SHARPENED FORMS OF A VON NEUMANN INEQUALITY FOR ρ-CONTRACTIONS

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Abstract

The purpose of this paper is to give some sharpened forms of the von Neumann inequality for strict ρ-contractions which were obtained in [6], [7]. Also, some sharpened forms of the Schwarz inequality for strict ρ-contractions will be given, and as applications, corresponding inequalities for strict contractions and for uniformly stable operators will be derived. In particular, we recover the results of K. Fan [12], [13] covering the strict contractions, and in the scalar context we find an improved form of the interior Schwarz inequality quoted by R. Osserman [19].

1. Introduction and preliminaries

Let H be a complex Hilbert space and $B(H)$ be the C^*-algebra of all bounded linear operators on H. For any scalar $\rho > 0$, we denote by $C_\rho(H)$ the set of all operators $T \in B(H)$ which admit a unitary ρ-dilation in the sense of Nagy-Foias [21], [22]. This means that there exists a Hilbert space $\mathscr{H} \supseteq H$ and a unitary operator U acting on $\mathscr{H}$ such that

$$T^n h = \rho P_H U^n h \qquad (h \in H, n \geq 1),$$

where P_H is the orthogonal projection of $\mathscr{H}$ onto H. We know that $C_1(H)$ consists of all contractions on H [20], that is the operators T on H with $\|T\| \leq 1$, and that $T \in C_2(H)$ if and only if the numerical range of T is contained in the closed unit disc [2].

According to J. Holbrook [15] and J. Williams [23] we define the ρ-numerical radius of an operator $T \in B(H)$ by the formula

$$w_\rho(T) = \inf \left\{ \mu : \mu > 0, \ \frac{1}{\mu} T \in C_\rho(H) \right\}.$$

Clearly, an operator T belongs to $C_\rho(H)$ if and only if $w_\rho(T) \leq 1$. Hence the operators in $C_\rho(H)$ are contractions with respect to the ρ-numerical radius, and according to this fact, any operator $T \in C_\rho(H)$ will be called a ρ-contraction on H (as in [4], [6]–[8], [9], [16], for instance).

Let us remark that $w_1(T) = \|T\|$ and $w_2(T) = w(T)$ this being the *numerical radius* of T. Hence $C_2(H)$ consists of all operators with numerical radius less or equal to one ([2], [3], [21]). Also, since the classes $C_\rho(H)$ increase with ρ ([3], [5], [21], [22]), one has $w_{\rho_1}(T) \geq w_{\rho_2}(T)$ if $\rho_1 \leq \rho_2$ for $T \in B(H)$, and

$$\lim_{\rho \to +\infty} w_\rho(T) = r(T)$$

where $r(T)$ is the spectral radius of T.

Recall that if T is an operator with $r(T) < 1$ and if f is an analytic function on the open unit disc D of the complex plane, then the operator $f(T)$ on H is well-defined by the Riesz-Dunford integral [10]. Clearly, if $w_\rho(T) < 1$ that is, T is a *strict ρ-contraction* (see [6], [7]), then $r(T) < 1$ and therefore $f(T)$ is well-defined for f as above. Also, if $r(T) < 1$ then $w_\rho(T) < 1$ for $\rho > 1$ large enough. Thus, many facts on the uniformly stable operators can be obtained from the strict ρ-contractions [6], [8].

It is well-known that an operator T with $r(T) \leq 1$ belongs to $C_\rho(H)$ if and only if the following von Neumann equality holds

$$(1.1) \qquad \|f(T)\| \leq \sup_{|z| \leq 1} |\rho f(z) + (1 - \rho) f(0)|$$

for any analytic function f on an open neighborhood of $\overline{\mathsf{D}}$. In fact, the inequality (1.1) for $T \in C_\rho(H)$ was essentially given by Sz.-Nagy-Foias [21], [22] and the converse assertion that the inequality (1.1) ensures $T \in C_\rho(H)$ was proved by D. Gaspar [14]. Clearly, the inequality (1.1) holds also for any continuous function f on $\overline{\mathsf{D}}$ which is analytic on D (notation $f \in A(\mathsf{D})$) if $T \in C_\rho(H)$, where $f(T)$ is defined by setting $f(T) = \lim_{r \to 1^-} f(rT)$ in $B(H)$ ((1.1) clearly implies the convergence of $f(rT)$ with respect to the norm of $B(H)$))

An equivalent inequality to (1.1) can be obtained, namely

$$(1.2) \qquad w_\rho(f(T)) \leq \|f\|_\infty := \sup_{|z| \leq 1} |f(z)|$$

for $T \in C_\rho(H)$ and f as in (1.1) with $f(0) = 0$. Indeed, if (1.2) holds for $T \in B(H)$ with $r(T) \leq 1$ and for any function f as above, then for $f(z) = z$, (1.2) implies $w_\rho(T) \leq 1$ that is, $T \in C_\rho(H)$ and consequently (1.1) holds for such a T. The converse implication is a consequence of the following version for ρ-contractions of the classical von Neumann theorem [18] for contractions, which was proved in [6].

THEOREM 1.1 (Mapping theorem). *Let $f \in A(\mathsf{D})$ be a non-constant func-*

tion with $f(\mathsf{D}) \subseteq \mathsf{D}$. Then, for $T \in C_\rho(H)$ we have $f(T) \in C_{\rho_f}(H)$ where

$$
(1.3) \qquad \rho_f = \begin{cases} 1 + (\rho - 1)\dfrac{1 - |f(0)|}{1 + |f(0)|} & \text{if } \rho \le 1 \\[3mm] 1 + (\rho - 1)\dfrac{1 + |f(0)|}{1 - |f(0)|} & \text{if } \rho \ge 1. \end{cases}
$$

Moreover, if $\rho > 0$ is the smallest scalar such that $T \in C_\rho(H)$ then there exists a function f as above such that ρ_f is the smallest positive scalar for which $f(T) \in C_{\rho_f}(H)$.

It is clear from (1.3) that $\rho = 1$ implies $\rho_f = 1$ for any f. Also, when $\rho \ne 1$ we have $\rho_f = \rho$ if and only if $f(0) = 0$. Thus the inequality (1.2), and hence (1.1), can be derived from Theorem 1.1. In general one has $\rho \le \rho_f$, and $\rho_f \le 1$ if $\rho \le 1$.

Using the fact that for any $\rho > 0$ the mapping $S \to w_\rho(S)$ is norm continuous for $S \in B(H)$ (see [3]), we can derive (as in Corollary 8 [6]) the following useful result.

THEOREM 1.2. *Let $f \in A(\mathsf{D})$ be a non-constant function with $f(\mathsf{D}) \subseteq \mathsf{D}$. Then, the inequality*

$$
(1.4) \qquad\qquad\qquad w_{\rho_f}(T) < 1
$$

holds for every operator T with $w_\rho(T) < 1$, where ρ_f is as in (1.3).

In the case $\rho = 1$, from this theorem we deduce the well-known result of K. Fan [11] which is concerned with strict contractions. In [12], [13] K. Fan obtained some sharpened forms of von Neumann's inequality, as well as of Schwarz's inequality and of Schwarz-Pick inequality, for strict contractions.

In the present paper we sharpen the versions of Schwarz's inequality and of Schwarz-Pick's inequality given in [6], [7] for ρ-contractions. When $\rho = 1$ we recover the results of K. Fan [12], [13], and in addition, we complete some results of K. Fan in this case. We find also corresponding sharpened inequalities relative to the spectral radius, as consequences. Among others, we give an analogue of interior Schwarz inequality for ρ-contractions, which provides in the scalar case an improved form of the corresponding inequality proved by R. Osserman [19].

To obtain these inequalities, the following inequality proved in [6] (Theorem 5.8 and Remark 5.9) will be used more than once in our proofs. Namely, if f is a non-constant analytic self map of D, $\rho > 0$ and $0 < r < 1$, then we

have

$$(1.5) \qquad w_{\rho'}(f(T)) \leq \sup_{|z|=r} |f(z)| := M_f(r)$$

for every operator T on H with $w_\rho(T) \leq r$, ρ' being as in (1.3) with $z \to f(rz)/M_f(r)$ instead of f, that is $\rho' = \rho_f(r)$ where

$$\rho_f(r) = \begin{cases} 1 + (\rho - 1)\frac{M_f(r) - |f(0)|}{M_f(r) + |f(0)|} & \text{if } \rho \leq 1 \\[2mm] 1 + (\rho - 1)\frac{M_f(r) + |f(0)|}{M_f(r) - |f(0)|} & \text{if } \rho \geq 1. \end{cases}$$

Clearly, ρ_f is related to the value $f(0)$ (by (1.3)), but we also apply (1.5) relatively to any point $a \in \mathsf{D}$ with $\rho(a)$ instead of $\rho > 0$ and $\rho_f(a, r)$ instead of ρ_f, where

$$(1.6) \quad \rho(a) = \rho_{\varphi_a} = \begin{cases} 1 + (\rho - 1)\dfrac{1 - |a|}{1 + |a|} & \text{if } \rho \leq 1 \\[3mm] 1 + (\rho - 1)\dfrac{1 + |a|}{1 - |a|} & \text{if } \rho \geq 1, \end{cases}$$

$$(1.7) \qquad \rho_f(a, r) = \begin{cases} 1 + (\rho(a) - 1)\dfrac{r(1 - |f(a)|^2)}{r(1 + |f(a)|^2) + 2|f(a)|} & \text{if } \rho \leq 1 \\[3mm] 1 + (\rho(a) - 1)\dfrac{r(1 + |f(a)|^2) + 2|f(a)|}{r(1 - |f(a)|^2)} & \text{if } \rho \geq 1. \end{cases}$$

and φ_b for $b \in \mathsf{D}$ is the Möbius transformation given by

$$\varphi_b(z) = \frac{b - z}{1 - \bar{b}z} \qquad (z \in \mathsf{D}).$$

It is obvious that if $T \in C_\rho(H)$ with $\rho > 0$ then the operator

$$T_a = \varphi_a(T) = (aI - T)(I - \bar{a}T)^{-1}$$

is well-defined in $B(H)$ for any $a \in \mathsf{D}$. In fact, we have $T_a \in C_{\rho(a)}(H)$ (by Theorem 1.1), and $w_{\rho(a)}(T) < 1$ if $w_\rho(T) < 1$ (by Theorem 1.2).

2. Sharpened von Neumann inequality for strict ρ-contractions

Like in the case $\rho = 1$, which was studied by K. Fan [12], [13], the corresponding inequality (1.4) for $a \in \mathsf{D}$ can be sharpened by using an analogue of the Schwarz-Pick inequality for ρ-contractions.

THEOREM 2.1. *Let f be a non-constant analytic self map of D, and let $a \in \mathsf{D}$ and n be the the order of multiplicity of the zero a for the function $f - f(a)$. Then for every operator T on H with $w_\rho(T) < 1$ for some $\rho > 0$ we have*

$$(2.1) \qquad\qquad w_{\rho(a)}[\varphi_{f(a)}(f(T))] \leq w_{\rho(a)}(T_a)^n$$

and

$$(2.2) \qquad w_{\rho_f(a, w_{\rho(a)}(T_a)^n)}(f(T)) \leq \frac{w_{\rho(a)}(T_a)^n + |f(a)|}{1 + w_{\rho(a)}(T_a)^n |f(a)|},$$

where $\rho(a)$ and $\rho_f(a, w_{\rho(a)}(T_a)^n)$ are as in (1.6) and (1.7).

PROOF. Let f and a as above, and $g = \varphi_{f(a)} \circ f$. Then, we see that

$$g(z) = \frac{f(a) - f(z)}{1 - \overline{f(a)} f(z)} = \frac{a - z}{1 - \overline{a}z} h(z) \qquad (z \in \mathsf{D})$$

where h is an analytic function on D such that $h(\mathsf{D}) \subseteq \mathsf{D}$ (maximum principle). Since $\varphi_b^{-1} = \varphi_b$, for any operator T with $w_\rho(T) < 1$ we have

$$\varphi_{f(a)}(f(T)) = g(T) = \varphi_a(T)h(T) = \varphi_a(T)h_a(\varphi_a(T)) = G(T_a),$$

where $h_a = h \circ \varphi_a$ and $G(z) = zh_a(z)$ for $z \in \mathsf{D}$. As $T_a \in C_{\rho(a)}(H)$ and $G(0) = 0$, we have $G(T_a) \in C_{\rho(a)}(H)$. Since $r_a = w_{\rho(a)}(T_a) < 1$ (by (1.4)), we can apply (1.5) to obtain

$$w_{\rho(a)}(G(T_a)) \leq M_G(r_a).$$

But we have

$$M_G(r_a) = \sup_{|z|=r_a} |G(z)| = \sup_{|z|=r_a} |zh(\varphi_a(z))|$$

$$= r_a \sup_{|z|=r_a} \left| \frac{f(a) - f(\varphi_a(z))}{1 - \overline{f(a)} f(\varphi_a(z))} \frac{1 - \overline{a}\varphi_a(z)}{a - \varphi_a(z)} \right|$$

$$= \sup_{|z|=r_a} |\varphi_{f(a)}(f(\varphi_a(z)))| \leq \sup_{|z|=r_a} |\varphi_a(\varphi_a(z))|^n \leq r_a^n,$$

the inequality being based on Schwarz-Pick's lemma for derivatives (Lemma 2.1, [1], for instance). Thus, we obtain

$$w_{\rho(a)}[\varphi_{f(a)}(f(T))] \leq r_a^n = w_{\rho(a)}(T_a)^n,$$

which is just inequality (2.1).

Now let us write $S = \varphi_{f(a)}(f(T))$. Then $f(T) = \varphi_{f(a)}(S)$, and since (2.1) yields

$$w_{\rho(a)}(S) \leq w_{\rho(a)}(T_a)^n := r < 1,$$

from (1.5) we infer

$$w_{\rho_f(a, w_{\rho(a)}(T_a)^n)}(f(T)) = w_{\rho_f(a, w_{\rho(a)}(T_a)^n)}(\varphi_{f(a)}(S)) \leq \sup_{|z|=r} |\varphi_{f(a)}(z)|$$

$$= \frac{r + |f(a)|}{1 + r|f(a)|} = \frac{w_{\rho(a)}(T_a)^n + |f(a)|}{1 + w_{\rho(a)}(T_a)^n|f(a)|},$$

that is the inequality (2.2). The proof is finished.

The sharpened form of the inequality (1.4) with $\rho_f = \rho_f(0)$ (defined by (1.3)) is obtained in the following

COROLLARY 2.2. *For f and T as in Theorem 2.1 we have*

(2.3) $$w_\rho[\varphi_{f(0)}(f(T))] \leq w_\rho(T)^n$$

and

(2.4) $$w_{\rho_f(0, w_\rho(T)^n)}(f(T)) \leq \frac{w_\rho(T)^n + |f(0)|}{1 + |f(0)|w_\rho(T)^n},$$

where either $n = 1$, or $n \geq 2$ such that $f'(0) = \cdots = f^{(n-1)}(0) = 0$.

PROOF. One applies (2.1) and (2.2) in the case $a = 0$.

COROLLARY 2.3. *Let f, a and n be as in Theorem 2.1. Then, for any operator T on H with $\|T\| < 1$ we have*

(2.5) $$\|\varphi_{f(a)}(f(T))\| \leq \|\varphi_a(T)\|^n$$

and

(2.6) $$\|f(T)\| \leq \frac{\|\varphi_a(T)\|^n + |f(a)|}{1 + |f(a)|\|\varphi_a(T)\|^n}.$$

In particular, for n as in Corollary 2.2 one has

(2.7) $$\|f(T)\| \leq \frac{\|T\|^n + |f(0)|}{1 + |f(0)|\|T\|^n}.$$

We remark that these inequalities for $n = 1$ (and $\rho = 1$) were obtained by K. Fan [12], but his inequalities are weaker than the above ones if $n \geq 2$. We

also observe that (2.6) yields

$$(2.8) \qquad |f(z)| \leq \frac{|\varphi_a(z)|^n + |f(a)|}{1 + |f(a)|\,|\varphi_a(z)|^n} \qquad (z \in \mathsf{D}),$$

if f, a and n are as in Theorem 2.1.

Now, in the case $\rho = 2$ we derive from Corollary 2.2 the following

COROLLARY 2.4. *Let f and n be as in Corollary 2.2. If $T \in B(H)$ with $w(T) < 1$ then*

$$(2.9) \qquad w_{\rho_f(0,w(T)^n)}(f(T)) \leq \frac{w(T)^n + |f(0)|}{1 + |f(0)|w(T)^n}.$$

PROOF. If $\rho = 2$ then $\rho_f(0, w_\rho(T)^n) = \rho_f(0, w(T)^n)$, and when $\rho_f(0, w_\rho(T)^n) = 2$ we have $\rho = 2(w_\rho(T)^n + |f(0)|)[w_\rho(T)^n(1 + |f(0)|^2) + 2|f(0)|]^{-1}$, so (2.9) obtained from (2.4).

The version for spectral radius of Theorem 2.1 can be also derived.

COROLLARY 2.5. *Let T, a, and n be as in Theorem 2.1. Then, for any operator T on H with $r(T) < 1$ we have*

$$(2.10) \qquad r[\varphi_{f(a)}(f(T))] \leq r(T_a)^n$$

and

$$(2.11) \qquad r(f(T)) \leq \frac{r(T_a)^n + |f(a)|}{1 + |f(a)|r(T_a)^n}.$$

PROOF. Since $r(T) < 1$, there is $\rho_0 > 1$ such that $w_\rho(T) < 1$ for $\rho \geq \rho_0$. Then we have $\rho_f(a, w_{\rho(a)}(T_a)^n) \geq \rho(a) \geq \rho$, hence $\rho(a) \to +\infty$ and $\rho_f(a, w_{\rho(a)}(T_a)^n) \to +\infty$ if $\rho \to +\infty$. Thus (2.10) and (2.11) are inferred from (2.1) and (2.2) by letting $\rho \to +\infty$.

3. An interior Schwarz inequality for strict ρ-contractions

Under the hypotheses of Theorem 2.1, the inequality (2.3) in the case $f(0) = 0$ becomes

$$(3.1) \qquad w_\rho(f(T)) \leq w_\rho(T)^n$$

if $w_\rho(T) < 1$. But this is the Schwarz inequality for ρ-contractions in the generalized version for derivatives. In the sequel we will obtain some sharpened forms of this inequality.

THEOREM 3.1. *Let f be a non-constant analytic function such that $f(\mathsf{D}) \subseteq \mathsf{D}$ and $f(0) = f'(0) = \cdots = f^{(n-1)}(0) = 0$ for some integer $n \geq 1$. Then, for any operator T on H with $w_\rho(T) < 1$, for some $\rho > 0$, and for any $a \in \mathsf{D} \setminus \{0\}$, we have the inequalities*

$$(3.2) \quad w_\rho(f(T)) \leq w_\rho(T)^n \frac{(|a| + w_\rho(T)) + |a^{-n}f(a)|(1 + |a|w_\rho(T))}{(1 + |a|w_\rho(T)) + |a^{-n}f(a)|(|a| + w_\rho(T))}$$

and

$$(3.3) \qquad w_\rho(f(T)) \leq w_\rho(T)^n \frac{w_\rho(T) + \frac{1}{n!}|f^{(n)}(0)|}{1 + \frac{1}{n!}|f^{(n)}(0)|w_\rho(T)}$$

PROOF. If $f(z) = \lambda z^n$ with $|\lambda| = 1$ then each of two inequalities one reduces to the power inequality : $w_\rho(T^n) \leq w_\rho(T)^n$, which can also be seen as a particular case of (2.4). Now suppose that f does not have the form $f(z) = \lambda z^n$ with $|\lambda| = 1$. Define the function g on D by setting

$$g(0) = \frac{1}{n!}f^{(n)}(0) \qquad \text{and} \qquad g(z) = \frac{f(z)}{z^n} \quad \text{for } z \in \mathsf{D} \setminus \{0\}.$$

Then g is an analytic function on D and $g(\mathsf{D}) \subseteq \mathsf{D}$ because $f(z) \neq \lambda z^n$. Thus, if $T \in B(H)$ with $\|T\| < 1$ then applying (2.2) for g and T with $\rho = 1$ and $a \neq 0$, we obtain

$$\|f(T)\| = \|T^n g(T)\| \leq \|T^n\| \frac{\|T_a\| + |g(a)|}{1 + |g(a)|\|T_a\|}$$

$$\leq \|T^n\| \frac{\left(\frac{|a| + \|T\|}{1 + |a|\|T\|}\right) + |a^{-n}f(a)|}{1 + |a^{-n}f(a)|\left(\frac{|a| + \|T\|}{1 + |a|\|T\|}\right)}$$

$$= \|T^n\| \frac{(|a| + \|T\|) + |f(a)|(|a^{-1}| + \|T\|)}{(1 + |a|\|T\|) + |f(a)|(1 + |a^{-1}|\|T\|)}.$$

Letting $a \to 0$, one also finds that

$$\|f(T)\| \leq \|T^n\| \frac{\|T\| + |a_n|}{1 + |a_n|\|T\|}$$

where $a_n = (1/n!)f^{(n)}(0)$. Consequently, the inequalities (3.2) and (3.3) have been proved in the case $\rho = 1$. In addition, the above inequalities show for $T = zI$ with $z \in \mathsf{D}$ that

$$(3.4) \qquad |f(z)| \leq |z|^n \frac{(|a| + |z|) + |a^{-n}f(a)|(1 + |az|)}{(1 + |az|) + |a^{-n}f(a)|(|a| + |z|)}$$

for $a \in \mathsf{D} \setminus \{0\}$, and

$$(3.5) \qquad |f(z)| \leq |z|^n \frac{|z| + |a_n|}{1 + |z||a_n|}.$$

Now let $T \in B(H)$ with $w_\rho(T) < 1$ for some $\rho > 0$, $\rho \neq 1$. Since $f(0) = 0$, we have $\rho' = \rho$, and for $a \in \mathsf{D} \setminus \{0\}$ we infer using (1.5) and (3.4) that

$$w_\rho(f(T)) \leq \sup_{|z|=w_\rho(T)} |f(z)|$$

$$\leq w_\rho(T)^n \frac{(|a| + w_\rho(T)) + |a^{-n} f(a)|(1 + |a|w_\rho(T))}{(1 + |a|w_\rho(T)) + |a^{-n} f(a)|(|a| + w_\rho(T))}$$

Also, using (1.5) and (3.5) we obtain

$$w_\rho(f(T)) \leq \sup_{|z|=w_\rho(T)} |f(z)| \leq w_\rho(T)^n \frac{w_\rho(T) + |a_n|}{1 + |a_n|w_\rho(T)}.$$

We conclude that the inequalities (3.2) and (3.4) hold, and this ends the proof.

We remark that (3.3) can be derived from (3.2) taking $a \to 0$. Both (3.2) and (3.3) are sharpened forms of (3.1) because the factors of the right-hand side of (3.2) and (3.3) are stricly less than 1, and they are equal to 1 if $f(z) = \lambda z^n$ with $|\lambda| = 1$.

Having in mind that the inequality (3.5) for $n = 1$ is just the interior Schwarz lemma for analytic functions on the unit disc which appears in K. Fan [12], P. Mercer [17] and R. Osserman [19], we can also consider (3.2) and (3.3) as being sharpened forms of the interior Schwarz inequality for ρ-contractions.

Now, in the case $n = 1$ we obtain from Theorem 3.1 the following result.

COROLLARY 3.2. *Let f be an analytic function D with $f(\mathsf{D}) \subseteq \mathsf{D}$ and $f(0) = 0$. Then, for any operator strict ρ-contraction and for $0 \neq a \in \mathsf{D}$ we have*

$$(3.6) \qquad w_\rho(f(T)) \leq w_\rho(T) \frac{(1 + |f(a)|)w_\rho(T) + |a| + |a^{-1}f(a)|}{(|a| + |a^{-1}f(a)|)w_\rho(T) + 1 + |f(a)|}$$

and

$$(3.7) \qquad w_\rho(f(T)) \leq w_\rho(T) \frac{w_\rho(T) + |f'(0)|}{1 + |f'(0)|w_\rho(T)}.$$

In the case $\rho = 1$, (3.7) gives just the corresponding inequality of K. Fan [12] for $a = 0$, but for $a \in \mathsf{D} \setminus \{0\}$ the inequality of K. Fan is obtained in the

proof of Theorem 3.1, namely

$$(3.8) \qquad \|f(T)\| \le \|T^n\| \frac{\|T_a\| + |a^{-1} f(a)|}{1 + |a^{-1} f(a)| \|T_a\|}$$

if f is as in Corollary 3.2. Clearly, this inequality is stronger than (3.6) in this case. Also, for $\rho = 1$ we infer from the proof of Theorem 3.1 the following

COROLLARY 3.3. *Let f and $n \ge 2$ be as in Theorem 3.1. Then, for any operator T on H with $\|T\| < 1$ and $a \in D \setminus \{0\}$ we have*

$$(3.9) \qquad \|f(T)\| \le \|T\|^n \frac{(|a| + \|T\|) + |a^{-n} f(a)|(1 + |a| \|T\|)}{(1 + |a| \|T\|) + |a^{-n} f(a)|(|a| + \|T\|)}$$

and

$$(3.10) \qquad \|f(T)\| \le \|T\|^n \frac{\|T\| + \frac{1}{n!}|f^{(n)}(0)|}{1 + \frac{1}{n!}|f^{(n)}(0)| \|T\|}$$

Finally, letting $\rho \to +\infty$ in (3.2) and (3.3), we can obtain sharpened forms of the inequality $r(f(T)) \le r(T)^n$ provided by (2.11) when $a = 0$ and $f(0) = 0$. So we have the following versions for interior Schwarz lemma relatively to the spectral radius.

COROLLARY 3.4. *Let f and n be as in Theorem 3.1. Then, for any operator T on H with $r(T) < 1$ and $0 \ne a \in D$ we have*

$$(3.11) \qquad r(f(T)) \le r(T)^n \frac{(|a| + r(T)) + |a^{-n} f(a)|(1 + |a| r(T))}{(1 + |a| r(T)) + |a^{-n} f(a)|(|a| + r(T))}$$

and

$$(3.12) \qquad r(f(T)) \le r(T)^n \frac{r(T) + \frac{1}{n!}|f^{(n)}(0)|}{1 + \frac{1}{n!}|f^{(n)}(0)| r(T)}.$$

4. An improved form of the sharpened von Neumann inequality for strict ρ-contractions

Like in the case of contractions [12], [13], the inequalities (2.1) and (2.2) can be sharpened as follows.

THEOREM 4.1. *Let f be a non-constant analytic function on D such that $f(D) \subseteq D$ and let $a \in D$ and $n \ge 1$ be a integer less than or equal to*

the multiplicity of the zero a for the function $f - f(a)$. Then for any strict ρ-contraction T on H for some $\rho > 0$ we have

(4.1) $$w_{\rho(a)}[\varphi_{f(a)}(f(T))] \leq \beta w_{\rho(a)}(T_a)^n$$

and

(4.2) $$w_{\rho_f(a, \beta w_{\rho(a)}(T_a)^n)}(f(T)) \leq \frac{\beta w_{\rho(a)}(T_a)^n + |f(a)|}{1 + \beta |f(a)| w_{\rho(a)}(T_a)^n}.$$

Here β can be taken either as $\beta = \beta_\rho(f, T, a, n, b)$ with $b \in \mathsf{D}$, $b \neq a$, or as $\beta = \beta_\rho(f, T, a, n)$, where

$$\beta_\rho(f, T, a, n, b)$$
$$= \frac{(|\varphi_a(b)| + w_{\rho(a)}(T_a)) + |\varphi_a(b)^{-n}\varphi_{f(a)}(f(b))|(1 + |\varphi_a(b)|w_{\rho(a)}(T_a))}{(1 + |\varphi_a(b)|w_{\rho(a)}(T_a)) + |\varphi_a(b)^{-n}\varphi_{f(a)}(f(b))|(|\varphi_a(b)| + w_{\rho(a)}(T_a))}$$

and

$$\beta_\rho(f, T, a, n) = \frac{w_{\rho(a)}(T_a) + \frac{1}{n!}(1 - |a|^2)(1 - |f(a)|^2)^{-1}|f^{(n)}(a)|}{1 + \frac{1}{n!}(1 - |a|^2)(1 - |f(a)|^2)^{-1}|f^{(n)}(a)|w_{\rho(a)}(T_a)}$$
$$= \lim_{b \to a} \beta_\rho(f, T, a, n, b).$$

PROOF. Let us write $F = \varphi_{f(a)} \circ f \circ \varphi_a$. Then F is an analytic function on D such that $F(\mathsf{D}) \subseteq \mathsf{D}$ and $F \neq 0$ because f is a non constant function on D. Also, the multiplicity of the zero $z = 0$ for F is equal to the multiplicity of the zero $z = a$ for $f - f(a)$. So, by the assumption on integer $n \geq 1$ we have $F(0) = F'(0) = \cdots = F^{(n-1)}(0) = 0$, and F is not a constant function on D.

Let T be a strict ρ-contraction for some $\rho > 0$. Then one has $w_{\rho(a)}(T_a) < 1$ and $\rho_F(a) = \rho(a)$ because $F(0) = 0$. We have also $\varphi_{f(a)}(f(T)) = F(T_a)$. Now let $b \in \mathsf{D}$, $b \neq a$ and $\alpha = \varphi_a(b)$. Applying the inequality (3.2) with F, T_a, α and $\rho(a)$ instead of f, T, a and ρ respectively, we get

$$w_{\rho(a)}[\varphi_{f(a)}(f(T))] = w_{\rho(a)}(F(T_a))$$

$$\leq w_{\rho(a)}(T_a)^n \frac{(|\alpha| + w_{\rho(a)}(T_a)) + |\alpha^{-n}F(a)|(1 + |\alpha|w_{\rho(a)}(T_a))}{(1 + |\alpha|w_{\rho(a)}(T_a)) + |\alpha^{-n}F(a)|(|\alpha| + w_{\rho(a)}(T_a))}$$

$$= \beta_\rho(f, T, a, n, b)w_{\rho(a)}(T_a)^n$$

where $\beta_\rho(f, T, a, n, b)$ is as in Theorem 4.1. On the other hand, applying the inequality (3.3) with F, T_a and $\rho(a)$ instead of f, T and ρ respectively, we

find

$$w_{\rho(a)}[\varphi_{f(a)}(f(T))] = w_{\rho(a)}(F(T_a))$$

$$\leq w_{\rho(a)}(T_a)\frac{w_{\rho(a)}(T_a) + \frac{1}{n!}|F^{(n)}(0)|}{1 + \frac{1}{n!}|F^{(n)}(0)|w_{\rho(a)}(T_a)}$$

$$= w_{\rho(a)}(T_a)\frac{w_{\rho(a)}(T_a) + \frac{1}{n!}(1-|a|^2)(1-|f(a)|^2)^{-1}|f^{(n)}(a)|}{1 + \frac{1}{n!}(1-|a|^2)(1-|f(a)|^2)^{-1}|f^{(n)}(a)|w_{\rho(a)}(T_a)}$$

$$= \beta_\rho(f, T, a, n)w_{\rho(a)}(T_a)^n.$$

Here we used the known fact (see [1], p. 18) that

$$F^{(n)}(0) = (-1)^{n+1}\frac{1-|a|^2}{1-|f(a)|^2}f^{(n)}(a).$$

Thus we proved the inequality (4.1), that is, we have

$$w_{\rho(a)}(F(T_a)) \leq \beta w_{\rho(a)}(T_a)^n := r_a < 1,$$

where β can be chosen either as $\beta = \beta_\rho(f, T, a, n, b)$, or as $\beta = \beta_\rho(f, T, a, n)$. Since $f(T) = \varphi_{f(a)}(F(T_a))$ and $f(a) = \varphi_{f(a)}(0)$ (which yields $\rho_f(a) = \rho_{\varphi_{f(a)}}(0)$), we obtain by (1.5)

$$w_{\rho_f(a,\beta w_{\rho(a)}(T_a)^n)}(f(T)) = w_{\rho_f(a,\beta w_{\rho(a)}(T_a)^n)}[\varphi_{f(a)}(F(T_a))] \leq \sup_{|z|=r_a}|\varphi_{f(a)}(z)|$$

$$= \frac{r_a + |f(a)|}{1 + |f(a)|r_a} = \frac{\beta w_{\rho(a)}(T_a)^n + |f(a)|}{1 + |f(a)|\beta w_{\rho(a)}(T_a)^n}.$$

Hence the inequality (4.2) holds with the quoted values for β.

Now we remark that since one has (see [1], p. 20)

$$\lim_{z\to 0}\frac{F(z)}{z^n} = \frac{F^{(n)}(0)}{n!}$$

and taking into account the above expressions of $F^{(n)}(0)$, $\beta_\rho(f, T, a, n)$ and $\beta_\rho(f, T, a, n, b)$ with $a \neq b \in \mathsf{D}$, it follows that

$$\lim_{b\to a}\beta_\rho(f, T, a, n, b) = \frac{w_{\rho(a)}(T_a) + \frac{1-|a|^2}{1-|f(a)|^2}\frac{|f^{(n)}(a)|}{n!}}{1 + \frac{1-|a|^2}{1-|f(a)|^2}\frac{|f^{(n)}(a)|}{n!}w_{\rho(a)}(T_a)} = \beta_\rho(f, T, a, n).$$

This concludes the proof.

Clearly $\beta < 1$ in (4.1) and (4.2), hence these inequalities are sharper than the corresponding ones from Theorem 2.1.

In the case $\rho = 1$, which implies $\rho(a) = 1$, we infer from Theorem 4.1 the following result which completes Theorem 4 in [12].

COROLLARY 4.2. *Let f, a and n be as in Theorem 4.1. Then for any strict contraction T acting on H, we have*

$$(4.3) \qquad \|\varphi_{f(a)}(f(T))\| \leq \beta_n \|\varphi_a(T)\|^n$$

and

$$(4.4) \qquad \|f(T)\| \leq \frac{\beta_n \|\varphi_a(T)\|^n + |f(a)|}{1 + \beta_n |f(a)| \|\varphi_a(T)\|^n},$$

where $\beta_n = \beta_1(f, T, a, n)$.

Note that for $n = 1$, the inequalities (4.3) and (4.4) are respectively the inequalities (20) and (22) from [12], where $\beta_1 = \beta_1(f, T, a, 1)$.

Analogous versions of (4.1) and (4.2) for spectral radius can be derived as follows

COROLLARY 4.3. *Let f, a and n be as in Theorem 4.1. Then for any operator T on H with $r(T) < 1$ we have*

$$(4.5) \qquad r[\varphi_{f(a)}(f(T))] \leq \beta_\infty r(T_a)^n$$

and

$$(4.6) \qquad r(f(T)) \leq \frac{\beta_\infty r(T_a)^n + |f(a)|}{1 + \beta_\infty |f(a)| r(T_a)^n},$$

where β_∞ can be taken either as $\beta_\infty = \lim_{\rho \to +\infty} \beta_\rho(f, T, a, n, b)$ with $a \neq b \in \mathsf{D}$, or as $\beta_\infty = \lim_{\rho \to +\infty} \beta_\rho(f, T, a, n)$.

When $r(T) < 1$ and $f(T) = 0$, we can derive from (4.5) a lower bound for $r(T_a)^n$ in terms of $f(a)$ and $f^{(n)}(a)$ for $a \in \mathsf{D}$. This gives also a lower bound for $\|T_a\|^n$, and this result in the case $n = 1$ one reduces to Corollary 3 [12].

COROLLARY 4.4. *Let f, a and n be as in Theorem 4.1. Let T be an operator on H such that $f(T) = 0$ and $r(T) < 1$. Then we have*

$$(4.7) \qquad r(T_a)^n \geq \gamma(f, a, n) \geq |f(a)|,$$

where

$$\gamma(f, a, n) = \left[|f(a)| + \left(\frac{(1 - |a|^2)|f^{(n)}(a)|}{2n!(1 + |f(a)|)} \right)^2 \right]^{\frac{1}{2}} - \frac{(1 - |a|^2)|f^{(n)}(a)|}{2n!(1 + |f(a)|)}.$$

In particular, if $T \in B(H)$ is a strict ρ-contraction ($\rho > 0$) such that $f(T) = 0$, then we have

(4.8) $$w_{\rho(a)}(T_a)^n \geq \gamma(f, a, n).$$

In addition, the second inequality in (4.7) becomes an equality only when either $f(a) = 0$, or f is an n-automorphism of D.

PROOF. Since $f(T) = 0$ one has $r[\varphi_{f(a)}(f(T))] = |f(a)|$, and so from (4.5) we obtain

$$|f(a)| \leq r(T_a)^n \frac{r(T_a)^n + \frac{1-|a|^2}{1-|f(a)|^2} \frac{|f^{(n)}(a)|}{n!}}{1 + \frac{1-|a|^2}{1-|f(a)|^2} \frac{|f^{(n)}(a)|}{n!} r(T_a)^n}.$$

This means that

$$|f(a)| \left[1 + \frac{1-|a|^2}{1-|f(a)|^2} \frac{|f^{(n)}(a)|}{n!} r(T_a)^n \right]$$
$$\leq r(T_a)^n \left[r(T_a)^n + \frac{1-|a|^2}{1-|f(a)|^2} \frac{|f^{(n)}(a)|}{n!} \right]$$

or equivalently

$$|f(a)| + \left(\frac{1-|a|^2}{2(1+|f(a)|)} \frac{|f^{(n)}(a)|}{n!} \right)^2 \leq \left[r(T_a)^n + \frac{1-|a|^2}{2(1+|f(a)|)} \frac{|f^{(n)}(a)|}{n!} \right]^2,$$

whence the first inequality in (4.7) follows immediately. The second inequality in (4.7) is equivalent to

$$|f(a)| \left(1 - \frac{1-|a|^2}{1-|f(a)|^2} \frac{|f^{(n)}(a)|}{n!} \right) \geq 0.$$

Here, equality occurs only if either $f(a) = 0$ or

$$|G(0)| = \frac{(1-|a|^2)|f^{(n)}(a)|}{n!(1-|f(a)|^2)} = 1$$

where G is the analytic function from D into D such that $G(z) = F(z)/z^n$ for $0 \neq z \in$ D. In this last case, the maximum modulus principle tells us that $G(z) = \lambda$ for some constant λ with $|\lambda| = 1$, hence $F(z) = \lambda z^n$, and consequently $f = \varphi_{f(a)} \circ F \circ \varphi_a$, that is f is an n-automorphism of D.

Now if $T \in B(H)$ is a strict ρ-contraction for some $\rho > 0$ then we have $r(T) \leq w_\rho(T) < 1$ and also $r(T_a) \leq w_{\rho(a)}(T_a) < 1$. Thus (4.8) follows from (4.7) and the proof is finished.

Next we will show that the inequality (3.1) can also be sharpened in some conditions, independent of (3.3).

THEOREM 4.5. *Let $f \neq 0$ be an analytic function on D such that $f(\mathsf{D}) \subseteq \mathsf{D}$ and $f(0) = f'(0) = \cdots = f^{(n-1)}(0) = 0$ for some integer $n \geq 1$. Suppose that f does not have the form $f(z) = \lambda z^n$ with λ a unimodular constant. Then for any strict ρ-contraction T on H for some $\rho > 0$ we have*

$$(4.9) \qquad w_\rho(f(T)) \leq w_\rho(T)^n \frac{\gamma w_\rho(T) + \frac{1}{n!}|f^{(n)}(0)|}{1 + \frac{\gamma}{n!}|f^{(n)}(0)|w_\rho(T)}$$

where $\gamma = \gamma_\rho(f, T, n)$ is given by

$$(4.10) \qquad \gamma = \frac{w_\rho(T) + \frac{n!}{n+1}\left[(n!)^2 - |f^{(n)}(0)|^2\right]^{-1}|f^{(n+1)}(0)|}{1 + \frac{n!}{n+1}\left[(n!)^2 - |f^{(n)}(0)|^2\right]^{-1}|f^{(n+1)}(0)|w_\rho(T)}.$$

Moreover, we have $\gamma < 1$ unless either $w_\rho(T) = 0$ or $f(z)/z^n$ is an automorphism of D.

PROOF. Define the function g on D by setting

$$g(0) = \frac{1}{n!}f^{(n)}(0) \qquad \text{and} \qquad g(z) = \frac{f(z)}{z^n} \quad \text{if} \quad z \in \mathsf{D} \setminus \{0\}.$$

Then g is a non-constant analytic function on D such that $g(\mathsf{D}) \subset \mathsf{D}$. So, applying the inequality (4.2) to the function g when $T = zI$ with $z \in \mathsf{D}$, $a = 0, n = 1$ and $\rho = 1$ one obtains

$$|g(z)| \leq \frac{\beta_g(|z|)|z| + |g(0)|}{1 + |g(0)|\beta_g(|z|)|z|} = \frac{\beta_g(|z|)|z| + \frac{1}{n!}|f^{(n)}(0)|}{1 + \frac{1}{n!}|f^{(n)}(0)|\beta_g(|z|)|z|},$$

where $\beta_g(|z|) = \beta_1(g, zI, 0, 1)$ is given by Theorem 4.1, that is

$$\beta_g(|z|) = \frac{|z| + (1 - |g(0)|^2)^{-1}|g'(0)|}{1 + (1 - |g(0)|^2)^{-1}|g'(0)||z|} = \frac{|z| + \frac{n!}{n+1}\frac{|f^{(n+1)}(0)|}{(n!)^2 - |f^{(n)}(0)|^2}}{1 + \frac{n!}{n+1}\frac{|f^{(n+1)}(0)|}{(n!)^2 - |f^{(n)}(0)|^2}|z|}.$$

Since we have

$$\frac{n!|f^{(n+1)}(0)|}{(n+1)[(n!)^2 - |f^{(n)}(0)|^2]} = \frac{|g'(0)|}{1 - |g(0)|^2} = |G(0)|$$

where G is the analytic function on D defined by

$$G(z) = \frac{\varphi_{g(0)}(g(z))}{z}, \qquad z \in \mathsf{D} \setminus \{0\},$$

by the maximum modulus principle we infer that $|G(0)| < 1$ unless G is a constant function, or equivalently g is an automorphism of D. Thus, if f does not have the form $f(z) = z^n g(z)$ where g is an automorphism of D, then we have $|G(0)| < 1$ which yields $\beta_g(|z|) < 1$ for any $z \in \mathsf{D}$.

Now let $T \in B(H)$ be a strict ρ-contraction for some $\rho > 0$. Since $f(0) = 0$ one has $\rho' = \rho_f = \rho$ so that $f(T) \in C_\rho(H)$, and using the above estimation of $|g(z)|$ and the inequality (1.5) one obtains

$$w_\rho(f(T)) \leq \sup_{|z|=w_\rho(T)} |f(z)| = \sup_{|z|=w_\rho(T)} |z^n g(z)| = w_\rho(T)^n \sup_{|z|=w_\rho(T)} |g(z)|$$

$$\leq w_\rho(T)^n \frac{\beta_g(w_\rho(T))w_\rho(T) + \frac{1}{n!}|f^{(n)}(0)|}{1 + \frac{1}{n!}|f^{(n)}(0)|\beta_g(w_\rho(T))w_\rho(T)}$$

$$= w_\rho(T)^n \frac{\gamma w_\rho(T) + \frac{1}{n!}|f^{(n)}(0)|}{1 + \frac{1}{n!}|f^{(n)}(0)|\gamma w_\rho(T)},$$

where $\gamma = \beta_g(w_\rho(T))$, that is

$$\gamma = \frac{w_\rho(T) + \frac{n!}{n+1}\frac{|f^{(n+1)}(0)|}{(n!)^2-|f^{(n)}(0)|^2}}{1 + \frac{n!}{n+1}\frac{|f^{(n+1)}(0)|}{(n!)^2-|f^{(n)}(0)|^2}w_\rho(T)}.$$

In addition, taking into account the above remark, we conclude that $\gamma < 1$ unless either f has the form $f(z) = z^n g(z)$ with g an automorphism of D, or $w_\rho(T) = 0$. This ends the proof.

COROLLARY 4.6. *Let f and n be as in Theorem 4.5. Then, for any operator T on H with $\|T\| < 1$ we have*

$$(4.11) \qquad \|f(T)\| \leq \|T^n\| \frac{\gamma_1\|T\| + \frac{1}{n!}|f^{(n)}(0)|}{1 + \frac{1}{n!}|f^{(n)}(0)|\gamma_1\|T\|},$$

where $\gamma_1 := \gamma_1(f, T, n)$ is given by (4.10). In addition, $\gamma_1 < 1$ unless either $T = 0$, or $f(z)/z^n$ is an automorphism of D.

We remark that in the case $n = 1$, (4.11) is just the inequality (36) from [12].

Finally, we can also obtain a version for spectral radius of (4.11) as follows.

COROLLARY 4.7. *Let f and n be as in Theorem 4.5. Then, for any operator T on H with $r(T) < 1$ we have*

$$(4.12) \qquad r(f(T)) \leq r(T)^n \frac{\gamma_\infty r(T) + \frac{1}{n!}|f^{(n)}(0)|}{1 + \frac{1}{n!}|f^{(n)}(0)|\gamma_\infty r(T)}$$

where
(4.13)
$$\gamma_\infty := \lim_{\rho \to +\infty} \gamma_\rho(f, T, n) = \frac{r(T) + \frac{n!}{n+1}\left[(n!)^2 - |f^{(n)}(0)|^2\right]^{-1}|f^{(n+1)}(0)|}{1 + \frac{n!}{n+1}\left[(n!)^2 - |f^{(n)}(0)|^2\right]^{-1}|f^{(n+1)}(0)|r(T)}.$$

In addition, $\gamma_\infty < 1$ unless either $r(T) = 0$, or $f(z)/z^n$ is an automorphism of $\mathbb{D}$.

PROOF. Since $r(T) < 1$ we have $w_\rho(T) < 1$ for sufficiently large $\rho > 1$. So, (4.12) and (4.13) follow from (4.9) and (4.10) respectively by letting $\rho \to +\infty$.

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