ON SPECTRA AND BROWN'S SPECTRAL MEASURES OF ELEMENTS IN FREE PRODUCTS OF MATRIX ALGEBRAS

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Abstract

We compute spectra and Brown measures of some non self-adjoint operators in $(M_2(\mathsf{C}), \frac{1}{2}\operatorname{Tr}) *$ $(M_2(\mathsf{C}), \frac{1}{2}\operatorname{Tr})$, the reduced free product von Neumann algebra of $M_2(\mathsf{C})$ with $M_2(\mathsf{C})$. Examples include AB and $A + B$, where A and B are matrices in $(M_2(\mathsf{C}), \frac{1}{2}\operatorname{Tr}) * 1$ and $1 * (M_2(\mathsf{C}), \frac{1}{2}\operatorname{Tr})$, respectively. We prove that AB is an R-diagonal operator (in the sense of Nica and Speicher [12]) if and only if $\operatorname{Tr}(A) = \operatorname{Tr}(B) = 0$. We show that if $X = AB$ or $X = A + B$ and A, B are not scalar matrices, then the Brown measure of X is not concentrated on a single point. By a theorem of Haagerup and Schultz [9], we obtain that if $X = AB$ or $X = A + B$ and $X \neq \lambda 1$, then X has a nontrivial hyperinvariant subspace affiliated with $(M_2(\mathsf{C}), \frac{1}{2}\operatorname{Tr}) * (M_2(\mathsf{C}), \frac{1}{2}\operatorname{Tr})$.

1. Introduction

In 1983, L. G. Brown [2] introduced a spectral distribution measure for non-normal elements in a finite von Neumann algebra with respect to a fixed normal faithful tracial state, which is called the Brown measure of the operator. Recently, U. Haagerup and H. Schultz [9] proved a remarkable result which states that if the support of Brown measure of an operator in a type II_1 factor contains more than two points, then the operator has a non-trivial hyperinvariant subspace affiliated with the type II_1 factor. In general cases, the computation of Brown measures of non-normal operators are nontrivial. The first essential result was given by Haagerup and F. Larsen. In [8], Haagerup and Larsen computed the spectrum and Brown measure of R-diagonal operators in a finite von Neumann algebra, in terms of the distribution of its radial part. Brown measures of some non-normal and non-R-diagonal operators, examples include $u_n + u_\infty$, where u_n and u_∞ are the generators of Z_n and Z respectively, in the free product $\mathsf{Z}_n * \mathsf{Z}$, and elements of the form $S_\alpha + i S_\beta$, where S_α and S_β are free semi-circular elements of variance α and β, are computed by P. Biane and F. Lehner in [1]. The purpose of this paper is to compute the spectra and Brown measures of some non hermitian operators in $(M_2(\mathsf{C}), \frac{1}{2}\operatorname{Tr}) * (M_2(\mathsf{C}), \frac{1}{2}\operatorname{Tr})$, the reduced free product von Neumann algebra of $M_2(\mathsf{C})$ with $M_2(\mathsf{C})$ (cf [3]). Examples

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include AB and $A + B$, where A and B are matrices in $(M_2(\mathsf{C}), \frac{1}{2}\operatorname{Tr}) * 1$ and $1 * (M_2(\mathsf{C}), \frac{1}{2}\operatorname{Tr})$, respectively. This paper is organized as follows.

In section 2 we recall some preliminary facts about Brown measures, R-diagonal operators, Haagerup and Larsen's result on Brown measures of R-diagonal operators and some notation used in this paper. In section 3, we provide some results on the spectra and spectral radius of operators in $M_2(\mathsf{C}) * M_2(\mathsf{C})$, the universal free product C*-algebra of $M_2(\mathsf{C})$ with $M_2(\mathsf{C})$. Firstly we compute the spectral radius of AB for two normal matrices $A \in M_2(\mathsf{C}) * 1$ and $B \in 1 * M_2(\mathsf{C})$ relative to $M_2(\mathsf{C}) * M_2(\mathsf{C})$. As a corollary, we also get the spectrum radius of AB for normal matrices $A \in (M_2(\mathsf{C}), \frac{1}{2}\operatorname{Tr}) * 1$ and $B \in 1 * (M_2(\mathsf{C}), \frac{1}{2}\operatorname{Tr})$, relative to the reduced free product von Neumann algebra of $M_2(\mathsf{C})$ with $M_2(\mathsf{C})$. Then we obtain the following result: Let A, B be matrices in $M_2(\mathsf{C}) * 1$ and $1 * M_2(\mathsf{C})$, respectively, such that $\operatorname{Tr}(A) = \operatorname{Tr}(B) = 0$. Then $\sigma(AB)$, the spectrum of AB, relative to $M_2(\mathsf{C}) * M_2(\mathsf{C})$, is the closure of the annulus centered at 0 with inner radius $\|A^{-1}\|^{-1}\|B^{-1}\|^{-1}$ and outer radius $\|A\|\|B\|$, where we use the convention $\infty^{-1} = 0$ and if A is not invertible then $\|A^{-1}\| := \infty$.

In section 4 we prove that AB is an R-diagonal operator if and only if $\operatorname{Tr}(A) = \operatorname{Tr}(B) = 0$, where $A \in (M_2(\mathsf{C}), \frac{1}{2}\operatorname{Tr}) * 1$ and $B \in 1 * (M_2(\mathsf{C}), \frac{1}{2}\operatorname{Tr})$. As a corollary, we explicitly compute the spectrum and Brown measure of AB $(\operatorname{Tr}(A) = \operatorname{Tr}(B) = 0)$ in terms of S-transform of A^*A and B^*B.

In section 5, we develop algebraic techniques used in [4]. Let $X \in 1 * (M_2(\mathsf{C}), \frac{1}{2}\operatorname{Tr}) * 1$. With respect to the matrix units of $(M_2(\mathsf{C}), \frac{1}{2}\operatorname{Tr}) * 1$, $X = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$. By [4], $(M_2(\mathsf{C}), \frac{1}{2}\operatorname{Tr}) * (M_2(\mathsf{C}), \frac{1}{2}\operatorname{Tr}) \cong L(\mathsf{F}_3) \otimes M_2(\mathsf{C})$. So $x_1, x_2, x_3, x_4 \in L(\mathsf{F}_3)$. In section 5, we find $*$-free generators h, u, v of $L(\mathsf{F}_3)$ (different from the free generators given in [4]) so that we may explicitly write out x_1, x_2, x_3, x_4 in terms of h, u, v.

In section 6, we compute miscellaneous examples of Brown measures of operators $A + B$ and AB, where $A \in (M_2(\mathsf{C}), \frac{1}{2}\operatorname{Tr}) * 1$ and $B \in 1 * (M_2(\mathsf{C}), \frac{1}{2}\operatorname{Tr})$. As a corollary, we show that $A + B$ is an R-diagonal operator if and only if $A + B = 0$.

In section 7, we prove the following result: Let $A \in (M_2(\mathsf{C}), \frac{1}{2}\operatorname{Tr}) * 1$ and $B \in 1 * (M_2(\mathsf{C}), \frac{1}{2}\operatorname{Tr})$. if $X = A + B$ or $X = AB$ and A, B are not scalar matrices, then the Brown measure of X is not concentrated on a single point. As a corollary of Theorem 7.1 of [9], we prove that if $X = A + B$ or $X = AB$ and $X \neq \lambda 1$, then X has a nontrivial hyperinvariant subspace affiliated with $(M_2(\mathsf{C}), \frac{1}{2}\operatorname{Tr}) * (M_2(\mathsf{C}), \frac{1}{2}\operatorname{Tr})$.

Many concrete examples of spectra and Brown measures are given in this paper. For some interesting applications, we refer to [5].

2. Preliminaries

2.1. *Fuglede-Kadison determinant and Brown's spectral measure*

Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful normal tracial state τ. The *Fuglede-Kadison determinant* [6], $\Delta : \mathcal{M} \to [0, \infty[$, is given by

$$\Delta(T) = \exp\{\tau(\log|T|)\}, \qquad T \in \mathcal{M},$$

with $\exp\{-\infty\} := 0$. For an arbitrary element T in $\mathcal{M}$ the function $\lambda \to \log \Delta(T - \lambda 1)$ is subharmonic on $\mathbf{C}$, and its Laplacian

$$d\mu_T(\lambda) := \frac{1}{2\pi} \nabla^2 \log \Delta(T - \lambda 1),$$

in the distribution sense, defines a probability measure μ_T on $\mathbf{C}$, called the *Brown's measure*[2] of T. From the definition, Brown measure μ_T only depends on the joint distribution of T and T^*.

If T is normal, μ_T is the trace τ composed with the spectral projections of T. If $\mathcal{M} = M_n(\mathbf{C})$ and $\tau = \frac{1}{n} \operatorname{Tr}$ is the normalized trace on $M_n(\mathbf{C})$, then μ_T is the normalized counting measure $\frac{1}{n} (\delta_{\lambda_1} + \delta_{\lambda_2} \cdots + \delta_{\lambda_n})$, where $\lambda_1, \lambda_2 \cdots, \lambda_n$ are the eigenvalues of T repeated according to root multiplicity.

The Brown measure has the following properties (see [2], [10]): μ_T is the unique compactly supported measure on $\mathbf{C}$ such that $\log \Delta(T - \lambda 1) = \int_{\mathbf{C}} \log|z - \lambda| d\mu_T(z)$ for all $\lambda \in \mathbf{C}$. The support of μ_T is contained in $\sigma(T)$, the spectrum of T. $\mu_{ST} = \mu_{TS}$ for arbitrary S, T in $\mathcal{M}$, and if $f(z)$ is analytic in a neighborhood of $\sigma(T)$, $\mu_{f(T)} = (\mu_T)_f$, the push-forward measure of μ_T under the map $\lambda \to f(\lambda)$. If $E \in \mathcal{M}$ is a projection such that $E \in \operatorname{Lat} T$, then with respect to $E, I - E$ we can write

$$T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix},$$

where $A = ETE$ and $C = (I - E)T(I - E)$ are elements of $\mathcal{M}_1 = E\mathcal{M}E$ and $\mathcal{M}_2 = (I - E)\mathcal{M}(I - E)$, respectively. Let μ_A and μ_C be the Brown measures of A and C computed relative to $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively. Then $\mu_T = \alpha\mu_A + (1 - \alpha)\mu_C$, where $\alpha = \tau(E)$.

For a generalization of Brown measures of sets of commuting operators in a type II_1 factor, we refer to [15].

2.2. R-*diagonal operators*

In 1995, A. Nica and S. Speicher [12] introduced the class of R-diagonal operators in non-commutative probability spaces. Recall that an operator T

in a non-commutative probability space is an R-diagonal operator if the R-transform $R_{\mu(T,T^*)}$ of the joint distribution $\mu(T, T^*)$ of T, T^* is of the form

$$R_{\mu(T,T^*)}(z_1, z_2) = \sum_{n=1}^{\infty} \alpha_n (z_1 z_2)^n + \sum_{n=1}^{\infty} \alpha_n (z_2 z_1)^n.$$

Nica and Speicher [12] proved that T is an R-diagonal operator if and only if T has same $*$-distribution as product UH, where U and H are $*$-free random variables in some tracial non commutative probability space, U is a Haar unitary operator and H is positive. If T is an R-diagonal operator, then the $*$-distribution of T is uniquely determined by the distribution of $T^*T = |T|^2$. If T is an R-diagonal operator and S is $*$-free with T, then both ST and TS are R-diagonal operators (see [12]). If T is an R-diagonal operator and $n \in \mathsf{N}$, then T^n is also an R-diagonal operator (see [8], [11]). For other important properties of R-diagonal operators, we refer to [8], [11], [12], [13].

2.3. Brown measures of R-diagonal operators

In [8], Haagerup and Larson explicitly computed the Brown measures of R-diagonal operators in a finite von Neumann algebra.

THEOREM 2.1 (Theorem 4.4 of [8]). *Let U, H be $*$-free random variables in a noncommutative probability space $(\mathcal{M}, \tau)$, with U a Haar unitary operator and H a positive operator such that the distribution μ_H of H is not a Dirac measure. Then the Brown measure μ_{UH} of UH can be computed as the following.*

(1) *μ_{UH} is rotation invariant and its support is the annulus with inner radius $\|H^{-1}\|_2^{-1}$ and outer radius $\|H\|_2$.*

(2) *$\mu_{UH}(\{0\}) = \mu_H(\{0\})$ and for $t \in \,]\mu_H(\{0\}), 1]$,*

$$\mu_{UH}\big(\mathsf{B}\big(0, (S_{\mu_{H^2}}(t-1))^{-1/2}\big)\big) = t,$$

where $S_{\mu_{H^2}}$ is the S-transform of H^2 and $\mathsf{B}(0, r)$ is the open disc with center 0 and radius r;

(3) *μ_{UH} is the only rotation invariant symmetric probability measure satisfying (2).*

Furthermore, if H is invertible, then $\sigma(UH) = \operatorname{supp} \mu_{UH}$; if H is not invertible, then $\sigma(UH) = \overline{\mathsf{B}(0, \|H\|_2)}$.

2.4. Some Notation

The following notation will be used in the rest of the paper

- $(\mathcal{M}, \tau) = (M_2(\mathbf{C}), \frac{1}{2}\operatorname{Tr}) * (M_2(\mathbf{C}), \frac{1}{2}\operatorname{Tr})$ denotes the reduced free product von Neumann algebra of $M_2(\mathbf{C})$ with $M_2(\mathbf{C})$ with the unique tracial state τ;

- $M_2(\mathbf{C})_{(1)} := (M_2(\mathbf{C}), \frac{1}{2}\operatorname{Tr}) * 1$ and $M_2(\mathbf{C})_{(2)} := 1 * (M_2(\mathbf{C}), \frac{1}{2}\operatorname{Tr})$;

- $\{E_{ij}\}_{i,j=1,2}, \{F_{ij}\}_{i,j=1,2}$ are matrix units of $M_2(\mathbf{C})_{(1)}$ and $M_2(\mathbf{C})_{(2)}$, respectively;

- $P = E_{11}$ and $Q = F_{11}$;

- $\mathcal{M} \cong \mathcal{N} \otimes M_2(\mathbf{C})_{(1)} \cong \mathcal{N} \otimes M_2(\mathbf{C})_{(2)}$. For $X \in \mathcal{M}$, $X = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}_{(1)} = \begin{pmatrix} x_1' & x_2' \\ x_3' & x_4' \end{pmatrix}_{(2)}$ means the decomposition is with respect to above matrix units of $M_2(\mathbf{C})_{(1)}$ and $M_2(\mathbf{C})_{(2)}$, respectively.

- $W_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{(1)}$, $W_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{(1)}$, $W_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}_{(1)}$, $W_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{(1)}$;

- $V_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{(2)}$, $V_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{(2)}$, $V_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}_{(2)}$, $V_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{(2)}$;

- $A, A_1, \ldots, A_n$ denote elements in $M_2(\mathbf{C})_{(1)}$, $B, B_1, \ldots, B_n$ denote elements in $M_2(\mathbf{C})_{(2)}$, X, Y, Z denote general elements in $\mathcal{M}$;

- An element X in $\mathcal{M}$ is called *centered* if $\tau(X) = 0$.

We end this section with the following lemma. The proof is an easy exercise.

LEMMA 2.2. $V_1 M_2(\mathbf{C})_{(1)} V_1$ *is free with* $M_2(\mathbf{C})_{(1)}$.

3. Spectra of elements in the universal free product of $M_2(\mathbf{C})$ and $M_2(\mathbf{C})$

Let $\mathring{A} = M_2(\mathbf{C}) * M_2(\mathbf{C})$ denote the universal free product C*-algebra of $M_2(\mathbf{C})$ with $M_2(\mathbf{C})$. Then there is a * homomorphism π from $\mathring{A}$ onto the reduced free product C*-algebra of $M_2(\mathbf{C})$ and $M_2(\mathbf{C})$, the C*-subalgebra generated by $M_2(\mathbf{C})_{(1)}$ and $M_2(\mathbf{C})_{(2)}$ in $\mathcal{M}$. Since $\sigma(\pi(a)) \subseteq \sigma(a)$ for $a \in \mathring{A}$, it is useful to obtain some information of spectrum of AB, where $A \in M_2(\mathbf{C}) * 1$ and $B \in 1 * M_2(\mathbf{C})$.

3.1. "Free product" of normal matrices

LEMMA 3.1. *Let* $A \in M_2(\mathsf{C}) * 1$ *and* $B \in 1 * M_2(\mathsf{C})$ *be normal matrices. Then* $r(AB) = \|A\| \cdot \|B\|$ *relative to* $\AA$.

PROOF. $r(AB) \leq \|AB\| \leq \|A\| \cdot \|B\|$. We need only to prove $r(AB) \geq \|A\| \cdot \|B\|$. Since A is a normal matrix, there is a unitary matrix $U_1 \in M_2(\mathsf{C}) * 1$ such that $U_1 A U_1^* = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \beta_1 \end{pmatrix}$ and $\|\alpha_1\| = \|A\|$. Similarly, there is a unitary matrix $U_2 \in 1 * M_2(\mathsf{C})$ such that $U_2 B U_2^* = \begin{pmatrix} \alpha_2 & 0 \\ 0 & \beta_2 \end{pmatrix}$ and $\|\alpha_2\| = \|B\|$. Let $\pi_1(X) = U_1 X U_1^*$ and $\pi_2(Y) = U_2 Y U_2^*$ be $*$-representations of $M_1(\mathsf{C}) * 1$ and $1 * M_2(\mathsf{C})$ to $M_2(\mathsf{C})$, respectively. Then there is a $*$-representation $\pi = \pi_1 * \pi_2$ from $\AA$ to $M_2(\mathsf{C})$ and $\pi(AB) = \begin{pmatrix} \alpha_1 \alpha_2 & 0 \\ 0 & \beta_1 \beta_2 \end{pmatrix}$. Therefore, $\alpha_1 \alpha_2 \in \sigma(\pi(AB)) \subseteq \sigma(AB)$. So $r(AB) \geq |\alpha_1 \alpha_2| = \|A\| \cdot \|B\|$.

COROLLARY 3.2. *Let* $A \in M_2(\mathsf{C})_{(1)}$ *and* $B \in M_2(\mathsf{C})_{(2)}$ *be normal matrices. Then* $r(AB) = \|A\| \cdot \|B\|$ *relative to* $\mathcal{M}$.

PROOF. We may assume that A and B are diagonal matrices. Then we can treat AB as an operator in the full free product $C^*(\mathsf{Z}_2 * \mathsf{Z}_2)$. Same technique used in the previous lemma gives the corollary.

3.2. "Free product" of non-normal matrices

It is well-known that two matrices X, Y in $M_2(\mathsf{C})$ are unitarily equivalent if and only if $\mathrm{Tr}(X) = \mathrm{Tr}(Y)$, $\mathrm{Tr}(X^2) = \mathrm{Tr}(Y^2)$ and $\mathrm{Tr}(X^*X) = \mathrm{Tr}(Y^*Y)$. The proof of the following lemma now is an easy exercise.

LEMMA 3.3. *If* $A \in M_2(\mathsf{C})$ *and* $\mathrm{Tr}(A) = 0$, *then* A *is unitarily equivalent to a matrix of form* $\begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}$, *where* α, β *are complex numbers.*

REMARK 3.4. We have the following useful observations:

- $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \beta \\ \alpha & 0 \end{pmatrix}$.

- $\begin{pmatrix} 1 & 0 \\ 0 & e^{i(\theta_1 - \theta_2)/2} \end{pmatrix} \begin{pmatrix} 0 & |\alpha| e^{i\theta_1} \\ |\beta| e^{i\theta_2} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{-i(\theta_1 - \theta_2)/2} \end{pmatrix} =$

  $e^{i(\theta_1 + \theta_2)/2} \begin{pmatrix} 0 & |\alpha| \\ |\beta| & 0 \end{pmatrix}$.

LEMMA 3.5. *Let* $A \in M_2(\mathsf{C}) * 1$ *and* $B \in 1 * M_2(\mathsf{C})$ *be matrices such that* $\mathrm{Tr}(A) = \mathrm{Tr}(B) = 0$. *Then* $r(AB) = \|A\| \cdot \|B\|$ *relative to* $\AA$.

PROOF. We need only to prove $r(AB) \geq \|A\| \cdot \|B\|$. By Lemma 3.3 and Remark 3.4, there are unitary matrices U, V in $M_2(\mathbb{C})$ such that $UAU^* = \begin{pmatrix} 0 & \alpha_1 \\ \beta_1 & 0 \end{pmatrix}$ and $VBV^* = \begin{pmatrix} 0 & \alpha_2 \\ \beta_2 & 0 \end{pmatrix}$ and $|\alpha_1| = \|A\|, |\beta_2| = \|B\|$. Let $\pi_1(X) = UXU^*$ and $\pi_2(Y) = VYV^*$ be $*$-representations of $M_2(\mathbb{C})*1$ and $1*M_2(\mathbb{C})$ to $M_2(\mathbb{C})$, respectively. Let $\pi = \pi_1*\pi_2$ be the induced $*$-representation of $\mathring{A}$ to $M_2(\mathbb{C})$. Then $\sigma(AB) \supseteq \sigma(\pi(AB)) = \sigma(\pi_1(A)\pi_2(B)) = \{\alpha_1\beta_2, \alpha_2\beta_1\}$. Therefore, $r(AB) \geq |\alpha_1\beta_2| = \|A\| \cdot \|B\|$.

THEOREM 3.6. *Let* $A \in M_2(\mathbb{C}) * 1$ *and* $B \in 1 * M_2(\mathbb{C})$ *be matrices such that* $\mathrm{Tr}(A) = \mathrm{Tr}(B) = 0$. *Then*

$$\sigma(AB) = [\|A^{-1}\|^{-1}\|B^{-1}\|^{-1}, \|A\|\|B\|] \times_p [0, 2\pi],$$

where $\times_p$ *denotes the polar set product* $\{re^{i\theta} : r \in [\|A^{-1}\|^{-1}\|B^{-1}\|^{-1}, \|A\|\|B\|], \theta \in [0, 2\pi]\}$.

PROOF. We will prove the theorem for two cases.

Case 1. Either A or B is not invertible. We may assume that A is not invertible. By $\mathrm{Tr}(A) = 0$, Lemma 3.3 and Remark 3.4, A is unitarily equivalent to $\begin{pmatrix} 0 & \alpha_1 \\ 0 & 0 \end{pmatrix}$. Without loss of generality, we assume that $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in M_2(\mathbb{C}) * 1$. By Lemma 3.3 and Remark 3.4, we may also assume that $B = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix} \in 1 * M_2(\mathbb{C})$ and $\beta \geq \alpha \geq 0$. We need to prove that $\sigma(AB)$ is the closed disc of complex plane with center 0 and radius β. Since A is unitarily equivalent to $e^{i\theta}A$ in $M_2(\mathbb{C}) * 1$, $\sigma(AB)$ is rotation invariant. For $\theta \in [0, 2\pi]$, let $U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. Let $\pi_1(X) = X$ and $\pi_2(Y) = UYU^*$ be $*$-representations of $M_2(\mathbb{C}) * 1$ and $1 * M_2(\mathbb{C})$ to $M_2(\mathbb{C})$, respectively. Let $\pi = \pi_1 * \pi_2$ be the induced $*$-representation of $\mathring{A}$ to $M_2(\mathbb{C})$. Then

$$\pi(AB) = AUBU^* = \begin{pmatrix} -\alpha\sin^2\theta + \beta\cos^2\theta & -(\alpha+\beta)\sin\theta\cos\theta \\ 0 & 0 \end{pmatrix}.$$

So $\sigma(\pi(AB)) = \{-\alpha\sin^2\theta + \beta\cos^2\theta, 0\}$. Since $[0, \beta] \subseteq [-\alpha, \beta] = \{-\alpha\sin^2\theta + \beta\cos^2\theta : \theta \in [0, 2\pi]\}$, $[0, \beta] \subseteq \sigma(AB)$. Since $\sigma(AB)$ is rotation invariant, $\sigma(AB)$ contains the closed disc with center 0 and radius β. By Lemma 3.5, $\sigma(AB)$ is the closed disc of complex plane with center 0 and radius β.

Case 2. Both A and B are invertible. By Lemma 3.3 and Remark 3.4, we may assume that $A = \begin{pmatrix} 0 & 1 \\ \beta_1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ \beta_2 & 0 \end{pmatrix}$ such that $\beta_1, \beta_2 \geq 1$. Then

$A^{-1} = \begin{pmatrix} 0 & \beta_1^{-1} \\ 1 & 0 \end{pmatrix}$ and $B^{-1} = \begin{pmatrix} 0 & \beta_2^{-1} \\ 1 & 0 \end{pmatrix}$. We need to prove that $\sigma(AB) =$ $[1, \beta_1\beta_2] \times_p [0, 2\pi]$. By Lemma 3.5, $r(AB) = \beta_1\beta_2$ and $r((AB)^{-1}) = 1$. This implies that $\sigma(AB) \subseteq [1, \beta_1\beta_2] \times_p [0, 2\pi]$. So we need only to prove $\sigma(AB) \supseteq [1, \beta_1\beta_2] \times_p [0, 2\pi]$.

For $\phi, \psi \in [0, 2\pi]$, let $U = \begin{pmatrix} \cos\psi & e^{i\phi}\sin\psi \\ -\sin\psi & e^{i\phi}\cos\psi \end{pmatrix}$. Then U is a unitary matrix. Let $\pi_1(X) = UXU^*$ and $\pi_2(Y) = Y$ be $*$-representations of $M_2(\mathbf{C}) * 1$ and $1 * M_2(\mathbf{C})$ to $M_2(\mathbf{C})$, respectively. Let $\pi = \pi_1 * \pi_2$ be the induced $*$-representation of $\mathring{A}$ to $M_2(\mathbf{C})$. Then

$$\pi(AB) = \begin{pmatrix} -\beta_1\beta_2 e^{i\phi}\sin^2\psi + \beta_2 e^{-i\phi}\cos^2\psi & * \\ * & \beta_1 e^{i\phi}\cos^2\psi - e^{-i\phi}\sin^2\psi \end{pmatrix}.$$

Let $\lambda_1(\phi, \psi), \lambda_2(\phi, \psi)$ be the eigenvalues of $\pi(AB)$. Then

(3.1) $\lambda_1(\phi, \psi)\lambda_2(\phi, \psi) = \det(\pi(AB)) = \det(A)\det(B) = \beta_1\beta_2,$

(3.2) $\lambda_1(\phi, \psi) + \lambda_2(\phi, \psi)$
$$= (\beta_1 e^{i\phi} + \beta_2 e^{-i\phi})\cos^2\psi - (\beta_1\beta_2 e^{i\phi} + e^{-i\phi})\sin^2\psi.$$

Note that $\sigma(AB) \supseteq \{\lambda_1(\phi, \psi) : \phi, \psi \in [0, 2\pi]\}$. We only need to prove that $\{\lambda_1(\phi, \psi) : \phi, \psi \in [0, 2\pi]\} \supseteq [1, \beta_1\beta_2] \times_p [0, 2\pi]$. For this purpose, we need to show for every $r \in [1, \beta_1\beta_2]$, $\theta \in [0, 2\pi]$, there are $\phi, \psi \in [0, 2\pi]$ such that

(3.3) $re^{i\theta} + \dfrac{\beta_1\beta_2}{r}e^{-i\theta} = (\beta_1 e^{i\phi} + \beta_2 e^{-i\phi})\cos^2\psi - (\beta_1\beta_2 e^{i\phi} + e^{-i\phi})\sin^2\psi.$

Let $\alpha = \cos^2\psi$. Simple computations show that equation (3.3) is equivalent to the following

$\left(r + \frac{\beta_1\beta_2}{r}\right)\cos\theta + i\left(r - \frac{\beta_1\beta_2}{r}\right)\sin\theta = (\alpha(1 + \beta_1)(1 + \beta_2) - (1 + \beta_1\beta_2))\cos\phi$
$$+ i(\alpha(\beta_1 - 1)(\beta_2 + 1) + (1 - \beta_1\beta_2))\sin\phi.$$

Let

$\Omega_1 = \left\{\left(r + \frac{\beta_1\beta_2}{r}\right)\cos\theta + i\left(r - \frac{\beta_1\beta_2}{r}\right)\sin\theta : r \in [1, \beta_1\beta_2], \theta \in [0, 2\pi]\right\},$

$\Omega_2 = \big\{(\alpha(1 + \beta_1)(1 + \beta_2) - (1 + \beta_1\beta_2))\cos\phi$
$\qquad + i(\alpha(\beta_1 - 1)(\beta_2 + 1) + (1 - \beta_1\beta_2))\sin\phi : \alpha \in [0, 1], \phi \in [0, 2\pi]\big\}.$

Now we need only to prove $\Omega_1 = \Omega_2$. Note that Ω_1 is the union of a family of ellipses with center the origin point and semimajor axis and semiminor axis

$\left| r + \frac{\beta_1 \beta_2}{r} \right|$ and $\left| r - \frac{\beta_1 \beta_2}{r} \right|$, $1 \le r \le \beta_1 \beta_2$, respectively. Similarly, Ω_2 is the union of a family of ellipses with center the origin point and semimajor axis and semiminor axis $|\alpha(1+\beta_1)(1+\beta_2) - (1+\beta_1\beta_2)|$ and $|\alpha(\beta_1-1)(\beta_2+1) + (1 - \beta_1\beta_2)|$, $0 \le \alpha \le 1$, respectively. Note that the "largest" ellipse in Ω_1 is with semimajor axis and semiminor axis $|1 + \beta_1\beta_2|$ and $|\beta_1\beta_2 - 1|$, respectively; the "smallest" ellipse in Ω_1 is with semimajor axis and semiminor axis $2\sqrt{|\beta_1\beta_2|}$ and 0, respectively. The "largest" ellipse in Ω_2 is with semimajor axis and semiminor axis $|1 + \beta_1\beta_2|$ and $|\beta_1\beta_2 - 1|$, respectively; the "smallest" ellipse in Ω_2 is with semimajor axis and semiminor axis 0 and $\frac{2\beta_1(\beta_2-1)}{\beta_1+1}$. So both Ω_1 and Ω_2 are the closure of the domain enclosed by the ellipse with center the origin point and semimajor axis and semiminor axis $|1 + \beta_1\beta_2|$ and $|\beta_1\beta_2 - 1|$, respectively. Thus $\Omega_1 = \Omega_2$.

4. R-diagonal operators in $\mathcal{M}$

In this section, we prove the following result. We will use the notation introduced in section 2.4.

THEOREM 4.1. *In $\mathcal{M}$, let $A \in M_2(\mathsf{C})_{(1)}$ and $B \in M_2(\mathsf{C})_{(2)}$. Then AB is an R-diagonal operator if and only if $\tau(A) = \tau(B) = 0$.*

To prove Theorem 4.1, we need the following lemmas.

LEMMA 4.2. $\{W_1, V_1, W_3 V_3\}'' \cong L(\mathsf{Z}_2) * L(\mathsf{Z}_2) * L(\mathsf{Z})$.

PROOF. Let $U = W_3 V_3$. Then U is a Haar unitary operator. We need only to prove that U is * free with the von Neumann subalgebra generated by W_1 and V_1. Let $g_1 g_2 \cdots g_n$ be an alternating product of $\{U^n : n \ne 0\}$ and $\{W_1, V_1, W_1 V_1, V_1 W_1, W_1 V_1 W_1, V_1 W_1 V_1, \ldots\}$. By regrouping, it is an alternating product of $\{W_1, W_1 W_3, W_3^* W_1, W_3^* W_1 W_3, W_3, W_3^*\}$ and $\{V_1, V_3 V_1, V_1 V_3^*, V_3 V_1 V_3^*, V_3, V_3^*\}$. Thus the trace is 0.

LEMMA 4.3. $\begin{pmatrix} 0 & \alpha_1 \\ \beta_1 & 0 \end{pmatrix}_{(1)} \begin{pmatrix} 0 & \alpha_2 \\ \beta_2 & 0 \end{pmatrix}_{(2)}$ *is an R-diagonal operator.*

PROOF. Note that

$$\begin{pmatrix} 0 & \alpha_1 \\ \beta_1 & 0 \end{pmatrix}_{(1)} \begin{pmatrix} 0 & \alpha_2 \\ \beta_2 & 0 \end{pmatrix}_{(2)}$$
$$= \begin{pmatrix} \alpha_1 & 0 \\ 0 & \beta_1 \end{pmatrix}_{(1)} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{(1)} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{(2)} \begin{pmatrix} \beta_2 & 0 \\ 0 & \alpha_2 \end{pmatrix}_{(2)}.$$

By Lemma 4.2 and basic properties of R-diagonal operators given in 2.2, we prove the lemma.

LEMMA 4.4. *With the assumption of Theorem 4.1 and assume AB is an R-diagonal operator and $\tau(A^2) \neq 0$. Then $\tau(B) = 0$.*

PROOF. Since AB is an R-diagonal operator, $\tau(AB) = 0$. Since A, B are ∗-free, $\tau(A)\tau(B) = \tau(AB) = 0$. If $\tau(B) = 0$, then done. Otherwise, assume $\tau(A) = 0$. Then $0 = \tau(ABAB) = \tau(A^2B)\tau(B) = \tau(A^2)(\tau(B))^2$. By the assumption, $\tau(B) = 0$.

LEMMA 4.5. *Let $B \in M_2(\mathsf{C})_{(2)}$ and λ be a complex number. Then $\sigma(E_{12}B) = \sigma(E_{12}(\lambda + B))$.*

PROOF. By Jacobson's theorem,

$$\sigma(E_{12}(\lambda + B)) \cup \{0\} = \sigma(E_{11}E_{12}(\lambda + B)) \cup \{0\}$$
$$= \sigma(E_{12}(\lambda + B)E_{11}) \cup \{0\}$$
$$= \sigma(E_{12}BE_{11}) \cup \{0\} = \sigma(BE_{12}) \cup \{0\}.$$

PROOF OF THEOREM 4.1. If $\tau(A) = \tau(B) = 0$, then by Lemma 3.3 and Lemma 4.3, AB is an R-diagonal operator. Conversely, assume that AB is an R-diagonal operator. Then $0 = \tau(AB) = \tau(A) \cdot \tau(B)$. So either $\tau(A) = 0$ or $\tau(B) = 0$. Without loss of generality, we assume that $\tau(A) = 0$. If $\tau(A^2) \neq 0$, then $\tau(B) = 0$ by Lemma 4.4. If $\tau(A^2) = 0$, then A is unitary equivalent to αE_{12}. We may assume that $A = E_{12}$. By Theorem 2.1, if $E_{12}B$ is an R-diagonal operator, then $(r(E_{12}B))^2 = \tau(B^*E_{21}E_{12}B) = \tau(E_{21}E_{12}BB^*) = \|E_{12}\|_2^2 \cdot \|B\|_2^2$. Note that $E_{12}(B - \tau(B))$ is an R-diagonal operator, $(r(E_{12}(B - \tau(B)))^2 = \|E_{12}\|_2^2 \cdot \|B - \tau(B)\|_2^2$. By Lemma 4.5, $\|B\|_2^2 = \|B - \tau(B)\|_2^2$. This implies that $\tau(B) = 0$. This ends the proof.

Combining Theorem 4.1, Theorem 2.1 and the S-transform of Voiculescu (see [16], [17]), we have the following theorem (It is interesting to compare the following theorem and Theorem 3.6).

THEOREM 4.6. *Let $A \in M_2(\mathsf{C})_{(1)}$, $B \in M_2(\mathsf{C})_{(2)}$ and $\tau(A) = \tau(B) = 0$. Then*

(1) *μ_{AB} is rotation invariant;*

(2) *$\sigma(AB) = \operatorname{supp}\mu_{AB} = [\|A^{-1}\|_2^{-1}\|B^{-1}\|_2^{-1}, \|A\|_2\|B\|_2] \times_p [0, 2\pi]$;*

(3) *$\mu_{AB}(\{0\}) = \max\{\mu_{A^*A}(\{0\}), \mu_{B^*B}(\{0\})\}$ and*

$$\mu_{AB}(\mathsf{B}(0, (S_{\mu_{A^*A}} S_{\mu_{B^*B}}(t - 1))^{-1/2})) = t, \qquad for \quad t \in [\mu_{AB}(\{0\}), 1].$$

5. Algebraic techniques

For $X \in \mathcal{M}$, define

$$\Phi(X) = \begin{pmatrix} E_{11}XE_{11} & E_{11}XE_{21} \\ E_{12}XE_{11} & E_{12}XE_{21} \end{pmatrix}.$$

Then Φ is a $*$-isomorphism from $\mathcal{M}$ onto $E_{11}\mathcal{M}E_{11} \otimes M_2(\mathsf{C})_{(1)}$. We will identify $\mathcal{M}$ with $E_{11}\mathcal{M}E_{11} \otimes M_2(\mathsf{C})_{(1)}$ by the canonical isomorphism Φ. In [4], K. Dykema proved that $E_{11}\mathcal{M}E_{11} \cong L(\mathsf{F}_3)$. For $B \in M_2(\mathsf{C})_{(2)}$, we may write

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

with respect to matrix units in $\mathcal{M}_1$. Then $b_{ij} \in L(\mathsf{F}_3)$. In this section, we will develop the algebraic techniques used in [4]. Combining the matrix techniques, we may explicitly express b_{ij} in terms of free generators of $L(\mathsf{F}_3)$.

Let $\Lambda\{W_1, V_1\}$ be the set of words generated by W_1, V_1. Note that $W_1^2 = V_1^2 = 1$ and $\tau(W_1) = \tau(V_1) = 0$. The following observation is crucial in [4]. The proof is an easy exercise.

LEMMA 5.1. $\tau(g_1 g_2 \cdots g_n) = 0$ for an alternating product of $\Lambda\{W_1, V_1\} \setminus \{1, W_1\}$ and $\{E_{12}, E_{21}\}$.

Recall that $P = E_{11}$ and $Q = F_{11}$. Let W be the "polar" part of $(1-P)QP$ and $U = E_{12}W$. The following corollary is a special case of Theorem 3.5 of [4].

COROLLARY 5.2. U is a Haar unitary operator in $\mathcal{M}_P = P\mathcal{M}P$ and U, PQP are $*$-free in $\mathcal{M}_P$.

With the canonical identification of $\mathcal{M}$ with $\mathcal{M}_P \otimes M_2(\mathsf{C})_{(1)}$,

$$Q = \begin{pmatrix} PQP & \sqrt{PQP - (PQP)^2}U \\ U^*\sqrt{PQP - (PQP)^2} & U^*(1 - PQP)U \end{pmatrix}.$$

By [16], the distribution of PQP (relative to $\mathcal{M}_P$) is non-atomic and the density function is

(5.1) $$\rho(t) = \frac{1}{\pi} \frac{1}{\sqrt{\frac{1}{4} - \left(\frac{1}{2} - t\right)^2}}, \qquad 0 \le t \le 1.$$

By Corollary 5.2, the von Neumann subalgebra $\mathcal{M}_1$ generated by $M_2(\mathsf{C})_{(1)}$ and Q is $*$-isomorphic to $L(\mathsf{F}_2) \otimes M_2(\mathsf{C})_{(1)}$. Since $\mathcal{M}_1$ is also $*$-isomorphic to

$M_2(\mathbf{C}) * L(\mathbf{Z}_2)$, $M_2(\mathbf{C}) * L(\mathbf{Z}_2) \cong L(\mathbf{F}_2) \otimes M_2(\mathbf{C})$, which is proved by Dykema in [4].

Since $V_1 = 2Q - 1$,

$$V_1 = \begin{pmatrix} 2PQP - 1 & 2\sqrt{PQP - (PQP)^2}U \\ 2U^*\sqrt{PQP - (PQP)^2} & U^*(1 - 2PQP)U \end{pmatrix}.$$

Simple computation shows that the density function of $2PQP - 1$ is

$$\sigma(t) = \frac{1}{\pi}\frac{1}{\sqrt{1-t^2}}, \qquad -1 \le t \le 1.$$

Let $H = 2PQP - 1$, then

$$V_1 = \begin{pmatrix} H & \sqrt{1-H^2}U \\ U^*\sqrt{1-H^2} & -U^*HU \end{pmatrix}.$$

Let $H = V|H|$ be the polar decomposition of H. Since H is a symmetric selfadjoint operator, $V^2 = 1$ and V is independent with the von Neumann algebra generated by $|H|$ in the classical probability sense. Let $h = |H|, u = VU, v = UV$. Then u, v are Haar unitary operators and the distribution of h relative to $\mathcal{M}_P$ is non-atomic.

LEMMA 5.3. *h, u, v are $*$-free.*

PROOF. Let $g_1 g_2 \cdots g_n$ be an alternating product of elements of $\mathfrak{S} = \{|H|\}'' \ominus \mathbf{C}I, \{(VU)^n : n \ne 0\}, \{(UV)^n : n \ne 0\}$. By regrouping, it is an alternating product of elements of $\{\mathfrak{S}, V, V\mathfrak{S}, \mathfrak{S}V, V\mathfrak{S}V\}$ and $\{U^n : n \ne 0\}$. Since H and U are $*$-free, $\{\mathfrak{S}, V, V\mathfrak{S}, \mathfrak{S}V, V\mathfrak{S}V\}$ and $\{U^n : n \ne 0\}$ are free. Since V and $\mathfrak{S}$ are independent, $\tau(VS) = \tau(SV) = 0$ for $S \in \mathfrak{S}$. This implies that $\tau(g_1 g_2 \cdots g_n) = 0$.

By simple computations, we have the following:

$$(5.2) \quad V_1 E_{11} V_1 = \begin{pmatrix} h^2 & \sqrt{1-h^2}hu \\ u^*h\sqrt{1-h^2} & u^*(1-h^2)u \end{pmatrix},$$

$$(5.3) \quad V_1 E_{12} V_1 = \begin{pmatrix} HU^*\sqrt{1-H^2} & -HU^*HU \\ U^*\sqrt{1-H^2}U^*\sqrt{1-H^2} & -U^*\sqrt{1-H^2}U^*HU \end{pmatrix}$$

$$(5.4) \qquad = \begin{pmatrix} hv^*\sqrt{1-h^2} & -hv^*hu \\ u^*\sqrt{1-h^2}v^*\sqrt{1-h^2} & -u^*\sqrt{1-h^2}v^*hu \end{pmatrix}.$$

By Lemma 2.2, $\mathcal{M} \cong M_2(\mathbb{C})_{(1)} * (V_1 M_2(\mathbb{C})_{(1)} V_1) \cong \mathcal{M}_P \otimes M_2(\mathbb{C})_{(1)}$. With this isomorphism, $\mathcal{M}_P$ is the von Neumann algebra generated by h, u and v by (5.2) and (5.4). So $\mathcal{M}_P \cong L(\mathsf{F}_3)$. By simple computations, we have

$$V_1 \begin{pmatrix} \alpha & \beta \\ \gamma & \sigma \end{pmatrix}_{(1)} V_1 = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix},$$

where

$$b_{11} = \sigma + (\alpha - \sigma)h^2 + \gamma\sqrt{1 - h^2}\,vh + \beta h v^*\sqrt{1 - h^2},$$

$$b_{12} = (\alpha - \sigma)h\sqrt{1 - h^2}\,u + \gamma\sqrt{1 - h^2}\,v\sqrt{1 - h^2}\,u - \beta h v^* h u,$$

$$b_{21} = (\alpha - \sigma)u^* h\sqrt{1 - h^2} - \gamma u^* h v h + \beta u^*\sqrt{1 - h^2}\,v^*\sqrt{1 - h^2},$$

$$b_{22} = \alpha + (\sigma - \alpha)u^* h^2 u - \gamma u^* h v\sqrt{1 - h^2}\,u - \beta u^*\sqrt{1 - h^2}\,v^* h u.$$

THEOREM 5.4. $\mathcal{M} \cong L(\mathsf{F}_3) \otimes M_2(\mathbb{C})_{(1)}$; furthermore, let $B = \begin{pmatrix} \alpha & \beta \\ \gamma & \sigma \end{pmatrix}_{(2)}$ in $M_2(\mathbb{C})_{(2)}$, then with respect to the matrix units $\{E_{ij}\}_{i,j=1,2} \subset M_2(\mathbb{C})_{(1)}$, $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}_{(1)}$, where b_{ij} are given as above.

EXAMPLE 5.5. In Theorem 5.4, let $\beta = \gamma = 0$. Then we have

$$\begin{pmatrix} \alpha & 0 \\ 0 & \sigma \end{pmatrix}_{(2)} = \begin{pmatrix} \sigma + (\alpha - \sigma)h^2 & (\alpha - \sigma)h\sqrt{1 - h^2}\,u \\ (\alpha - \sigma)u^* h\sqrt{1 - h^2} & \alpha + (\sigma - \alpha)u^* h^2 u \end{pmatrix}_{(1)}.$$

EXAMPLE 5.6. In Theorem 5.4, let $\alpha = \sigma$ and $\gamma = 0$. Then we have

$$\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix}_{(2)} = \begin{pmatrix} \alpha + \beta h v^*\sqrt{1 - h^2} & -\beta h v^* h u \\ \beta u^*\sqrt{1 - h^2}\,v^*\sqrt{1 - h^2} & \alpha - \beta u^*\sqrt{1 - h^2}\,v^* h u \end{pmatrix}_{(1)}.$$

REMARK 5.7. By equation (5.2), the distribution of h^2 is the distribution of $E_{11} V_1 E_{11} V_1 E_{11}$ relative to M_P. So the distribution of h^2 is same as the distribution of PQP (relative to $\mathcal{M}_P$). By [16], the distribution of PQP (relative to $\mathcal{M}_P$) is non-atomic and the density function is

$$\rho(t) = \frac{1}{\pi}\frac{1}{\sqrt{\frac{1}{4} - \left(\frac{1}{2} - t\right)^2}}, \qquad 0 \le t \le 1.$$

6. Miscellaneous examples

EXAMPLE 6.1. We compute the Brown spectrum of $\alpha E_{12} + \beta F_{12}$. Let $F_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}_{(2)} = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}_{(1)}$. Then

$$(\alpha E_{12} + \beta F_{12})^2 = \alpha\beta(E_{12}F_{12} + F_{12}E_{12}) = \alpha\beta(E_{12} + F_{12})^2$$

$$= \alpha\beta \begin{pmatrix} b_3 & b_1 + b_4 \\ 0 & b_3 \end{pmatrix}_{(1)}.$$

So $\mu_{(\alpha E_{12} + \beta F_{12})^2} = \mu_{\alpha\beta b_3}$. By equation (5.3), the distribution of b_3 is same as the distribution of $(U^*\sqrt{1 - H^2})^2$. Since $U^*\sqrt{1 - H^2}$ is an R-diagonal operator, $(U^*\sqrt{1 - H^2})^2$ is also an R-diagonal operator. Since the distribution of $\alpha E_{12} + \beta F_{12}$ is rotation invariant, $\mu_{\alpha E_{12} + \beta F_{12}} = \mu_{\sqrt{|\alpha\beta|}b}$, where $b = U^*\sqrt{1 - H^2}$. Simple computations show that (or by Proposition 5.10 and Corollary 5.11 of [5])

$$d\mu_b(z) = \frac{1}{\pi} \frac{1}{(1 - r^2)^2} dr d\theta, \qquad 0 \le r \le \frac{1}{\sqrt{2}}.$$

Hence

$$d\mu_{\alpha E_{12} + \beta F_{12}}(z) = d\mu_{\sqrt{|\alpha\beta|}b}(z) = \frac{1}{\pi} \frac{|\alpha\beta|^{3/2}}{(|\alpha\beta| - r^2)^2} dr d\theta, \qquad 0 \le r \le \sqrt{\frac{|\alpha\beta|}{2}}.$$

and

$$\sigma(\alpha E_{12} + \beta F_{12}) = \mathsf{B}\left(0, \overline{\sqrt{\frac{|\alpha\beta|}{2}}}\right).$$

COROLLARY 6.2. $r(\alpha E_{12} + \beta F_{12}) = \sqrt{\frac{|\alpha\beta|}{2}}$.

COROLLARY 6.3. Let $A \in M_2(\mathsf{C})_{(1)}$ and $B \in M_2(\mathsf{C})_{(2)}$. Then $A + B$ is an R-diagonal operator if and only if $A + B = 0$.

PROOF. Indeed, if $A + B$ is an R-diagonal operator, then $\tau(A + B) = 0$. So we may assume that $\tau(A) = \tau(B) = 0$. Let $\lambda, -\lambda$ and $\eta, -\eta$ be the spectra of A and B, respectively. Then $0 = \tau((A + B)^2) = \tau(A^2) + \tau(B^2) = \lambda^2 + \eta^2$. By simple computation we have $0 = \tau((A + B)^4) = \tau(A^4) + \tau(B^4) + 4\tau(A^2)\tau(B^2) = \lambda^4 + \eta^4 + 4\lambda^2\eta^2 = (\lambda^2 + \eta^2)^2 + 2\lambda^2\eta^2 = 2\lambda^2\eta^2$. Thus $\lambda = \eta = 0$. This implies that A and B are unitary equivalent to αE_{12} and βF_{12}, respectively. By Corollary 6.2, $r(A + B) = \sqrt{\frac{|\alpha\beta|}{2}}$. On the other hand, since we assume that $A + B$ is an R-diagonal operator, by Theorem 2.1, $(r(A + B))^2 =$

$\tau((A^* + B^*)(A + B)) = \tau((\bar{\alpha} E_{21} + \bar{\beta} F_{21})(\alpha E_{12} + \beta F_{12})) = \frac{|\alpha|^2 + |\beta|^2}{2}$. So $|\alpha|^2 + |\beta|^2 = |\alpha\beta|$. This implies that $\alpha = 0$ and $\beta = 0$. Hence $A + B = 0$.

EXAMPLE 6.4.   We compute the spectrum and Brown spectrum of

$$X = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{(1)} \begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix}_{(2)}.$$

By Example 5.6, we have the following

$$X = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{(1)} \begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix}_{(2)}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{(1)} \begin{pmatrix} \alpha + \beta h v^* \sqrt{1-h^2} & -\beta h v^* h u \\ \beta u^* \sqrt{1-h^2} v \sqrt{1-h^2} & \alpha - \beta u^* \sqrt{1-h^2} v^* h u \end{pmatrix}_{(1)}$$

$$= \begin{pmatrix} \alpha + \beta h v^* \sqrt{1-h^2} & -\beta h v^* h u \\ 0 & 0 \end{pmatrix}_{(1)}.$$

So $\sigma(X) = \{0\} \cup \sigma(\alpha + \beta h v^* \sqrt{1-h^2})$ and $\mu_X = \frac{1}{2}\delta_0 + \frac{1}{2}\mu_{\alpha+\beta h v^* \sqrt{1-h^2}}$. Note that $\mu_{h v^* \sqrt{1-h^2}} = \mu_{v^* \sqrt{1-h^2} h}$ and $v^* \sqrt{1-h^2} h$ is an R-diagonal operator. We have the following computations:

$$\left\| \sqrt{1-h^2} h \right\|_2^2 = \tau_P((1-h^2)h^2) = \tau_P((1 - PQP)PQP)$$

$$= \int_0^1 \frac{1}{\pi} \frac{t(1-t)dt}{\sqrt{\frac{1}{4} - \left(\frac{1}{2} - t\right)^2}} = \frac{1}{8},$$

$$\left\| \left(\sqrt{1-h^2} h\right)^{-1} \right\|_2^2 = \tau_P(((1-h^2)h^2)^{-1}) = \tau_P(((1 - PQP)PQP)^{-1})$$

$$= \int_0^1 \frac{1}{\pi} \frac{dt}{t(1-t)\sqrt{\frac{1}{4} - \left(\frac{1}{2} - t\right)^2}} = \infty.$$

By Theorem 2.1,

$$\sigma(X) = \text{supp}\,\mu_X = \{0\} \cup \overline{B\left(\alpha, |\beta|/2\sqrt{2}\right)}.$$

EXAMPLE 6.5.   We compute the spectrum and Brown spectrum of

$$Y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}_{(1)} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}_{(2)}.$$

By Example 5.5, we have the following

$$
Y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}_{(1)} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}_{(2)}
$$

$$
= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}_{(1)} \begin{pmatrix} \beta + (\alpha - \beta)h^2 & (\alpha - \beta)h\sqrt{1 - h^2}u \\ (\alpha - \beta)u^*h\sqrt{1 - h^2} & \alpha + (\beta - \alpha)u^*h^2u \end{pmatrix}_{(1)}
$$

$$
= \begin{pmatrix} (\alpha - \beta)u^*h\sqrt{1 - h^2} & \alpha + (\beta - \alpha)u^*h^2u \\ 0 & 0 \end{pmatrix}_{(1)}.
$$

Since $u^*h\sqrt{1 - h^2}$ is an R-diagonal operator, similar computations as Example 6.4, we have

$$
\sigma(Y) = \operatorname{supp} \mu_Y = \overline{\mathsf{B}\big(0, |\alpha - \beta|/2\sqrt{2}\big)}.
$$

EXAMPLE 6.6.   We compute the spectrum and Brown spectrum of

$$
Z = (1 + \alpha E_{12})(1 + \beta F_{12}) = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}_{(1)} \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}_{(2)}.
$$

For $\lambda \in \mathsf{C}$, we have

$$
\begin{aligned}
Z - \lambda 1 &= (1 + \alpha E_{12})(1 + \beta F_{12}) - \lambda(1 + \alpha E_{12})(1 - \alpha E_{12}) \\
&= (1 + \alpha E_{12})(\lambda \alpha E_{12} + \beta F_{12} - (\lambda - 1)).
\end{aligned}
$$

This implies that $\lambda \in \sigma(Z)$ if and only if $\lambda - 1 \in \sigma(\lambda \alpha E_{12} + \beta F_{12})$. By Example 6.1, $\lambda - 1 \in \sigma(\lambda \alpha E_{12} + \beta F_{12})$ if and only if

$$
|\lambda - 1|^2 \leq \frac{|\alpha\beta||\lambda|}{2}.
$$

So

$$
\sigma(Z) = \left\{ \lambda \in \mathsf{C} : |\lambda - 1|^2 \leq \frac{|\alpha\beta||\lambda|}{2} \right\}.
$$

In the following, we will show that $\operatorname{supp} \mu_Z \supseteq \partial\sigma(Z) = \big\{ \lambda \in \mathsf{C} : |\lambda - 1|^2 = \frac{|\alpha\beta||\lambda|}{2} \big\}$. For this purpose, we need only to prove that $\operatorname{supp} \mu_{Z-1} \supseteq \partial\sigma(Z-1) = \big\{ \lambda \in \mathsf{C} : |\lambda|^2 = \frac{|\alpha\beta||\lambda+1|}{2} \big\}$.

Note that $\Delta(1 + \alpha E_{12}) = 1$. For $\lambda \in \mathsf{C}$, we have

$$\log \Delta((Z - 1) - \lambda)$$
$$= \log \Delta((1 + \alpha E_{12})(1 + \beta F_{12}) - (1 + \lambda)(1 + \alpha E_{12})(1 - \alpha E_{12}))$$
$$= \log \Delta(1 + \alpha E_{12}) + \log \Delta(1 + \beta F_{12} - \lambda + (1 + \lambda)\alpha E_{12})$$
$$= \log \Delta((1 + \lambda)\alpha E_{12} + \beta F_{12} - \lambda).$$

By Example 6.1, $\mu_{(1+\lambda)\alpha E_{12}+\beta F_{12}} = \mu_{\sqrt{|1+\lambda||\alpha\beta|}b}$. Hence,

$$\log \Delta((Z - 1) - \lambda)$$
$$= \log \Delta(\sqrt{|1 + \lambda||\alpha\beta|}b - \lambda)$$
$$= \log \Delta\left(b - \frac{\lambda}{\sqrt{|1 + \lambda||\alpha\beta|}}\right) - \log \frac{|\lambda|}{\sqrt{|1 + \lambda||\alpha\beta|}} + \log |\lambda|.$$

Since b is an R-diagonal operator, this implies that

$$(6.1) \quad \log \Delta\left(b - \frac{|\lambda|}{\sqrt{|1 + \lambda||\alpha\beta|}}\right)$$
$$= \log \frac{|\lambda|}{\sqrt{|1 + \lambda||\alpha\beta|}} - \log |\lambda| + \log \Delta((Z - 1) - \lambda).$$

Suppose $\lambda_0 \in \partial\sigma(Z - 1)$ and $\lambda_0 \notin \operatorname{supp} \mu_{Z-1}$. Then there is $\delta > 0$ such that $\mathsf{B}(\lambda_0, \delta) \subset \mathsf{C} \setminus \operatorname{supp} \mu_{Z-1}$. Now $\log \Delta((Z - 1) - \lambda)$ is a harmonic function on $\mathsf{B}(\lambda_0, \delta)$. Since $\tau((Z - 1)^n) = 0$ for all $n = 1, 2, \ldots$. By Lemma 4.3 of [8], for $\lambda \in \mathsf{C}$ such that $|\lambda| \geq r(Z - 1)$, $\log \Delta((Z - 1) - \lambda) = \log |\lambda|$. By the uniqueness of harmonic functions, we have $\log \Delta((Z - 1) - \lambda) = \log |\lambda|$ for $\lambda \in \mathsf{B}(\lambda_0, \delta)$. By equation (6.1), this implies that

$$(6.2) \qquad \log \Delta\left(b - \frac{|\lambda|}{\sqrt{|1 + \lambda||\alpha\beta|}}\right) = \log \frac{|\lambda|}{\sqrt{|1 + \lambda||\alpha\beta|}}.$$

Let $r = \frac{|\lambda|}{\sqrt{|1+\lambda||\alpha\beta|}}$. Then equation (6.2) implies that

$$\log \Delta (b - r) = \log r$$

for $r \in (s, t) \subseteq \left[0, \frac{1}{\sqrt{2}}\right]$. Since b is an R-diagonal operator, this implies that $\log \Delta(b - z)$ is harmonic on the annulus with inner radius s and outer radius t, $0 < s < t < \frac{1}{\sqrt{2}}$. By Theorem 2.1, $\operatorname{supp} \mu_b = \overline{\mathsf{B}\left(0, \frac{1}{\sqrt{2}}\right)}$. It is a contradiction.

7. Hyperinvariant subspaces for operators in $\mathcal{M}$

LEMMA 7.1. *For $X \in \mathcal{M}$, if $\operatorname{supp} \mu_X = \{\lambda\}$, then $\tau(X^n) = \lambda^n$ for $n = 1, 2, \ldots$.*

PROOF. $\tau(X^n) = \displaystyle\int_{\operatorname{supp} \mu_X} z^n \, d\mu_X(z) = \lambda^n$.

The converse of Lemma 7.1 is not true. Since for an R-diagonal operator X, we have $\tau(X^n) = 0$ for $n = 1, 2, \ldots$.

PROPOSITION 7.2. *Let $X = A + B$, where $A \in M_2(\mathsf{C})_{(1)}$ and $B \in M_2(\mathsf{C})_{(2)}$. If A, B are not scalar matrices, then $\operatorname{supp} \mu_X$ contains more than two points.*

PROOF. Suppose A, B are not scalar matrices. Since $A + B = \tau(A)1 + \tau(B)1 + (A - \tau(A)1) + (B - \tau(B)1)$, to show $\operatorname{supp} \mu_X$ contains more than two points, we need only to show that $\mu_{(A-\tau(A)1)+(B-\tau(B)1)}$ contains more than two points. So we may assume that $\tau(A) = \tau(B) = 0$ and $A, B \neq 0$. Assume that the spectra of A and B are $\lambda_1, -\lambda_1$ and $\lambda_2, -\lambda_2$. If $\tau(A^2) = \tau(B^2) = 0$, then A and B are unitarily equivalent to αE_{12} and βF_{12} in $M_2(\mathsf{C})_{(1)}$ and $M_2(\mathsf{C})_{(2)}$, respectively. Thus $\mu_X = \mu_{\alpha E_{12} + \beta F_{12}}$. By Example 6.1, $\operatorname{supp} \mu_X$ contains more than two points. Now suppose $\tau(A^2) \neq 0$ or $\tau(B^2) \neq 0$. Without loss of generality, we assume that $\lambda_1^2 = \tau(A^2) \neq 0$. Note that $\tau(A + B) = 0$ and $\tau((A + B)^2) = \tau(A^2) + \tau(B^2) = \lambda_1^2 + \lambda_2^2$. If $\lambda_1^2 + \lambda_2^2 \neq 0$, by Lemma 7.1, $\operatorname{supp} \mu_X$ contains more than two points. Suppose $\lambda_1^2 + \lambda_2^2 = 0$. Then $\tau(B^2) = \lambda_2^2 \neq 0$. Simple computations show that

$$\tau((A + B)^4) = \tau(A^4) + \tau(B^4) + 4\tau(A^2)\tau(B^2)$$

$$= \lambda_1^4 + \lambda_2^4 + 4\lambda_1^2\lambda_2^2 = (\lambda_1^2 + \lambda_2^2)^2 + 2\lambda_1^2\lambda_2^2 = 2\lambda_1^2\lambda_2^2 \neq 0.$$

Note that $\tau(A + B) = 0$. By Lemma 7.1, $\operatorname{supp} \mu_X$ contains more than two points.

PROPOSITION 7.3. *Let $X = AB$, where $A \in M_2(\mathsf{C})_{(1)}$ and $B \in M_2(\mathsf{C})_{(2)}$. If A, B are not scalar matrices, then $\operatorname{supp} \mu_X$ contains more than two points.*

PROOF. Suppose A, B are not scalar matrices. We consider the following cases:

Case 1. $\tau(A) = \tau(B) = 0$ and $A, B \neq 0$. By Theorem 4.1, $AB(\neq 0)$ is an R-diagonal operator. So $\operatorname{supp} \mu_X$ contains more than two points.

Case 2. $\tau(A) = 0, \tau(B) \neq 0$ or $\tau(A) \neq 0, \tau(B) = 0$. Without loss of generality, we assume that $\tau(A) = 0$ and $\tau(B) \neq 0$. Then $\tau(AB) = 0$ and $\tau(ABAB) = \tau(A^2)\tau(B)$. If $\tau(A^2) \neq 0$, then $\tau(ABAB) \neq 0$. By Lemma 7.1, $\operatorname{supp} \mu_X$ contains more than two points. If $\tau(A^2) = 0$, then A is unitarily equivalent to αE_{12} in $M_2(\mathsf{C})_{(1)}$. By Lemma 4.5, $\mu_X = \mu_{\alpha E_{12}(B - \tau(B))}$. Since

$\alpha E_{12}(B - \tau(B))(\neq 0)$ is an R-diagonal operator, supp μ_X contains more than two points.

Case 3. $\tau(A) \neq 0$ and $\tau(B) \neq 0$. We may assume that $\tau(A) = \tau(B) = 1$. Let $A = 1 + A_1$ and $B = 1 + B_1$. Then $\tau(A_1) = \tau(B_1) = 0$.

Subcase 3.1. $\tau(A_1^2) \neq 0$ or $\tau(B_1^2) \neq 0$. We may assume that $\tau(A_1^2) \neq 0$. Simple computation shows that $\tau(AB) = 1$, $\tau(ABAB) = 1 + \tau(A_1^2) + \tau(B_1^2)$ and $\tau((AB)^3) = 1 + 3(\tau(A_1^2) + \tau(B_1^2)) + 9\tau(A_1^2)\tau(B_1^2)$. If $\tau(A_1^2) + \tau(A_2^2) \neq 0$, then $\tau(ABAB) \neq 1$. By Lemma 7.1, supp μ_X contains more than two points. If $\tau(A_1^2) + \tau(A_2^2) = 0$, then $\tau(A_2^2) = -\tau(A_1^2) \neq 0$. So $\tau((AB)^3) \neq 1$. By Lemma 7.1 again, supp μ_X contains more than two points.

Subcase 3.2. $\tau(A_1^2) = \tau(A_2^2) = 0$. Then A_1 and A_2 are unitarily equivalent to αE_{12} and βF_{12} in $M_2(\mathsf{C})_{(1)}$ and $M_2(\mathsf{C})_{(2)}$, respectively. So $\mu_X = \mu_{(1+\alpha E_{12})(1+\beta F_{12})}$. We may assume that $A = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}_{(1)}$ and $B = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}_{(2)}$. By Example 6.6, supp μ_X contains more than two points.

COROLLARY 7.4. *Let $X = AB$ or $X = A + B$, where $A \in M_2(\mathsf{C})_{(1)}$ and $B \in M_2(\mathsf{C})_{(2)}$. If $X \neq \lambda 1$, then X has a nontrivial hyperinvariant subspace relative to $\mathcal{M}$.*

PROOF. If $X = A + B$ and $A = \lambda 1$ or $B = \lambda 1$, then $X = \lambda 1 + B$ or $X = \lambda 1 + A$. If X is not a scalar matrix and η is an eigenvalue of X, then $\ker(X - \eta 1)$ is a nontrivial hyperinvariant subspace of X. If $X = A + B$ and $A, B \neq \lambda 1$, then supp μ_X contains more than two points by Proposition 7.3. By [9], X has a nontrivial hyperinvariant subspace relative to $\mathcal{M}$. If $X = AB$ and $A = \lambda 1$ or $B = \lambda 1$, then $X = \lambda B$ or $X = \lambda A$. If X is not a scalar matrix and η is an eigenvalue of X, then $\ker(X - \eta 1)$ is a nontrivial hyperinvariant subspace of X. If $X = AB$ and $A, B \neq \lambda 1$, then supp μ_X contains more than two points. By [9], X has a nontrivial hyperinvariant subspace relative to $\mathcal{M}$.

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