+g(x))}\,dx,$$

which is an improvement of (3.1) and an analog of the local estimate (0.7) in [12].

Note that (3.8) is true for $g \in L^n$. Hence,

$$\int_{M(x)<t} J(f;x)\,dx \prec \int_{M(x)>t} |M(x)|^n\,dx$$

and taking the limit as $t \to \infty$, we obtain $J(f;x) = 0$. In particular, this means that inequality (3.1) is trivial if $J(f;x) \geq 0$. (see also Theorem 7.82 and Corollary 7.2.1 [11]) We can generalize this result replacing L^n by the Orlicz class L_Φ.

REMARK 3.3. Let the conditions of Theorem 3.1 be satisfied. If $g \in L_\Phi$ then $J(f;x) = 0$.

PROOF. In (3.10) we can replace $\Phi^{\frac{1}{1+\sigma}}$ by Φ and take the limit as $\sigma \to 0$. Hence $\int_{\mathbb{R}^n} J(f;x)\,dx = 0$.

If in Theorem 3.1 we choose $\Phi(t) = t^n$, then we get

COROLLARY 3.4.

$$\int_{\mathbb{R}^n} J(f;x)\,dx \prec \|g\|^n_{L^{n)}}. \tag{3.17}$$

Moreover, if $g \in L^{n-\varepsilon}$ for some small $\varepsilon \geq 0$, then

$$\int_{\mathbb{R}^n} J(f;x)\,dx \prec \sup_{0<t<\mu_g(1)} w(t) \int_t^{\mu_g(1)} [g^*(s)]^n\,ds, \tag{3.18}$$

where

$$w(t) \approx (1-\log t)^{-1} \text{ if } 0<t<1, \qquad w(t) \approx t^{\sigma_0} \text{ if } t>1. \tag{3.19}$$

Theorem 3.1 can be applied also to the case $\Phi(t) := t^n b^n(t)$, where b is slowly varying on $(0,\infty)$. Moreover, we shall show that in this case the corresponding grand Orlicz space $L^{\Phi)}$ can be replaced by the grand Lebesgue-Karamata space $L^{n)}_b$, where

$$\|g\|_{L^{n)}_b} := \sup_{0<\sigma<\sigma_0} [N_b(\sigma)]^{1/n} \|g\|_{L_b^{n/(1+\sigma),n}}$$

and

$$\frac{1}{N_b(\sigma)} := \int_2^\infty t^{-\frac{\sigma n}{1+\sigma}-1} b^{\frac{n}{1+\sigma}}(t)\,dt, \qquad \int_2^\infty b^n(t)t^{-1}\,dt = \infty.$$

THEOREM 3.5. *Let b be slowly varying on $(0,\infty)$, decreasing and such that $tb(t)$ is increasing, $b(t) \to 0$ as $t \to \infty$, $b(t) \approx b(t^a)$, a real number, and let N_b be tempered on $(0,\sigma_0)$, $\sigma_0 < 1/2$. If $g \in L^{n-\varepsilon}$ for some small $\varepsilon \geq 0$, then*

$$\int_{\mathsf{R}^n} J(f;x)\,dx \prec \|g\|^n_{L_b^{n)}}. \tag{3.20}$$

Moreover, the following characterization of the grand Lebesgue-Karamata space is also valid

$$\|g\|_{L_b^{n)}} := \sup_{0<t<1} \left(w_b(t) \int_t^1 [b(s)g^*(s)]^n\,ds \right)^{1/n} + \left(\int_1^\infty s^{\sigma_0}[b(s)g^*(s)]^n\,ds \right)^{1/n},$$

where

$$w_b(t) := \sup_{0<\sigma<\sigma_0} N_b(\sigma)t^\sigma. \tag{3.21}$$

PROOF. We start with the following estimate that can be proved analogously to (3.11), (3.12):

$$J_{2\sigma} A_g^\sigma \prec N_b(\sigma) \int_0^\infty t^\sigma [g^*(t)b(g^*(t))]^n\,dt = N_b(\sigma)(I + II),$$

for all $0 < \sigma < \sigma_g$, where A_g is a positive functional, depending on g and I is the part of the integral over $(0,1)$. Our goal is to replace $b(g^*(t))$ by $b(t)$, using the fact that b is slowly varying.

To estimate I, we represent $(0,1)$ as a union $A \cup B$, such that $g^*(t) > t^{-1/2n}$ on A and $g^*(t) \leq t^{-1/2n}$ on B. Then

$$I \prec \int_A t^\sigma [g^*(t)b(g^*(t))]^n\,dt + I_1, \qquad I_1 := \int_B t^\sigma [g^*(t)b(g^*(t))]^n\,dt.$$

Since b is slowly varying and $b(t^a) \approx b(t)$, a real number, we see that $g^*(t) \leq t^{-1/2n}$ implies $\{[g^*]^\delta(t)b(g^*(t)\}^n \leq c_\delta t^{-\delta/2} b^n(t)$ for all $\delta > 0$. Hence

$$I_1 \prec c_\delta \int_B t^\sigma [g^*(t)]^{n(1-\delta)} t^{-\delta/2} b^n(t)\,dt$$

and using Hölder's inequality, we get

$$I_1 \prec c_\delta \left(\int_0^1 t^\sigma [g^*(t)b(t)]^n\,dt\right)^{1-\delta} \left(\int_0^1 t^{\sigma-1/2}[b(t)]^n\,dt\right)^\delta,$$

or

$$I \prec \|g\|^n_{L_b^{n/(1+\sigma),n}} + c_\delta c^\delta \|g\|^{n(1-\delta)}_{L_b^{n/(1+\sigma),n}}.$$

Analogously we estimate the integral II, representing $(1,\infty)$ as the union $A\cup B$, where $g^*(t) > t^{-2/n}$ on A and $g^*(t) \le t^{-2/n}$ on B. Thus

$$II \prec \|g\|^n_{L_b^{n/(1+\sigma),n}} + c_\delta c^\delta \|g\|^{n(1-\delta)}_{L_b^{n/(1+\sigma),n}}.$$

Hence

$$J_{2\sigma} A_g^\sigma \prec N_b(\sigma)\left[\|g\|^n_{L_b^{n/(1+\sigma),n}} + c_\delta c^\delta \|g\|^{n(1-\delta)}_{L_b^{n/(1+\sigma),n}}\right].$$

To get rid of δ, we first replace f by λf, whence

$$J_{2\sigma} A_g^\sigma \prec N_b(\sigma)\left[\|g\|^n_{L_b^{n/(1+\sigma),n}} + c_\delta c^\delta \lambda^{-\delta n} \|g\|^{n(1-\delta)}_{L_b^{n/(1+\sigma),n}}\right],$$

and choosing λ such that $\lambda^{-\delta n} c_\delta = 1$, we get

$$J_{2\sigma} A_g^\sigma \prec N_b(\sigma)\left[\|g\|^n_{L_b^{n/(1+\sigma),n}} + c^\delta \|g\|^{n(1-\delta)}_{L_b^{n/(1+\sigma),n}}\right],$$

for all $\delta > 0$. Taking the limit as $\delta \to 0$, we obtain

$$J_{2\sigma} A_g^\sigma \prec N_b(\sigma)\|g\|^n_{L_b^{n/(1+\sigma),n}}.$$

From here we get the estimate (3.20) using the same arguments as at the end of the proof of Theorem 3.1. The theorem is proved.

EXAMPLE 3.6. Let $b(t) := (1+|\log t|)^{-\alpha}$, $0 \le \alpha \le 1/n$. Since

$$\begin{aligned}\frac{1}{N_b(\sigma)} &:= \int_1^\infty t^{-\sigma n-1}(1+\log t)^{-\alpha n(1-\sigma)}\,dt \approx \int_1^\infty e^{-\sigma u n} u^{-\alpha n(1-\sigma)}\,du \\ &\approx \int_\sigma^\infty e^{-vn}\left(\frac{v}{\sigma}\right)^{-\alpha n(1-\sigma)}\,dv/\sigma \approx \sigma^{n\alpha-1}\int_\sigma^\infty e^{-vn} v^{-\alpha n(1-\sigma)}\,dv \\ &\approx \sigma^{n\alpha-1}\left(1+\int_\sigma^1 v^{-\alpha n(1-\sigma)}\,dv\right) \approx \sigma^{n\alpha-1}, \qquad \alpha < 1/n,\end{aligned}$$

we have

$$N_b(\sigma) \approx \sigma^{1-n\alpha}, \qquad \text{if} \quad \alpha < 1/n$$

and

$$N_b(\sigma) \approx |\log \sigma|^{-1}, \qquad \text{if} \quad \alpha = 1/n.$$

Therefore in this case for $0 < t < 1$,

$$w_b(t) \approx (1 - \log t)^{n\alpha - 1} \qquad \text{if} \quad 0 \leq \alpha < 1/n,$$

and

$$w_b(t) \approx (\log(1 - \log t))^{-1} \qquad \text{if} \quad \alpha = 1/n.$$

Note that, in general, the class GL^{Φ} introduced by Koskela and Zhong [14] and used to obtain local integrability of the Jacobian, is larger than $L_{\Phi)}$. For example, suppose that $\Phi(t) = t^n(1 + |\log t|)^{-\alpha}$, $0 < \alpha < 1/n$. Then for large M,

$$\tilde{\Phi}(M) \approx (\log M)^{1-\alpha n}$$

and $N_{\Phi}(\sigma) \approx \sigma^{1-\alpha n}$ for small σ. Let $\sigma = (\log M)^{-1}$. Then

$$\frac{1}{\tilde{\Phi}(M)} \int_{g(x)<M} \Phi(g(x))\, dx \prec N_{\Phi}(\sigma) \int_{\Omega} [\Phi(g(x))]^{\frac{1}{1+\sigma}}\, dx.$$

Since the volume of Ω is finite, the assertion follows from here and Theorem 2.7.

REFERENCES

1. Berg, J., and Löfström, J., *Interpolation spaces*, Springer-Verlag, 1976.
2. Bennett, C., and Rudnick, K., *On Lorentz-Zygmund spaces*, Dissertationes Math. (Rozprawy Mat.) 175 (1980), 1–72.
3. Bennett, C., and Sharpley, R., *Interpolation of Operators*, Academic Press, 1988.
4. Capone, C., and Fiorenza, A., *On small Lebesgue spaces*, J. Funct. Spaces Appl. 3 (1) 73–89 (2005).
5. Fiorenza, A., *Duality and reflexivity in grand Lebesgue spaces*, Collect. Math. 51 (2000), 131–148.
6. Fiorenza, A., and Rakotoson, J. M., *New properties of small Lebesgue spaces and their applications*, Math. Ann. 326 (2003) 543–561.
7. Fiorenza, A., and Karadzhov, G. E., *Grand and Small Lebesgue Spaces and Their Analogs*, Zeit. Anal. Anwendungen 23 (2004), 657–681.
8. Greco, L., *Sharp integrability of nonnegative Jacobians*, Rend. Mat. Appl. 18 (1998), 585–600.
9. Jawerth, B., and Milman, M., *Extrapolation theory with applications*, Mem. Amer. Math. Soc. 440 (1991).
10. Iwaniec, T., Koskela, P., and Onninen, J., *Mappings of finite distortion: Monotonicity and Continuity*, Invent. Math. 144 (2001), 507–531.
11. Iwaniec, T., Martin, G., *Geometric Function Theory and Non-linear Analysis*, Oxford Science Publications, 2001.

12. Iwaniec, T., and Sbordone, C., *On the integrability of the Jacobian under minimal hypotheses*, Arch. Rational Mech. Anal. 119 (1992), 129–143.
13. Karadzhov, G. E., and Milman, M., *Extrapolation theory: new results and applications*, J. Approx. Theory 133 (2005), 38–99.
14. Koskela, P., and Zhong, X., *Minimal assumptions for the integrability of the Jacobian*, Ricerche Mat. 51 (2) (2002), 297–311.
15. Müller, S., *Higher integrability of determinants and weak convergence in L^1*, J. Reine Angew. Math. 412 (1990), 20–34.
16. Neves, J., *Extrapolation results on general Besov-Hölder-Lipschitz spaces*, Math. Nachr. 230 (2001), 117–141.
17. Rakotoson, J. M., *Some new applications of the pointwise relations for the relative rearrangement*, Adv. Differential Equations 7 (2002), 617–640.
18. Sbordone, C., *Grand Sobolev spaces and their applications to variational problems*, Matematiche (Catania) 51 (1996), 335–347.

C.N.R. - ISTITUTO PER LE APPLICAZIONI DEL
CALCOLO "MAURO PICONE", SEZIONE DI NAPOLI
CONSIGLIO NAZIONALE DELLE RICERCHE
VIA PIETRO CASTELLINO, 111
I-80131 NAPOLI
ITALY
E-mail: capone@na.iac.cnr.it, fiorenza@unina.it

DIPARTIMENTO DI COSTRUZIONI E METODI
MATEMATICI IN ARCHITETTURA
UNIVERSITÀ DI NAPOLI "FEDERICO II"
VIA MONTEOLIVETO, 3
I-80134 NAPOLI
ITALY
E-mail: fiorenza@unina.it

INSTITUTE OF MATHEMATICS
BULGARIAN ACADEMY OF SCIENCES
1113 SOFIA
BULGARIA
E-mail: geremika@yahoo.com