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$\mathbb{Q}$-LINEAR FUNCTIONS, FUNCTIONS WITH DENSE GRAPH, AND EVERYWHERE SURJECTIVITY

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Abstract

Let L, S and D denote, respectively, the set of $\mathbb{Q}$-linear functions, the set of everywhere surjective functions and the set of dense-graph functions on $\mathbb{R}$. In this note, we show that the sets $D \setminus (S \cup L)$, $S \setminus L$, $S \cap L$ and $D \cap L \setminus S$ are lineable. Moreover, all these sets contain (omitting zero) a vector space of the biggest possible dimension, 2^c.

1. Introduction and preliminaries

Examples of functions verifying some kind of *pathological* property have been found and constructed in analysis (differentiable nowhere monotone functions, continuous nowhere differentiable functions, linear discontinuous functions, or *everywhere surjective* functions). Given such a special property, we say that the subset M of functions which satisfies it is *lineable* if $M \cup \{0\}$ contains an infinite dimensional vector space. At times, we will be more specific, referring to the set M as μ-lineable if it contains a vector space of dimension μ. Also, we let $\lambda(M)$ be the maximum cardinality (if it exists) of such a vector space. This terminology of lineable was first introduced in [1], [2], [4].

Some of these so called special properties are not isolated phenomena. In [6], [7] Lebesgue constructed an *everywhere surjective* function, i.e. a function $f : \mathbb{R} \to \mathbb{R}$ such that $f((a,b)) = \mathbb{R}$ for every nonvoid interval (a,b). Clearly, these everywhere surjective functions have dense graph in $\mathbb{R}^2$. In [2] it was shown that the set of all everywhere surjective functions is 2^c-lineable ($c = \mathrm{card}\,(\mathbb{R})$). Moreover, in [3] it was proved that there exists an infinitely generated algebra every nonzero element of which is an everywhere surjective function on $\mathbb{C}$. These results suggest that everywhere surjective functions seem to appear more often that one could expect.

Also, in [5] it was shown that the set of $\mathbb{Q}$-linear discontinuous functions on $\mathbb{R}$ is lineable.

Let us denote by L, S and D, respectively, the set of $\mathbb{Q}$-linear functions, the set of everywhere surjective functions and the set of dense-graph functions on

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$\mathbb{R}$. As mentioned, $S \subseteq D$. The motivation of this note is the following: The set $S \cap L$ is not empty, moreover we will see that it is lineable. We will also prove that

$$\lambda(S \cap L) = \lambda(D \cap L \setminus S) = \lambda(S \setminus L) = \lambda(D \setminus (S \cup L)) = 2^{\mathfrak{c}},$$

obtaining vector spaces of the biggest possible dimension, and improving some results from [2] and [5] since we are adding extra pathologies to our functions.

2. A key vector space

In this section, we recall an infinite dimensional vector space which will be frequently used in what follows.

LEMMA 2.1. *There exists a vector space V_0 of dimension $2^{\mathfrak{c}}$ whose nonzero elements are discontinuous surjective functions.*

PROOF. Let $\varphi : \mathbb{R} \longrightarrow \mathbb{R}^{\mathbb{N}}$ be a bijection mapping the whole interval (0, 1) onto the set of sequences whose first term is 0. For every $A \subseteq \mathbb{R}$, let us consider the map H_A defined as

$$\begin{aligned} \mathbb{R}^{\mathbb{N}} &\longrightarrow \mathbb{R} \\ (y, x_1, x_2, \ldots) &\longmapsto y \cdot \prod_{i=1}^{\infty} \chi_A(x_i). \end{aligned}$$

In [2] it is proved that the vector space $V_0 = \operatorname{span}\{H_A \circ \varphi : A \subseteq \mathbb{R}\}$ has dimension $2^{\mathfrak{c}}$ and its nonzero elements are discontinuous surjective functions.

3. The lineability of $D \setminus (S \cup L)$

In this section, we will construct an infinite dimensional vector space whose nonzero elements are dense-graph functions that are neither everywhere surjective nor $\mathbb{Q}$-linear.

THEOREM 3.1. *The set $D \setminus (S \cup L)$ is lineable. Moreover, there exists a vector space $U \subset (D \setminus (S \cup L)) \cup \{0\}$ of the biggest possible dimension, i.e. $\lambda(D \setminus (S \cup L)) = 2^{\mathfrak{c}}$.*

PROOF. Let W be a vector space satisfying $\dim W = 2^{\mathfrak{c}}$ and $W \subseteq S \cup \{0\}$ (an example was constructed in [2]). Let $\Phi : \mathbb{R} \to \mathbb{R} \setminus \{1\}$ be a bijection. Consider

$$U = \{\Phi \circ g : g \in W\},$$

clearly $\dim U = 2^{\mathfrak{c}}$ and every nonzero element of U is a function which maps every interval onto $\mathbb{R} \setminus \{1\}$; every such a function has dense graph and is neither everywhere surjective nor $\mathbb{Q}$-linear. Therefore U is the desired vector space.

4. The lineability of $S \cap L$

In this section, we construct an infinite dimensional vector space whose nonzero elements are everywhere surjective $\mathbb{Q}$-linear functions. We will start by characterizing this kind of functions. Clearly, from the above definition, a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is everywhere surjective if and only if $f^{-1}(t)$ is dense for every $t \in \mathbb{R}$. It is well known too that every 1-dimensional $\mathbb{Q}$-subspace of $\mathbb{R}$ is dense, therefore also is every nonzero $\mathbb{Q}$-subspace of $\mathbb{R}$.

THEOREM 4.1. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a $\mathbb{Q}$-linear function. The following conditions are equivalent:*

(1) *f is everywhere surjective.*

(2) *f is surjective and not injective.*

PROOF. If f is everywhere surjective, then f is surjective and not injective. Assume then that f is surjective but not injective. We only need to check whether $f^{-1}(t)$ is dense for every $t \in \mathbb{R}$. Now, f is not injective, therefore $\ker(f)$ is not zero, so it is dense. Next, f is surjective, that is, $f^{-1}(t)$ is nonempty for every $t \in \mathbb{R}$. As a consequence, this last fact along the density of the kernel of f gives the density of every bundle $f^{-1}(t)$.

THEOREM 4.2. *The set $S \cap L$ is lineable. Moreover, there exists a vector space $U \subset (S \cap L) \cup \{0\}$ of the biggest possible dimension, i.e. $\lambda(S \cap L) = 2^c$.*

PROOF. Let I be a $\mathbb{Q}$-basis of $\mathbb{R}$. Let $\Phi : I \to \mathbb{R}$ be a bijection. Consider

$$W = \{g \circ \Phi : g \in V_0\},$$

clearly $\dim W = 2^c$ and every nonzero element of W is a surjective function $f : I \to \mathbb{R}$ that can be uniquely extended to a $\mathbb{Q}$-linear function $\bar{f} : \mathbb{R} \to \mathbb{R}$, which must be surjective and not injective. Therefore

$$U = \{\bar{f} : f \in W\}$$

is the desired vector space.

5. The lineability of $S \setminus L$

In this section, we will construct an infinite dimensional vector space whose nonzero elements are everywhere surjective functions that are not $\mathbb{Q}$-linear.

LEMMA 5.1. *Every element of V_0 is not a $\mathbb{Q}$-linear function.*

PROOF. Every $f \in V_0$ is a surjective function satisfying $f((0,1)) = \{0\}$, therefore f is neither injective nor everywhere surjective. According to theorem 4.1, f cannot be $\mathbb{Q}$-linear.

THEOREM 5.2. *The set $S \setminus L$ is lineable. Moreover, there exists a vector space $W \subset (S \setminus L) \cup \{0\}$ of the biggest possible dimension, i.e.* $\lambda(S \setminus L) = 2^c$.

PROOF. Take any everywhere surjective $\mathbb{Q}$-linear function f and consider $W = \{g \circ f : g \in V_0\}$. We have that W is a vector space isomorphic to V_0 whose nonzero elements are everywhere surjective functions. In particular, it has dimension 2^c. Let $g \in V_0 \setminus \{0\}$. Since g is not $\mathbb{Q}$-linear, there must exist $x, y \in \mathbb{R}$ so that $g(x + y) \neq g(x) + g(y)$. Now, there are $a, b \in \mathbb{R}$ such that $f(a) = x$ and $f(b) = y$. Now,

$$\begin{aligned}(g \circ f)(a + b) &= g(f(a) + f(b)) \\ &= g(x + y) \\ &\neq g(x) + g(y) \\ &= (g \circ f)(a) + (g \circ f)(b).\end{aligned}$$

Therefore, $g \circ f$ is not $\mathbb{Q}$-linear.

6. The lineability of $D \cap L \setminus S$

Now we construct an infinite dimensional vector space whose nonzero elements are dense-graph, $\mathbb{Q}$-linear functions that are not surjective (and, in particular, not everywhere surjective).

THEOREM 6.1. *The set $D \cap L \setminus S$ is lineable. Moreover, there exists a vector space $U \subset (D \cap L \setminus S) \cup \{0\}$ of the biggest possible dimension, i.e.* $\lambda(D \cap L \setminus S) = 2^c$.

PROOF. Let I be a $\mathbb{Q}$-basis of $\mathbb{R}$. Choose any $i \in I$ and take a bijection $\phi : I \to I \setminus \{i\}$, which can be uniquely extended to a $\mathbb{Q}$-linear injection $\Phi : \mathbb{R} \to \mathbb{R}$. It is not difficult to verify that $\Phi(\mathbb{R})$ is a dense proper subset of $\mathbb{R}$.

Let W be a vector space with $\dim W = 2^c$ and $W \subseteq (S \cap L) \cup \{0\}$. Consider

$$U = \{\Phi \circ g : g \in W\},$$

clearly $\dim U = 2^c$ and every nonzero element of U is a nonsurjective, $\mathbb{Q}$-linear function $f : \mathbb{R} \to \mathbb{R}$. In addition, f maps every interval onto $\Phi(\mathbb{R})$. Therefore U is the desired vector space.

7. What about $L \setminus D$?

The reader may have guessed, or perhaps already knew, what happens with $L \setminus D$. We believe that the following result is known but, for the sake of completeness, we include a short proof here. This result will settle the question.

PROPOSITION 7.1. *Every* Q*-linear function is discontinuous if and only if it has dense graph.*

PROOF. Suppose $f : \mathsf{R} \to \mathsf{R}$ is Q-linear and discontinuous. Then f is not linear, i.e., there exist nonzero $x, y \in \mathsf{R}$ satisfying $\frac{f(x)}{x} \neq \frac{f(y)}{y}$. This implies

$$\overline{\mathrm{Gr}\, f} \supseteq \overline{\{q_1(x, f(x)) + q_2(y, f(y)) : q_1, q_2 \in \mathsf{Q}\}}$$
$$= \{r_1(x, f(x)) + r_2(y, f(y)) : r_1, r_2 \in \mathsf{R}\} = \mathsf{R}^2$$

COROLLARY 7.2. $\lambda(L \setminus D) = 1$.

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