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# AN EXAMPLE OF A BOUNDED $\mathsf{C}$-CONVEX DOMAIN WHICH IS NOT BIHOLOMORPHIC TO A CONVEX DOMAIN

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## Abstract

We show that the symmetrized bidisc is a $\mathsf{C}$-convex domain. This provides an example of a bounded $\mathsf{C}$-convex domain which cannot be exhausted by domains biholomorphic to convex domains.

## 1. Introduction

Recall that a domain $D$ in $\mathsf{C}^n$ is called $\mathsf{C}$-convex if any non-empty intersection with a complex line is contractible (cf. [2], [9]). A consequence of the fundamental Lempert theorem (see [12]) is the fact that any bounded $\mathsf{C}$-convex domain $D$ with $C^2$ boundary has the following property (see [8]):

$(*)$ *The Carathéodory distance and the Lempert function of D coincide.*

Any convex domain can be exhausted by smooth bounded convex ones (which are obviously $\mathsf{C}$-convex); therefore, any convex domain satisfies $(*)$, too. To extend this phenomenon to bounded $\mathsf{C}$-convex domains (see Problem 4' in [14]), it is sufficient to give a positive answer to one of the following questions:

(a) *Can any bounded $\mathsf{C}$-convex domain be exhausted by $C^2$-smooth $\mathsf{C}$-convex domains*? (See Problem 2 in [14] and Remark 2.5.20 in [2].)

(b) *Is any bounded $\mathsf{C}$-convex domain biholomorphic to a convex domain*? (See Problem 4 in [14].)

The main aim of this note is to give a negative answer to the question (b).

Denote by $\mathsf{G}_2$ the so-called symmetrized bidisc, that is, the image of the bidisc under the mapping whose components are the two elementary symmetric functions of two complex variables. $\mathsf{G}_2$ serves as the first example of a

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bounded pseudoconvex domain in $\mathbb{C}^2$ with the property $(*)$ which cannot be exhausted by domains biholomorphic to convex domains (see [3], [6]). We shall show that $\mathbb{G}_2$ is a $\mathbb{C}$-convex domain. This fact gives a counterexample to the question (b) and simultaneously, it supports the conjecture that (cf. Problem 4' in [14]) *any bounded* $\mathbb{C}$*-convex domain has property* $(*)$. Note that the answer to the question (a) for $\mathbb{G}_2$ is not known. The positive answer to this question would imply an alternative (to that of [4] and [1]) proof of the equality of the Carathéodory distance and Lempert function on $\mathbb{G}_2$ whereas the negative answer would solve Problem 2 in [14].

Some additional properties of $\mathbb{C}$-convex domains and symmetrized polydiscs are also given in the paper.

## 2. Background and results

Recall that a domain $D$ in $\mathbb{C}^n$ is called (cf. [9], [2]):

- $\mathbb{C}$*-convex* if any non-empty intersection with a complex line is contractible (i.e. $D \cap L$ is connected and simply connected for any complex affine line $L$ such that $L \cap D$ is not empty);
- *linearly convex* if its complement in $\mathbb{C}^n$ is a union of affine complex hyperplanes;
- *weakly linearly convex* if for any $a \in \partial D$ there exists an affine complex hyperplane through $a$ which does not intersect $D$.

Note that the following implications hold:

$$\mathbb{C}\text{-convexity} \Rightarrow \text{linear convexity} \Rightarrow \text{weak linear convexity.}$$

Moreover, these three notions coincide in the case of bounded domains with $C^1$ boundary (cf. [2], [9]).

Let $\mathbb{D}$ denote the unit disc in $\mathbb{C}$. Let $\pi_n = (\pi_{n,1}, \dots, \pi_{n,n}) : \mathbb{C}^n \to \mathbb{C}^n$ be defined as follows:

$$\pi_{n,k}(\mu) = \sum_{1 \le j_1 < \cdots < j_k \le n} \mu_{j_1} \dots, \mu_{j_k}, \quad 1 \le k \le n, \;\; \mu = (\mu_1, \dots, \mu_n) \in \mathbb{C}^n.$$

The set $\mathbb{G}_n := \pi_n(\mathbb{D}^n)$ is called *the symmetrized n-disc* (cf. [1], [11]).

Recall that $\mathbb{G}_2$ is the first example of a bounded pseudoconvex domain with the property $(*)$, which cannot be exhausted by domains biholomorphic to convex ones (see [3], [6]). On the other hand, $\mathbb{G}_n$, $n \ge 3$, does not satisfy the property $(*)$ (see [13]). In particular, it cannot be exhausted by domains biholomorphic to convex domains, either.

In this note we shall show the following additional properties of domains $\mathbb{G}_n, n \geq 2$.

THEOREM 1. (i) $\mathbb{G}_2$ *is a* $\mathbb{C}$*-convex domain.*
(ii) $\mathbb{G}_n$*,* $n \geq 3$*, is a linearly convex domain which is not* $\mathbb{C}$*-convex.*

Theorem 1 (i) together with a result of [3] and [6] gives a negative answer to the following question posed by S. V. Znamenskiĭ (cf. Problem 4 in [14]):

*Is any bounded* $\mathbb{C}$*-convex domain biholomorphic to a convex domain?*

Moreover, it seems to us that Theorem 1 (ii) gives the first example of a linearly convex domain homeomorphic to $\mathbb{C}^n$, $n \geq 3$, which is not $\mathbb{C}$-convex, is not a Cartesian product and does not satisfy property $(*)$. To see that $\mathbb{G}_n$ is homeomorphic to $\mathbb{C}^n$, observe that $\rho_\lambda(z) := (\lambda z_1, \lambda^2 z_2, \dots, \lambda^n z_n) \in \mathbb{G}_n$ if $z \in \mathbb{G}_n$ and $\lambda \in \mathbb{C}$. Then setting $h(z) = \max_{1 \leq j \leq n} \{|\mu_j| : \pi_n(\mu) = z\}$ and $g(z) = \frac{1}{1-h(z)}$, it is easy to see that the function $\mathbb{G}_n \ni z \mapsto \rho_{g(z)}(z) \in \mathbb{C}^n$ is the desired homeomorphism.

These remarks also show that $\mathbb{G}_n$ is close, in some sense, to a balanced domain, that is, a domain $D$ in $\mathbb{C}^n$ such that $\lambda z \in D$ for any $z \in D$ and $\lambda \in \overline{\mathbb{D}}$. On the other hand, in spite of the properties of $\mathbb{G}_n$, one has the following.

PROPOSITION 2. *Any weakly linearly convex balanced domain is convex.*

This proposition is a simple extension of Example 2.2.4 in [2], where it is shown that any $\mathbb{C}$-convex complete Reinhardt domain is convex.

We may also prove some general property of $\mathbb{C}$-convex domains showing that all *non-degenerate* $\mathbb{C}$-convex domains, that is, containing *no* complex lines, are $c$-finitely compact. For definitions of the Carathéodory distance $c_D$ of the domain $D$, $c$-finite compactness, $c$-completeness and basic properties of these notions we refer the Reader to consult [10].

Observe that a degenerate linearly convex domain $D$ is linearly equivalent to $\mathbb{C} \times D'$ (cf. Proposition 4.6.11 in [9]). Indeed, we may assume that $D$ contains the $z_1$-line. Since the complement ${}^cD$ of $D$ is a union of complex hyperplanes disjoint from this line, then ${}^cD = \mathbb{C} \times G$ and hence $D = \mathbb{C} \times {}^cG$. On the other hand, we have

PROPOSITION 3. *Any non-degenerate* $\mathbb{C}$*-convex domain is biholomorphic to a bounded domain and c-finitely compact. In particular, it is c-complete and hyperconvex.*

REMARKS. (i) In virtue of Proposition 3, we claim that one may conjecture more than the question (a) (see [15]), namely, any $\mathbb{C}$-convex domain containing no complex hyperplanes can be exhausted by bounded $C^2$-smooth $\mathbb{C}$-convex domains (this is not true in general without the above assumption); then the Carathéodory pseudodistance and Lempert function will coincide on any $\mathbb{C}$-convex domain.

(ii) The hyperconvexity of $\mathbb{G}_n$ is simple and well-known (see [7]). The above proposition implies more in dimension two. Namely, it implies that the symmetrized bidisc is $c$-finitely compact. Although the symmetrized polydiscs in higher dimensions are not $\mathbb{C}$-convex the conclusion of the above proposition, that is, the $c$-finite compactness of the symmetrized $n$-disc $\mathbb{G}_n$, holds for any $n \geq 2$. In fact, it is a straightforward consequence of Corollary 3.2 in [5].

(iii) Finally, we mention that, for $n \geq 2$, $\mathbb{G}_n$ is starlike with respect to the origin if and only if $n = 2$. This observation gives the next difference in the geometric shape of the 2-dimensional and higher dimensional symmetrized discs. Recall that the fact that $\mathbb{G}_2$ is starlike is contained in [1]. For the converse just take the point $(3, 3, 1, 0, \dots, 0)$.

## 3. Proofs

Proof of Theorem 1 (i). We shall make use of the following description of $\mathbb{C}$-convex domains. For $a \in \partial D$, denote by $\Gamma(a)$ the set of all hyperplanes through $a$ and disjoint from $D$. Then a bounded domain $D$ in $\mathbb{C}^n$, $n > 1$, is $\mathbb{C}$-convex if and only if for any $a \in \partial D$ the set $\Gamma(a)$ is non-empty and connected as a set in $\mathbb{CP}^n$ (cf. Theorem 2.5.2 in [2]).

So we have to check that $\Gamma(a)$ is non-empty and connected for any $a \in \partial \mathbb{G}_2$.

Let us first consider a regular point of $\partial \mathbb{G}_2$, that is, a point of the form $\pi_2(\mu)$, where $|\mu_1| = 1$, $|\mu_2| < 1$ (or vice versa). Then the complex tangent line to $\partial D$ at $a$ is of the form $\{\pi_2(\mu_1, \lambda) : \lambda \in \mathbb{C}\}$, which is obviously disjoint from $\mathbb{G}_2$. So $\Gamma(a)$ is a singleton.

Now we fix a non-regular point of $\partial \mathbb{G}_2$, that is, a point of the form $\pi_2(\mu)$, where $|\mu_1| = |\mu_2| = 1$.

After a rotation we may assume that $\mu_1 \mu_2 = 1$, that is, $\mu_2 = \bar{\mu}_1$. Then $\mu_1 + \mu_2 = 2 \operatorname{Re} \mu_1 =: 2x$, where $x \in [-1, 1]$.

We shall find all the possible directions of complex lines passing simultaneously through $\pi_2(\mu)$ and an element of $\mathbb{G}_2$. Any such line is of the form $\pi_2(\mu) + \mathbb{C}(\pi_2(\mu) - \pi_2(\lambda))$, where $\lambda \in \mathbb{D}^2$. So

$$A := {}^c\Gamma(\pi_2(\mu)) = \left\{ \frac{\lambda_1 + \lambda_2 - 2x}{\lambda_1 \lambda_2 - 1} : \lambda_1, \lambda_2 \in \mathbb{D} \right\}.$$

In particular, $\Gamma(\pi_2(\mu)) \neq \emptyset$.

To show the connectedness of $\Gamma(\pi_2(\mu))$, we shall check the simple-connectedness of $A$. Let us recall that the mapping $\frac{z-\alpha}{z-\beta}$, where $|\beta| > 1$, maps the unit disc $\mathbb{D}$ onto the disc $\triangle\left(\frac{1-\alpha\bar\beta}{1-|\beta|^2}, \frac{|\alpha-\beta|}{|\beta|^2-1}\right)$, so

$$\left\{\frac{\lambda+\lambda_1-2x}{\lambda\lambda_1-1} : \lambda\in\mathbb{D}\right\} = \triangle\left(\frac{2x-2\operatorname{Re}\lambda_1}{1-|\lambda_1|^2}, \frac{|2x\lambda_1-\lambda_1^2-1|}{1-|\lambda_1|^2}\right) =: A_{\lambda_1}.$$

Consequently the set $A=\bigcup_{\lambda_1\in\mathbb{D}} A_{\lambda_1}\subset\mathbb{C}$ is simply connected.

Proof of Theorem 1 (ii). For the proof of the linear convexity of $\mathbb{G}_n$ consider the point $z=\pi_n(\lambda)\in\mathbb{C}^n\setminus\mathbb{G}_n$. We may assume that $|\lambda_1|\ge 1$. Then the set

$$B := \{\pi_n(\lambda_1,\mu_1,\dots,\mu_{n-1}) : \mu_1,\dots,\mu_{n-1}\in\mathbb{C}\}$$

is disjoint from $\mathbb{G}_n$. On the other hand, it is easy to see that

$$B=\{(\lambda_1+z_1, \lambda_1 z_1+z_2,\dots,\lambda_1 z_{n-2}+z_{n-1}, \lambda_1 z_{n-1}) : z_1,\dots,z_{n-1}\in\mathbb{C}\},$$

so $B$ is a complex affine hyperplane. Hence $\mathbb{G}_n$ is linearly convex.

To show that $\mathbb{G}_n$ is not $\mathbb{C}$-convex for $n\ge 3$, consider the points

$$\begin{aligned} a_t &:= \pi_n(t,t,t,0,\dots,0) = (3t, 3t^2, t^3, 0,\dots,0),\\ b_t &:= \pi_n(-t,-t,-t,0,\dots,0) = (-3t, 3t^2, -t^3, 0,\dots,0), \quad t\in(0,1).\end{aligned}$$

Obviously $a_t, b_t\in\mathbb{G}_n$. Denote by $L_t$ the complex line passing through $a_t$ and $b_t$, that is,

$$L_t=\{c_{t,\lambda} := (3t(1-2\lambda), 3t^2, t^3(1-2\lambda), 0,\dots,0) : \lambda\in\mathbb{C}\}.$$

Assume that the set $\mathbb{G}_n\cap L_t$ is connected. Since $a_t=c_{t,0}$ and $b_t=c_{t,1}$, then $c_{t,\lambda}\in\mathbb{G}_n$ for some $\lambda=\frac12+i\tau$, $\tau\in\mathbb{R}$. It follows that

$$c_{t,\lambda}=(-6i\tau t, 3t^2, -2i\tau t^3, 0,\dots,0).$$

We may choose $\mu\in\mathbb{D}^n$ such that $\mu_j=0$, $j=4,\dots,n$, and $c_{t,\lambda}=\pi_n(\mu)$, $\mu\in\mathbb{D}^n$. Then $-36\tau^2t^2=(\mu_1+\mu_2+\mu_3)^2=\mu_1^2+\mu_2^2+\mu_3^2+6t^2$ and hence

$$t^2=\frac{|\mu_1^2+\mu_2^2+\mu_3^2|}{36\tau^2+6} < \frac{3}{36\tau^2+6}\le\frac12.$$

Therefore, $\mathbb{G}_n\cap L_t$ is not connected if $t\in\left[\frac{1}{\sqrt2},1\right)$ and so $\mathbb{G}_n$ is not a $\mathbb{C}$-convex domain.

Proof of Proposition 2. Set $D^* := \{w\in\mathbb{C}^n :< z,w >\neq 1,\ \forall z\in D\}$. We shall use the fact that a domain $D$ in $\mathbb{C}^n$ containing the origin is weakly linearly convex if and only if $D$ is a connected component of $D^{**}$ (cf. Proposition 2.1.4 in [2]).

Since our domain $D$ is balanced, it is easy to see that $D^*$ is balanced. We shall show $D^*$ is convex. Then, applying this fact to $D^*$, we conclude that $D^{**}$ is a convex balanced domain. On the other hand, it follows by our assumption that $D$ is a component of $D^{**}$ and hence $D^{**} = D$.

To see that $D^*$ is convex, suppose the contrary. Then we find points $w_1, w_2 \in D^*$, $z \in D$ and a number $t \in (0, 1)$ such that $\langle z, tw_1 + (1-t)w_2\rangle = 1$. We may assume that $|\langle z, w_1\rangle| \geq 1$. Since $D$ is balanced, we get $\tilde{z} := \frac{z}{\langle z, w_1\rangle} \in D$ and $\langle \tilde{z}, w_1\rangle = 1$, a contradiction.

PROOF OF PROPOSITION 3. Let $D$ be non-degenerate $\mathbb{C}$-convex domain in $\mathbb{C}^n$. For any point $z \in {}^cD$ consider a hyperplane $L_z$ through $z$ and disjoint from $D$. Let $l_z$ be the orthogonal line through 0 and orthogonal to $L_z$. Denote by $\pi_z$ the orthogonal projection of $\mathbb{C}^n$ onto $l_z$ and set $a_z = \pi_z(a)$. Observe that $D_z = \pi_z(D)$ is biholomorphic to $\mathbb{D}$, since it is connected, simply connected (cf. Theorem 2.3.6 in [2]) and $\pi_z(z) \notin \pi_z(D)$. Moreover, since $D$ is a non-degenerate linearly convex domain, it is easy to see that there are $n$ $\mathbb{C}$-independent $l_z$'s. We may assume that these $l_z$'s are the set $C$ of coordinate planes. Then $D \subset G := \prod_{l_z \in C} \pi_z(D)$ and $G$ is biholomorphic to the polydisc $\mathbb{D}^n$. In particular, $D$ is biholomorphic to a bounded domain, hence it is $c$-hyperbolic.

Further, we may assume that $0 \in D$. To see that $D$ is $c$-finitely compact, it is enough to show that $\lim_{a \to z} c_D(0; a) = \infty$ for any $z \in \partial D$ and, if $D$ is unbounded, $z = \infty$. But the last one follows by the fact that $G$ is $c$-finitely compact. On the other hand, if $a \to z \in \partial D$, then $a_z \to \pi_z(z) \in \partial D_z$ and hence $c_D(0; a) \geq c_{D_z}(0; a_z) \to \infty$.

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