MATH. SCAND. 103 (2008), 40–52

TOEPLITZ OPERATORS ON WEIGHTED SPACES OF HOLOMORPHIC FUNCTIONS

ANAHIT HARUTYUNYAN* and WOLFGANG LUSKY

Abstract

We define a notion of Toeplitz operator on certain spaces of holomorphic functions on the unit disk and on the complex plane which are endowed with a weighted sup-norm. We establish boundedness and compactness conditions, give norm estimates and characterize the essential spectrum of these operators for many symbols.

1. Introduction

We deal with holomorphic functions $h : \Omega \to \mathbf{C}$, where Ω is the open unit disk $\mathbf{D} = \{z \in \mathbf{C} : |z| < 1\}$ or $\Omega = \mathbf{C}$, which are subject to certain growth conditions. To this end we consider an arbitrary function $v : [0, a[\to \mathbf{R}_+$ which is continuous and non-increasing where $a = 1$ if $\Omega = \mathbf{D}$ and $a = \infty$ if $\Omega = \mathbf{C}$. If $a = 1$ we assume that $\lim_{r\to 1} v(r) = 0$ while for $a = \infty$ we assume that $\lim_{r\to\infty} r^n v(r) = 0$ for any $n \geq 0$. v is called a weight function. For fixed r we put

$$M_\infty(h, r) = \sup_{|z|=r} |h(z)| \qquad \text{and} \qquad \|h\|_v = \sup_{0\leq r<a} M_\infty(h, r)v(r)$$

and we define

$$Hv(\Omega) = \{h : \Omega \to \mathbf{C} \text{ holomorphic} : \|h\|_v < \infty\}$$

$Hv(\Omega)$ is a Banach space with the norm $\|\cdot\|_v$. We obtain $h \in Hv(\Omega)$ if and only if $M_\infty(h, r) = O\big(\frac{1}{v(r)}\big)$ as $r \to a$. The conditions on v ensure that $Hv(\Omega)$ contains all polynomials.

The complete isomorphic classification of the spaces $Hv(\Omega)$ is known ([1]). Indeed, $Hv(\Omega)$ is either isomorphic to l_∞ or to $H_\infty = \{h : \mathbf{D} \to \mathbf{C} \text{ holomorphic} : h \text{ bounded}\}$. To decide whether $Hv(\Omega)$ is isomorphic to l_∞ one needs to consider the functions $\gamma_n(r) = r^n v(r)$ for any $n > 0$. For each

* Supported by Deutsche Forschungsgemeinschaft 436 ARM 17/2/06.
Received December 31, 2006.

$n > 0$ pick a global maximum point r_n of γ_n. We easily see that $\lim_{n\to\infty} r_n = a$. v is said to satisfy condition (B) if

$$\forall b_1 > 1\ \exists b_2 > 1\ \exists c > 0\ \forall m, n > 0 :$$

$$\left(\frac{r_m}{r_n}\right)^m \frac{v(r_m)}{v(r_n)} \le b_1 \quad \text{and} \quad m, n, |m-n| \ge c \quad \Rightarrow \quad \left(\frac{r_n}{r_m}\right)^n \frac{v(r_n)}{v(r_m)} \le b_2$$

We have (see [2, Theorem 1.1])

$Hv(\Omega)$ is isomorphic to l_∞ if and only if v satisfies (B).

Examples of weights satisfying (B) include all normal weights on $[0, 1[$ (see [3]), in particular $v(r) = (1-r)^\alpha$ for any $\alpha > 0$. Moreover $\exp(-1/(1-r))$, $\exp(-\exp(1/(1-r)))$, ... satisfy (B).

If $a = \infty$ then $\exp(-r^\rho)$ for any $\rho > 0$, $\exp(-\log^\tau r)$ for any $\tau \ge 2$, $\exp(-\exp(r))$, $\exp(-\exp(\exp(r)))$, ... satisfy (B) (see [1] for details).

If v satisfies (B) then $Hv(\Omega)$ is complemented in any superspace. In this situation it is possible to give a meaningful definition of Toeplitz operator on $Hv(\Omega)$. At first, we use induction to find indices $0 < m_1 < m_2 < \cdots$ such that $r_{m_1} > 0$ and

$$3 \le \min\left(\left(\frac{r_{m_n}}{r_{m_{n+1}}}\right)^{m_n} \frac{v(r_{m_n})}{v(r_{m_{n+1}})}, \left(\frac{r_{m_{n+1}}}{r_{m_n}}\right)^{m_{n+1}} \frac{v(r_{m_{n+1}})}{v(r_{m_n})}\right) \le 4 \tag{1.1}$$

(This is possible since, by assumption, $\lim_{M\to\infty} r_M^n v(r_M) = 0$ for any $n > 0$, see [1, Lemma 5.1].)

Let $g : \Omega \to \mathbb{C}$ be a function such that $g|_{r\partial\mathbb{D}}$ is continuous for each $r \in]0, a[$. Then, for fixed r let $\sum_{j\in\mathbb{Z}} g_j(r) r^{|j|} e^{ij\varphi}$ be the Fourier series of g. For $0 \le m < n$ put

$$V_{n,m} g = \sum_{|j|\le m} g_j(r) r^{|j|} e^{ij\varphi} + \sum_{m<|j|\le n} \frac{[n]-|j|}{[n]-[m]} g_j(r) r^{|j|} e^{ij\varphi}$$

where $[c]$ is the largest integer $\le c$. In [2] the following theorem is shown (based on the results of [1])

THEOREM 1.1. *If v satisfies* (B) *and* (m_n) *are the preceding indices then there exist $d_1 > 0$ and $d_2 > 0$ such that, for any $h \in Hv(\Omega)$, we have*

$$d_1 \sup_k M_\infty((V_{m_{k+1},m_k} - V_{m_k,m_{k-1}})h, r_{m_k}) v(r_{m_k})$$
$$\le \|h\|_v \le d_2 \sup_k M_\infty((V_{m_{k+1},m_k} - V_{m_k,m_{k-1}})h, r_{m_k}) v(r_{m_k})$$

(Put $m_0 = 0$ and $V_{m_0,m_{-1}} = 0$.)

This gives rise to the following definition. Let g be as in the definition of $V_{n,m}$ and let $t_{k,j}$ be such that

$$(V_{m_{k+1},m_k} - V_{m_k,m_{k-1}})g = \sum_{m_{k-1}<|j|\leq m_{k+1}} t_{k,j} g_j(r) r^{|j|} e^{ij\varphi}.$$

Then define

$$W_k g = \sum_{m_{k-1}<j\leq m_{k+1}} t_{k,j} g_j(r_{m_k}) r^j e^{ij\varphi}, \quad k \in \mathsf{Z}_+, \quad \text{and} \quad Pg = \sum_k W_k g$$

provided the last definition makes sense (i.e. the preceding Fourier series represents a holomorphic function on Ω). One can show that $Pg \in Hv(\Omega)$ if $\sup_{0\leq r<a} M_\infty(g, r)v(r) < \infty$ ([2]). It is easily seen that $Pg = g$ if $g \in Hv(\Omega)$.

Now we define Toeplitz operators.

DEFINITION 1.2. Let $f : \Omega \to \mathsf{C}$ be such that, for all $r \in]0, a[$, $f|_{r\partial\mathsf{D}}$ is continuous. For $h \in Hv(\Omega)$ put

$$T_f(h) = P(fh)$$

(if this definition makes sense). Then T_f is called Toeplitz operator with symbol f.

Later on (Corollary 2.6.) we show that for many symbols f the definition of T_f is independent of the numbers m_k up to compact perturbations.

In section 2 we give boundedness and compactness conditions for T_f and discuss the case of functions $f : \overline{\mathsf{D}} \to \mathsf{C}$ which are continuous on $\overline{\mathsf{D}} \setminus \rho\mathsf{D}$ for some $0 < \rho < 1$. In particular we show that, for suitable harmonic g, $T_f - T_g$ is compact. In section 4 we determine the essential spectrum for T_f with respect to such functions f. Moreover, we show that, for harmonic $g : \mathsf{D} \to \mathsf{C}$, $\|T_g\|$ and $M_\infty(g, 1)$ are equivalent.

2. Continuity and compactness conditions for Toeplitz operators

Again in this section let $f : \Omega \to \mathsf{C}$ be such that

(2.1) $$f|_{r\partial r\mathsf{D}} \text{ is continuous for all } r \in]0, a[.$$

Then, for each $r \in]0, a[$, $f(re^{i\varphi})$ has a Fourier series $\sum_j f_j(r) r^{|j|} e^{ij\varphi}$.

For $0 < p$ let the Cesaro mean σ_p be defined by

$$\sigma_p f = \sum_{|j|\leq p} \frac{[p] - |j|}{[p]} f_j(r) r^{|j|} e^{ij\varphi}.$$

Note that

$$M_\infty(\sigma_p f, r) \le M_\infty(f, r) \qquad \text{and} \qquad \lim_{p\to\infty} M_\infty(f - \sigma_p f, r) = 0$$

for each r.

If m_k are the preceding indices then fix n_k such that

$$0 < \inf_k \left(\frac{m_{k+1} - m_k}{n_k}\right) \le \sup_k \left(\frac{m_{k+1} - m_k}{n_k}\right) < \infty.$$

We show

THEOREM 2.1. *Let* $f : \Omega \to \mathsf{C}$ *satisfy* (2.1) *and assume that*

$$\sup_k M_\infty(\sigma_{n_k}|f|, r_{m_k}) < \infty.$$

Then T_f *is a bounded operator* $Hv(\Omega) \to Hv(\Omega)$*. Moreover there is a constant* $c > 0$ *(independent of* f*) such that*

$$\|T_f\| \le c \sup_k M_\infty(\sigma_{n_k}|f|, r_{m_k}).$$

We shall prove Theorem 2.1 in section 3. Using the preceding theorem we easily find examples even of unbounded $f : \Omega \to \mathsf{C}$ where $T_f : Hv(\Omega) \to Hv(\Omega)$ is bounded.

EXAMPLE. Put

$$f(z) = \begin{cases} \frac{1}{z^n}, & z \ne 0 \\ 0, & z = 0 \end{cases}$$

for some integer $n > 0$. Then, according to Theorem 2.1, T_f is bounded and $\|T_f\| \le c r_{m_1}^{-n}$.

In section 3 we also show

THEOREM 2.2. *Let* $f : \Omega \to \mathsf{C}$ *satisfy* (2.1)*. If* $\lim_{k\to\infty} M_\infty(\sigma_{n_k}|f|, r_{m_k}) = 0$ *then* $T_f : Hv(\Omega) \to Hv(\Omega)$ *is compact.*

In the rest of this section we discuss some consequences of Theorem 2.2 for $\Omega = \mathsf{D}$.

PROPOSITION 2.3. *Let* $f : \overline{\mathsf{D}} \to \mathsf{C}$ *satisfy* (2.1) *and assume that, for some* $0 < \rho < 1$*,* f *is continuous on* $\overline{\mathsf{D}} \setminus \rho\mathsf{D}$*. Then*

$$\lim_{n\to\infty} \sup_{\rho \le r \le 1} M_\infty(f - \sigma_n f, r) = 0.$$

In particular, $\lim_{n\to\infty} \|T_f - T_{\sigma_n f}\| = 0$.

PROOF. Using the Weierstraß theorem we see that f can be uniformly approximated on $\overline{\mathsf{D}} \setminus \rho\mathsf{D}$ by functions g_m of the form

$$g_m(re^{i\varphi}) = \sum_{|k|\le m} g_{m,k}(r) r^{|k|} e^{ik\varphi}.$$

So fix $\epsilon > 0$ and g_m such that $M_\infty(f - g_m, r) < \epsilon$ for all $\rho \le r \le 1$. Hence

$$\begin{aligned}\sup_{\rho\le r\le 1} M_\infty(f - \sigma_n f, r) \le & \sup_{\rho\le r\le 1} M_\infty(f - g_m, r) + \sup_{\rho\le r\le 1} M_\infty(\sigma_n g_m - \sigma_n f, r) \\ & + \sup_{\rho\le r\le 1} M_\infty(g_m - \sigma_n g_m, r) \\ < & \, 3\epsilon \end{aligned}$$

for suitably large n. Then $\lim_{n\to\infty} \sup_{\rho\le r\le 1} M_\infty(f - \sigma_n f, r) = 0$.

Let k_0 be such that $r_{m_k} > \rho$ for all $k \ge k_0$. Theorem 2.1. implies

$$\begin{aligned}\|T_f - T_{\sigma_n f}\| = \|T_{f-\sigma_n f}\| &\le c \sup_k M_\infty(f - \sigma_n f, r_{m_k}) \\ &\le c \max\Bigl(\sup_{k\le k_0} M_\infty(f - \sigma_n f, r_{m_k}),\ \sup_{\rho\le r\le 1} M_\infty(f - \sigma_n f, r)\Bigr)\end{aligned}$$

In view of (2.1) this proves $\lim_{n\to\infty} \|T_f - T_{\sigma_n f}\| = 0$.

Let f satisfy the assumptions of Proposition 2.3. Then $f|_{\partial\mathsf{D}}$ is continuous and has a harmonic extension f_h on D. So, if

$$\gamma_k = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\varphi}) e^{-ik\varphi}\, d\varphi, \qquad k \in \mathsf{Z},$$

then $f_h(re^{i\varphi}) = \sum_{k\in\mathsf{Z}} \gamma_k r^{|k|} e^{ik\varphi}$, $r < 1$.

COROLLARY 2.4. *Let* $f : \overline{\mathsf{D}} \to \mathsf{C}$ *satisfy* (2.1) *and assume that* f *is continuous on* $\overline{\mathsf{D}} \setminus \rho\mathsf{D}$ *for some* $0 < \rho < 1$. *Then* $T_f - T_{f_h}$ *is compact.*

PROOF. In view of Proposition 2.3 it suffices to assume $f(re^{i\varphi}) = f_j(r) r^{|j|} e^{ij\varphi}$ for some $j \in \mathsf{Z}$. Then $|f - f_h|(re^{i\varphi}) = |f_j(r) - f_j(1)| r^{|j|}$. Using the numbers n_k and m_k of Theorems 2.1 and 2.2 we obtain

$$\lim_{k\to\infty} M_\infty(\sigma_{n_k} |f - f_h|, r_{m_k}) = \lim_{k\to\infty} |f_j(r_{m_k}) - f_j(1)| = 0.$$

Now Theorem 2.2 proves the corollary.

Let $f(re^{i\varphi}) = \sum_{|k|\le n} f_k(r)r^{|k|}e^{ik\varphi}$ where all f_k are continuous on $[\rho, 1]$ for some $0 < \rho < 1$. Put

$$h_f(z) = \sum_{|k|\le n} f_k(1)z^k.$$

Then h_f is continuous on $\overline{\mathsf{D}} \setminus \{0\}$ and $(h_f)_h = f_h$. Corollary 2.4 implies

LEMMA 2.5. *Let $f : \overline{\mathsf{D}} \to \mathsf{C}$ satisfy* (2.1) *and assume that f is continuous on $\overline{\mathsf{D}} \setminus \rho\mathsf{D}$ for some $0 < \rho < 1$. Then $T_{\sigma_n f} - T_{h_{\sigma_n f}}$ is compact for all $n > 0$.*

The Toeplitz operator T_f is defined via numbers m_k satisfying (1.1). Let us go over to numbers $\tilde{m}_k$ which also satisfy (1.1) and consider the resulting Toeplitz operator $\tilde{T}_f$. Quite often we obtain that $T_f - \tilde{T}_f$ is compact.

COROLLARY 2.6. *Let $f : \overline{\mathsf{D}} \to \mathsf{C}$ satisfy* (2.1) *and assume that f is continuous on $\overline{\mathsf{D}} \setminus \rho\mathsf{D}$ for some $0 < \rho < 1$. Then T_f is independent of the numbers m_k up to compact perturbations.*

PROOF. In view of Proposition 2.3 and Lemma 2.5 it suffices to asume $f(z) = \alpha z^n$ for some $\alpha \in \mathsf{C}$ and $n \in \mathsf{Z}$. If $h \in Hv(\mathsf{D})$ is such that $h(z) = \sum_{k>|n|} \beta_k z^k$ then fh is holomorphic. Hence if T_f and $\tilde{T}_f$ are the Toeplitz operators with respect to the numbers m_k and $\tilde{m}_k$ then $(T_f - \tilde{T}_f)h = 0$. It follows that $T_f - \tilde{T}_f$ has finite rank and, therefore, is compact.

3. Proofs of Theorems 2.1 and 2.2

The proof follows from some lemmas. At first, let $f : \Omega \to \mathsf{C}$ be such that $f|_{r\partial\mathsf{D}} \in L_1(r\partial\mathsf{D})$ for all r and the Fourier series of f for fixed r is $\sum_{j\in\mathsf{Z}} f_j(r)r^{|j|}e^{ij\varphi}$. Let R be the Riesz projection, i.e. Rf has the Fourier series $\sum_{j\ge 0} f_j(r)r^{|j|}e^{ij\varphi}$. Finally, for $k \in \mathsf{Z}$, define the translation operator U_k by $U_k f = e^{ik\varphi} f$.

In [1, Lemma 3.3], it was shown that

$$M_\infty(R(V_{p,n} - V_{n,m})f, r) \le \delta M_\infty(f, r)$$

for all r and $m, n, p \in \mathsf{Z}_+$ with $0 \le m \le n \le p$ where δ is a constant which depends only on

$$\frac{p - m}{\min(p - n, n - m, m)}$$

but not on f or r.

LEMMA 3.1. *Let $m, n, p \in \mathbb{Z}_+$ such that $0 \le m \le n \le p$ and fix $q \in \mathbb{Z}_+$. There is a universal constant $d > 0$ depending only on*

$$\frac{p-m}{\min(p-n, n-m, m, q)}$$

(but not on f) such that

$$M_\infty(R(V_{p,n} - V_{n,m})f, r) \le d \sup_{k\in\mathbb{Z}_+} M_\infty(U_k\sigma_q U_{-k} f, r) \qquad \text{for all} \quad r.$$

PROOF. For $k \ge q$ we have

$$U_k\sigma_q U_{-k} f = \sum_{j=k-q}^{k} f_j(r)\frac{j+q-k}{q} r^j e^{ij\varphi} + \sum_{j=k+1}^{k+q} f_j(r)\frac{q+k-j}{q} e^{ij\varphi}.$$

Hence we find $k_1, \dots, k_N$ with

$$R(V_{p,n} - V_{n,m})\sum_{l=1}^{N} U_{k_l}\sigma_q U_{-k_l} f = R(V_{p,n} - V_{n,m})f$$

where N depends only on $(p-m)/q$. This implies, with the previous constant δ,

$$M_\infty(R(V_{p,n} - V_{n,m})f, r) \le N\delta \sup_{k\in\mathbb{Z}_+} M_\infty(U_k\sigma_{p-n} U_{-k} f, r)$$

for any r which proves the lemma.

Now we return to the definition of W_k (preceding 1.2).

LEMMA 3.2. *There is a constant $c > 0$ such that, for any $f : \Omega \to \mathbb{C}$, $h \in Hv(\Omega)$ and any $l \in \mathbb{Z}_+$, we have*

$$\Big\| \sum_{k\ge l} W_k(fh) \Big\|_v \le c \sup_{k\ge l} M_\infty(W_k(fh), r_{m_k}) v(r_{m_k}).$$

PROOF. By definition we have

$$(V_{m_{j+1},m_j} - V_{m_j,m_{j-1}})W_k = 0 \qquad \text{if} \quad |k-j| > 1.$$

Since v satisfies (B) then either

$$0 < \inf_k \frac{m_{k+1} - m_k}{m_k - m_{k-1}} \le \sup_k \frac{m_{k+1} - m_k}{m_k - m_{k-1}} < \infty$$

or $\sup_k(m_{k+1} - m_{k-1}) < \infty$ ([1, Proposition 4.1]). According to [1, Lemma 3.3], we obtain that

$$M_\infty((V_{m_{j+1},m_j} - V_{m_j,m_{j-1}})g, r) \le dM_\infty(g, r)$$

for any g and any j where d is a universal constant. Theorem 1.1 yields

$$\begin{aligned}
&\Big\|\sum_{k\ge l} W_k(fh)\Big\|_v \\
&\quad \le d_2 \sup_j M_\infty((V_{m_{j+1},m_j} - V_{m_j,m_{j-1}})\Big(\sum_{k\ge l} W_k\Big)(fh), r_{m_j})v(r_{m_j}) \\
&\quad = d_2 \sup_j M_\infty((V_{m_{j+1},m_j} - V_{m_j,m_{j-1}})(W_{j-1} + W_j + W_{j+1})(fh), r_{m_j})v(r_{m_j}) \\
&\quad \le 3d_2 d \sup_{j\ge l} \max(M_\infty(W_j(fh), r_{m_{j-1}})v(r_{m_{j-1}}), \\
&\qquad M_\infty(W_j(fh), r_{m_j})v(r_{m_j}), M_\infty(W_j(fh), r_{m_{j+1}})v(r_{m_{j+1}}))
\end{aligned}$$

$W_j(fh)$ is a polynomial of the form $\sum_{k=m_{j-1}}^{m_{j+1}} \alpha_k z^k$. According to [1, Lemma 3.1], we infer

$$\begin{aligned}
&M_\infty(W_j(fh), r_{m_{j-1}})v(r_{m_{j-1}}) \\
&\qquad \le 2\left(\frac{r_{m_{j-1}}}{r_{m_j}}\right)^{m_{j-1}} \frac{v(r_{m_{j-1}})}{v(r_{m_j})} M_\infty(W_j(fh), r_{m_j})v(r_{m_j})
\end{aligned}$$

and

$$\begin{aligned}
&M_\infty(W_j(fh), r_{m_{j+1}})v(r_{m_{j+1}}) \\
&\qquad \le 2\left(\frac{r_{m_{j+1}}}{r_{m_j}}\right)^{m_{j+1}} \frac{v(r_{m_{j+1}})}{v(r_{m_j})} M_\infty(W_j(fh), r_{m_j})v(r_{m_j})
\end{aligned}$$

This yields the lemma.

To finish the proof of Theorem 2.1 consider, for $l > 0$, the Fejer kernel

$$F_l(\varphi) = \sum_{|j|\le l} \frac{[l] - |j|}{[l]} e^{ij\varphi}.$$

It is well-known that $F_l(\varphi) \ge 0$ for all φ. We have

$$(\sigma_l f)(re^{i\varphi}) = \frac{1}{2\pi}\int_0^{2\pi} F_l(\varphi - \psi) f(re^{i\psi})\, d\psi.$$

Hence, if $h \in Hv(\Omega)$ we obtain

$$M_\infty(U_k\sigma_l U_{-k}(fh), r) \le M_\infty(\sigma_l|f|, r)M_\infty(h, r)$$

for any $k \in \mathbb{Z}$ and any $r > 0$.

Conclusion of the proof of Theorem 2.1

We take into account that, for arbitrary $0 < n < p$, we have $V_{p,n} = V_{[p],[n]}$. Assume $\sup_k M_\infty(\sigma_{n_k}|f|, r_{m_k}) < \infty$. With Lemma 3.1 we obtain constants d_k such that, for any $h \in Hv(\Omega)$,

$$\begin{aligned} M_\infty(W_k(fh), r_{m_k}) &\le d_k \sup_{l\in\mathbb{Z}_+} M_\infty(U_l\sigma_{n_k}U_{-l}(fh), r_{m_k}) \\ &\le d_k M_\infty(\sigma_{n_k}|f|, r_{m_k})M_\infty(h, r_{m_k}). \end{aligned}$$

Since v satisfies (B) then either

$$0 < \inf_k \frac{m_{k+1} - m_k}{m_k - m_{k-1}} \le \sup_k \frac{m_{k+1} - m_k}{m_k - m_{k-1}} < \infty$$

or $\sup_k(m_{k+1} - m_{k-1}) < \infty$ ([1, Proposition 4.1]).

Hence, in view of Lemma 3.1, the d_k are uniformly bounded. According to Lemma 3.2 with $l = 1$ we obtain

$$\|T_f h\|_v \le (\sup_k d_k)(\sup_k M_\infty(\sigma_{n_k}|f|, r_{m_k}))\|h\|_v.$$

Since $T_f h$ is a holomorphic function on Ω we conclude $T_f h \in Hv(\Omega)$.

Conclusion of the proof of Theorem 2.2

Assume that $\lim_{k\to\infty} M_\infty(\sigma_{n_k}|f|, r_{m_k}) = 0$. By the same argument as in the preceding proof, in view of Lemma 3.2, we see that, for any $\epsilon > 0$, there is l such that $\|\sum_{k>l} W_k(fh)\|_v \le \epsilon\|h\|_v$ for any $h \in Hv(\Omega)$. Hence $\|T_f(h) - \sum_{k=1}^l W_k(fh)\|_v \le \epsilon\|h\|_v$ for all $h \in Hv(\Omega)$. This means that T_f is the limit of a sequence of finite rank operators. Hence T_f is compact.

4. Some consequences

Here we give more applications of the results of section 2 for the case $\Omega = \mathbb{D}$. At first we show

LEMMA 4.1. *Let $h_n(z) = z^n$, $n \in \mathbb{Z}_+$. Then $h_n/\|h_n\|$ tends to* 0 *weakly (in $Hv(\Omega)$).*

PROOF. Consider

$$(Hv)_0(\Omega) = \left\{h : \Omega \to \mathsf{C} : h \text{ holomorphic}, \lim_{r\to a} \sup_{|z|=r} |h(z)|v(r) = 0\right\}$$

and take the indices m_k of Theorem 1.1 Put

$$H_k = \operatorname{span}\{z^j : m_{k-1} \le j \le m_{k+1}\} \subset (Hv)_0(\Omega).$$

Then Theorem 1.1 yields that $(Hv)_0(\Omega) \subset \left(\sum_{k=1}^{\infty} \oplus H_k\right)_0$ (endowed with the norm $\|(g_k)\| = \sup_k \|g_k\|_v$). Since $(Hv)_0(\Omega) \subset Hv(\Omega)$ this implies Lemma 4.1. (Notice, both inclusions are inclusions as closed subspaces.)

Now we show the central

THEOREM 4.2. *Let $f : \mathsf{D} \to \mathsf{C}$ be harmonic. Then f is bounded if and only if T_f is bounded. In this case there are universal constants $d_1, d_2 > 0$ (independent of f) such that*

$$d_1 M_\infty(f, 1) \le \inf\{\|T_f + K\| : K : Hv(\mathsf{D}) \to Hv(\mathsf{D}) \textit{ linear, compact}\}$$
$$\le \|T_f\| \le d_2 M_\infty(f, 1)$$

PROOF. At first let f be bounded. Then f has $L_\infty(\partial\mathsf{D})$-boundary values. According to Theorem 2.1 we obtain

$$\|T_f\| \le c \sup_k M_\infty(\sigma_{n_k}|f|, r_{m_k}) = cM_\infty(|f|, 1) = cM_\infty(f, 1).$$

Conversely, let T_f be bounded. Put $f(re^{i\varphi}) = \sum_{k\in\mathsf{Z}} \alpha_k r^{|k|} e^{ik\varphi}$. Fix $n \in \mathsf{Z}_+$. Then we have with $z = re^{i\varphi}$,

$$z^n f(z) = \sum_{k=-\infty}^{-n-1} \alpha_k r^{2n} r^{|k|-n} e^{-i(|k|-n)\varphi} + \sum_{k=-n}^{-1} \alpha_k r^{2|k|} r^{n+k} e^{i(n+k)\varphi} + \sum_{k=0}^{\infty} \alpha_k r^{k+n} e^{i(k+n)\varphi}$$

Definition 1.2 implies, with $h_n(z) = z^n$,

$$T_f(h_n) = \sum_{k=-n}^{-1} \alpha_k \gamma_k(n) r^{n+k} e^{i(n+k)\varphi} + \sum_{k=0}^{\infty} \alpha_k r^{k+n} e^{i(k+n)\varphi},$$

where, for $[m_j] \le n + k < [m_{j+1}]$ and $k < 0$,

$$\gamma_k(n) = \frac{[m_{j+1}] - (n+k)}{[m_{j+1}] - [m_j]} r_{m_j}^{2|k|} + \frac{n+k-[m_j]}{[m_{j+1}] - [m_j]} r_{m_{j+1}}^{2|k|}.$$

Since $\lim_{j\to\infty} r_{m_j} = 1$ we obtain $\lim_{n\to\infty} \gamma_k(n) = 1$ for each k. This implies

$$\begin{aligned}(V_{2n,n} - V_{n,0})T_f(h_n) \\ &= \sum_{k=-n}^{-1} \alpha_k \frac{n+k}{n} \gamma_k(n) r^{n+k} e^{i(n+k)\varphi} + \sum_{k=0}^{n} \alpha_k \frac{n-k}{n} r^{k+n} e^{i(k+n)\varphi} \\ &= z^n \left(\sum_{j=1}^{n} \alpha_{-j} \frac{n-j}{n} \gamma_{-j}(n) r^{-j} e^{-ij\varphi} + \sum_{k=0}^{n} \alpha_k \frac{n-k}{n} r^k e^{ik\varphi} \right).\end{aligned}$$

We have $\|h_n\|_v = r_n^n v(r_n)$. Let $K : Hv(\mathsf{D}) \to Hv(\mathsf{D})$ be linear and compact. Then we obtain a universal constant $c > 0$ (independent of K, f and n) such that

$$\begin{aligned}\|T_f + K\| &\geq \left\| T_f \left(\frac{h_n}{\|h_n\|_v} \right) \right\|_v - \left\| K \left(\frac{h_n}{\|h_n\|_v} \right) \right\|_v \\ &\geq c \frac{\|(V_{2n,n} - V_{n,0})T_f(h_n)\|_v}{\|h_n\|_v} - \left\| K \left(\frac{h_n}{\|h_n\|_v} \right) \right\|_v \\ &\geq c \frac{M_\infty((V_{2n,n} - V_{n,0})T_f(h_n), r_n)}{r_n^n v(r_n)} - \left\| K \left(\frac{h_n}{\|h_n\|_v} \right) \right\|_v \\ &= c \sup_\varphi \left| \sum_{k=1}^{n} \alpha_{-k} \frac{n-k}{n} \gamma_{-k}(n) r_n^{-k} e^{-ik\varphi} \right. \\ &\qquad \left. + \sum_{k=0}^{n} \alpha_k \frac{n-k}{n} r_n^k e^{ik\varphi} \right| - \left\| K \left(\frac{h_n}{\|h_n\|_v} \right) \right\|_v\end{aligned}$$

If we fix $m \in \mathsf{Z}_+$ and take $n \geq m$ then we also have

$$\begin{aligned}\|T_f + K\| \geq c \sup_\varphi &\left| \sum_{k=1}^{m} \alpha_{-k} \frac{n-k}{n} \frac{m-k}{m} \gamma_{-k}(n) r_n^{-k} e^{-ik\varphi} \right. \\ &\qquad \left. + \sum_{k=0}^{m} \alpha_k \frac{n-k}{n} \frac{m-k}{m} r^k e^{ik\varphi} \right| - \left\| K \left(\frac{h_n}{\|h_n\|_v} \right) \right\|_v\end{aligned}$$

Letting $n \to \infty$ Lemma 4.1 implies $\lim_{n\to\infty} \|K(h_n/\|h_n\|_v)\|_v = 0$ since K is compact. We arrive at

$$\|T_f + K\| \geq c \sup_\varphi \left| \sum_{k=1}^{m} \alpha_{-k} \frac{m-k}{m} e^{-ik\varphi} + \sum_{k=0}^{m} \alpha_k \frac{m-k}{m} e^{ik\varphi} \right| = cM_\infty(\sigma_m f, 1)$$

and hence
$$cM_\infty(f,1) = c\sup_m M_\infty(\sigma_m f,1) \le \|T_f + K\|$$

This proves

$$cM_\infty(f,1) \le \inf\{\|T_f + K\| : K : Hv(\mathsf{D}) \to Hv(\mathsf{D}) \text{ linear and compact}\}.$$

Theorem 2.1 yields $\|T_f\| \le cM_\infty(f,1)$ in view of the maximum principle.

COROLLARY 4.3. *Let* $f : \mathsf{D} \to \mathsf{C}$ *be harmonic such that* T_f *is compact. Then* $f(z) = 0$ *for all* $z \in \mathsf{D}$.

Recall that, according to Corollary 2.4, for many f, we can replace T_f by T_g up to compact perturbations where g is harmonic. (In the terminology of Corollary 2.4, $g = f_h$.)

LEMMA 4.4. *Let* $f, g : \overline{\mathsf{D}} \to \mathsf{C}$ *satisfy* (2.1) *and assume that* f *and* g *are continuous on* $\overline{\mathsf{D}} \setminus \rho\mathsf{D}$ *for some* $0 < \rho < 1$. *Then* $T_f T_g - T_{fg}$ *is compact.*

PROOF. In view of Proposition 2.3 and Lemma 2.5 it suffices to assume that $f(z) = \alpha z^n$ and $g(z) = \beta z^m$ for some $\alpha, \beta \in \mathsf{C}$ and $m, n \in \mathsf{Z}$. If $h \in Hv(\mathsf{D})$ is such that $h(z) = \sum_{k \ge |m|+|n|} \alpha_k z^k$ for some α_k then $T_f T_g h - T_{fg} h = 0$. This means that $T_f T_g - T_{fg}$ has finite rank and hence is compact.

THEOREM 4.5. *Let* $f : \overline{\mathsf{D}} \to \mathsf{C}$ *satisfy* (2.1) *and assume that* f *is continuous on* $\overline{\mathsf{D}} \setminus \rho\mathsf{D}$ *for some* $0 < \rho < 1$. *Then the essential spectrum of* T_f *is equal to* $f(\partial\mathsf{D})$. *Moreover there are constants* $c, d > 0$ *(independent of* f*) such that*

$$\begin{aligned} cM_\infty(f,1) &\le \inf\{\|T_f + K\| : K : Hv(\mathsf{D}) \to Hv(\mathsf{D}) \textit{ linear, compact}\} \\ &\le dM_\infty(f,1) \end{aligned}$$

PROOF. Let

$$\mathscr{B} = \{T : Hv(\mathsf{D}) \to Hv(\mathsf{D}) : T \text{ linear and bounded}\}$$

and $\mathscr{K} = \{K \in \mathscr{B} : K \text{ compact}\}$. Then, by Lemma 4.4, the algebra $\mathscr{A}_f$ generated by $T_f + \mathscr{K}$ in $\mathscr{B}/\mathscr{K}$ is commutative. By Theorem 4.2 and Corollary 2.4 its norm is equivalent to $M_\infty(\cdot, 1)$. Hence $\mathscr{A}_f$ is a function algebra and the spectrum of $T_f + \mathscr{K}$ in $\mathscr{A}_f$ is equal to $f(\partial\mathsf{D})$.

REFERENCES

1. Lusky, W., *On the isomorphism classes of weighted spaces of harmonic and holomorphic functions*, Studia Math. 175 (1) (2006), 19–45.

2. Lusky, W., and Taskinen, J., *Bounded holomorphic projections for exponentially decreasing weights*, to appear in J. Funct. Spaces Appl.
3. Shields, A. L., and Williams, D. L., *Bounded projections, duality and multipliers in spaces of analytic functions*, Trans. Amer. Math. Soc. 162 (1971), 287–302.

FAC. FOR INF. AND APPL. MATH.
UNIVERSITY OF YEREVAN
ALEK MANUKIAN 1
YEREVAN 25
ARMENIA
E-mail: anahit@ysu.am

INST. FOR MATH.
UNIVERSITY OF PADERBORN
WARBURGER STR. 100
D-33098 PADERBORN
GERMANY
E-mail: lusky@uni-paderborn.de