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COMPOSITION, NUMERICAL RANGE AND ARON-BERNER EXTENSION

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Abstract

Given an entire mapping $f \in \mathcal{H}_b(X, X)$ of bounded type from a Banach space X into X, we denote by $\overline{f}$ the Aron-Berner extension of f to the bidual X^{**} of X. We show that $\overline{g \circ f} = \overline{g} \circ \overline{f}$ for all $f, g \in \mathcal{H}_b(X, X)$ if X is symmetrically regular. We also give a counterexample on l_1 such that the equality does not hold. We prove that the closure of the numerical range of f is the same as that of $\bar{f}$.

1. Introduction

Given complex Banach spaces X and Y, we denote by $\mathcal{P}(^n X, Y)$ the Banach space of bounded n-homogeneous polynomials of X into Y. When Y is the scalar field $\mathbf{C}$, we denote this space by $\mathcal{P}(^n X)$. We recall that a bounded n-homogeneous polynomial $P \in \mathcal{P}(^n X, Y)$ is the restriction to the diagonal of a continuous n-linear mapping A from X into Y, that is, $P(x) = A(x, \dots, x)$, $x \in X$. Each such P has a unique associated bounded symmetric n-linear mapping A from X into Y. Each bounded n-homogeneous polynomial P has a canonical extension $\overline{P} \in \mathcal{P}(^n X^{**}, Y^{**})$ to the bidual X^{**} of X, which is called the Aron-Berner extension of P ([2]) (see the next section for definitions). By [10, Theorem 3] (see also [2]), every entire mapping $f \in \mathcal{H}_b(X, Y)$ of bounded type extends in a canonical fashion to a mapping $\overline{f} \in \mathcal{H}_b(X^{**}, Y^{**})$ in the following way. Given the Taylor series expansion of f at 0, $f = \sum_{n=0}^{\infty} P_n$, $\overline{f}$ is defined as $\overline{f} = \sum_{n=0}^{\infty} \overline{P_n}$.

Our first interest in this paper is to verify if $\overline{g \circ f} = \overline{g} \circ \overline{f}$ for $f \in \mathcal{H}_b(X, Y)$ and $g \in \mathcal{H}_b(Y, Z)$. We are motivated by the following two problems: We consider the case $X = Y = Z$.

(1) The Aron-Berner extension is an isomorphism of the Fréchet space $\mathcal{H}_b(X, X)$ into the Fréchet space $\mathcal{H}_b(X^{**}, X^{**})$ and both spaces are Fréchet

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algebras under composition. Is it true that the Aron-Berner extension is an isomorphism into between Fréchet algebras ?

(2) Given $g \in \mathcal{H}_b(X, X)$ consider the composition operator $\varphi_g : \mathcal{H}_b(X, X) \to \mathcal{H}_b(X, X)$ defined by $\varphi_g(f) = g \circ f$. This composition operator φ_g is extended to the composition operator $\varphi_{\overline{g}} : \mathcal{H}_b(X^{**}, X^{**}) \to \mathcal{H}_b(X^{**}, X^{**})$. Does the following diagram commute?

$$\begin{array}{ccc} \mathcal{H}_b(X, X) & \longrightarrow & \mathcal{H}_b(X^{**}, X^{**}) \\ {\scriptstyle \varphi_g}\uparrow & & \uparrow{\scriptstyle \varphi_{\overline{g}}} \\ \mathcal{H}_b(X, X) & \longrightarrow & \mathcal{H}_b(X^{**}, X^{**}) \end{array}$$

The answer to our questions is, in general, negative. In Section 2 we show the existence of a 2-homogeneous continuous polynomial $P : \ell_1 \longrightarrow \ell_1$ such that $\overline{P \circ P} \neq \overline{P} \circ \overline{P}$.

Our second interest is to know if the Aron-Berner extension preserves numerical ranges. Lumer in 1961 ([14]) gave a theory of numerical range for bounded linear operators on Banach spaces. Harris in 1971 ([13]) developed a theory of numerical range and numerical radius for a holomorphic mapping. This theory has many applications. For example, he obtained an inequality ([13, Theorem 1]) which is a bound for each of the terms of the Taylor series expansion of a holomorphic mapping in terms of the numerical radius of the mapping. This inequality implies some results concerning the spectrum of holomorphic mappings ([13, Proposition 5]), the rotundity at the identity of the sup norm on holomorphic mappings ([12, Theorem 2]) and the extremal case of the Schwarz lemma ([11, Theorem 1]). We prove that the closure of the numerical range of $f \in \mathcal{H}_b(X, X)$ is the same as that of $\overline{f} \in \mathcal{H}_b(X^{**}, X^{**})$, which implies that the numerical radius of f is the same as that of $\overline{f}$.

2. Aron-Berner extension and composition

A bounded n-homogeneous polynomial $P \in \mathcal{P}(^n X, Y)$ has an extension $\overline{P} \in \mathcal{P}(^n X^{**}, Y^{**})$ to the bidual X^{**} of X, which is called the *Aron-Berner extension* of P. In fact, $\overline{P}$ is defined in the following way. Let A be the symmetric n-linear mapping associated to P, A can be extended to an n-linear mapping $\overline{A}$ from X^{**} into Y^{**} in such a way that for each fixed j, $1 \leq j \leq n$, and for each fixed $x_1, \dots, x_{j-1} \in X$ and $z_{j+1}, \dots, z_m \in X^{**}$, the linear mapping

$$z \to \overline{A}(x_1, \dots, x_{j-1}, z, z_{j+1}, \dots, z_n), \qquad z \in X^{**},$$

is (w^*, w^*)-continuous. In other words, we define $\overline{A}(x_1, \dots, x_{j-1}, z, z_{j+1}, \dots, z_n)$ to be the weak-star limit of the net $(\overline{A}(x_1, \dots, x_{j-1}, x_\alpha, z_{j+1}, \dots, z_n))$ for a weak-star convergent net $(x_\alpha) \subset X$ to z. By this (w^*, w^*)-continuity A

can be extended to an n-linear mapping $\overline{A}$ from X^{**} into Y^{**}, beginning with the last variable and working backwards to the first. Then the restriction

$$\overline{P}(z) = \overline{A}(z, \ldots, z)$$

is called the Aron-Berner extension of P. Given $z \in X^{**}$ and $w \in Y^*$, we have

$$\overline{P}(z)(w) = \overline{w \circ P}(z).$$

Actually this equality is often used as the definition of the vector-valued Aron-Berner extension based upon the scalar-valued Aron-Berner extension. Davie and Gamelin [10, Theorem 8] proved that $\|P\| = \|\overline{P}\|$. It is also worth remarking that $\overline{A}$ is not symmetric in general.

A complex Banach space X is called *symmetrically regular* if every continuous symmetric linear mapping $T : X \to X^*$ is weakly compact. Recall that T is symmetric means that $T(x)(y) = T(y)(x)$ for all $x, y \in X$. If X is symmetrically regular then, by [3, 8.3 Theorem], $\overline{A}$ is also symmetric and separately weak-star continuous on X^{**}, for all symmetric n-linear form $A : X \times \cdots \times X \to \mathbb{C}$.

THEOREM 2.1. *Let X, Y and Z be complex Banach spaces. If Y is symmetrically regular then $\overline{Q \circ (P_0 + P_1 + \cdots + P_m)} = \overline{Q} \circ (P_0 + \overline{P_1} + \cdots + \overline{P_m})$ for every $P_i \in \mathcal{P}(^i X, Y)$, for $i = 0, 1, \ldots, m$, $Q \in \mathcal{P}(^k Y, Z)$ and $m, k \geq 1$.*

PROOF. Let us denote $P = P_0 + P_1 + \ldots + P_m$, and let B be the symmetric k-linear form associated to Q. We put $\mathcal{J} = \{\mathbf{j} = (j_0, \ldots, j_m) \mid \sum_{h=0}^{m} j_h = k, 0 \leq j_h \leq k, h = 0, 1, \ldots, m\}$ and $|\mathbf{j}| = \sum_{h=0}^{m} h j_h$. We have

$$Q \circ P(x) = \sum_{(j_0,\ldots,j_m)\in\mathcal{J}} \binom{k}{j_0, \ldots, j_m} B(P_0^{j_0}, P_1^{j_1}(x), \ldots, P_m^{j_m}(x)),$$

for all $x \in X$, where $P_i^{j_i}$ means that the coordinate P_i is repeated j_i-times. The mapping $R_{\mathbf{j}}(x) = B(P_0^{j_0}, P_1^{j_1}(x), \ldots, P_m^{j_m}(x))$ is a continuous $|\mathbf{j}|$-homogeneous polynomial on X for all $\mathbf{j} \in \mathcal{J}$. Hence

$$\overline{Q \circ P}(z) = \sum_{\mathbf{j}=(j_0,\ldots,j_m)\in\mathcal{J}} \binom{k}{j_0, \ldots, j_m} \overline{R_{\mathbf{j}}}(z),$$

for all $z \in X^{**}$. On the other hand, as Y is symmetrically regular, $\overline{B}$ is symmetric and hence

$$\overline{Q} \circ \overline{P}(z) = \sum_{\mathbf{j}=(j_0,\ldots,j_m)\in\mathcal{J}} \binom{k}{j_0, \ldots, j_m} T_{\mathbf{j}}(z),$$

where $T_{\mathbf{j}}(z) = \overline{B}(P_0^{j_0}, \overline{P_1}^{j_1}(z), \ldots, \overline{P_m}^{j_m}(z))$ for all $z \in X^{**}$. If we prove that $\overline{R_{\mathbf{j}}} = T_{\mathbf{j}}$ for all $\mathbf{j} \in \mathscr{J}$ with $|\mathbf{j}| > 0$, then $\overline{Q \circ P} = \overline{Q} \circ \overline{P}$.

Recall that the differential of a polynomial $P \in \mathscr{P}(^k X, Y)$ is the $(k-1)$-homogeneous polynomial $D(P) : X \to \mathscr{L}(X, Y)$ given by $D(P)(x)(z) = kA(x, \ldots, x, z)$, $(x, z \in X)$, where A is the symmetric k-linear mapping associated to P.

Given $\mathbf{j} \in \mathscr{J}$ with $|\mathbf{j}| > 0$, we have $R_{\mathbf{j}}(x) = T_{\mathbf{j}}(x)$ for all $x \in X$, hence, by [7, Proposition 1.1] (see also [15, Theorem 2]), $\overline{R_{\mathbf{j}}} = T_{\mathbf{j}}$ if and only if the following two properties hold:

(a) For every $x \in X$, $D(T_{\mathbf{j}})(x) : X^{**} \to Z^{**}$ is (w^*, w^*)-continuous.

(b) For every $z \in X^{**}$ and every net $(x_\mu) \subset X$ such that (x_μ) converges weak-star to z, $D(T_{\mathbf{j}})(z)(x_\mu)$ converges weak-star to $D(T_{\mathbf{j}})(z)(z)$ in Z^{**}.

We consider $C_{\mathbf{j}} : X^{**} \longrightarrow Y^{**}$ the bounded $|\mathbf{j}|$-linear mapping defined by

$$\begin{aligned} &C_{\mathbf{j}}(z_1, \ldots, z_{|\mathbf{j}|}) \\ &\quad = \overline{B}\Big(P_0^{j_0}, \overline{A_1}(z_1), \ldots, \overline{A_1}(z_{j_1}), \overline{A_2}(z_{j_1+1}, z_{j_1+2}), \ldots, \overline{A_2}(z_{j_1+2j_2-1}, z_{j_1+2j_2}), \\ &\qquad \ldots, \overline{A_m}\big(z_{\sum_{h=1}^{m-1} hj_h+1}, \ldots, z_{\sum_{h=1}^{m-1} hj_h+m}\big), \ldots, \overline{A_m}\big(z_{|\mathbf{j}|-m+1}, \ldots, z_{|\mathbf{j}|}\big)\Big), \end{aligned}$$

where A_h is the symmetric h-linear mapping associated to P_h for $h = 1, \ldots, m$. Clearly $T_{\mathbf{j}}(z) = C_{\mathbf{j}}(z, \ldots, z)$ for all $z \in X^{**}$. If $SC_{\mathbf{j}}$ denotes the symmetrization of $C_{\mathbf{j}}$, we have that

$$SC_{\mathbf{j}}(z_1, \ldots, z_{|\mathbf{j}|}) = \frac{1}{|\mathbf{j}|!} \sum_{\sigma \in \mathsf{S}_{|\mathbf{j}|}} C_{\sigma\mathbf{j}}(z_1, \ldots, z_{|\mathbf{j}|}),$$

where $\mathsf{S}_{|\mathbf{j}|}$ stands for the group of permutations of $\{1, 2, \ldots, |\mathbf{j}|\}$ and

$$C_{\sigma\mathbf{j}}(z_1, \ldots, z_{|\mathbf{j}|}) = C_{\mathbf{j}}(z_{\sigma(1)}, \ldots, z_{\sigma(|\mathbf{j}|)}).$$

With this notation

$$D(T_{\mathbf{j}})(z)(w) = |\mathbf{j}| SC_{\mathbf{j}}(z, \ldots, z, w) = \frac{1}{(|\mathbf{j}|-1)!} \sum_{\sigma \in \mathsf{S}_{|\mathbf{j}|}} C_{\sigma\mathbf{j}}(z, \ldots, z, w),$$

for all $z, w \in X^{**}$.

We know that $\overline{B}$ is symmetric. On the other hand

$$\overline{A_h}(z, \ldots, z, x) = \overline{A_h}(z, \ldots, z, x, z) = \cdots = \overline{A_h}(x, z, \ldots, z)$$

for all $z \in X^{**}$, $x \in X$ and $h = 1, \ldots, m$. Thus, for fixed $\sigma \in \mathsf{S}_{|\mathbf{j}|}$ there exists a unique $h = 1, \ldots, m$ such that

$$C_{\sigma\mathbf{j}}(z, \ldots, z, x) = \overline{B}(P_0^{j_0}, \overline{P_1}^{j_1}(z), \ldots, \overline{P_{h-1}}^{j_{h-1}}(z), \overline{A_h}(x, z, \ldots, z), \overline{P_{h+1}}^{j_{h+1}}(z), \ldots, \overline{P_m}^{j_m}(z)).$$

The linear mapping $\overline{A_h}(-, z \ldots, z)$ is weak-star continuous on X^{**}. Since Y is symmetrically regular, $\overline{B}$ is weak-star separately continuous. Hence, if $(x_\mu) \subset X$ converges weak-star to z in X^{**}, then $C_{\sigma\mathbf{j}}(z, \ldots, z, x_\mu)$ converges weak-star to $T_{\mathbf{j}}(z)$. As an immediate consequence $D(T_{\mathbf{j}})(z)(x_\mu)$ converges to $|\mathbf{j}|T_{\mathbf{j}}(z) = D(T_{\mathbf{j}})(z)(z)$ for all $z \in X^{**}$ and property (b) holds for every $T_{\mathbf{j}}$.

Finally, given $x \in X$ and $w \in X^{**}$, we have $\overline{A_h}(x, \ldots, x, w) = \overline{A_h}(x, \ldots, x, w, x) = \cdots = \overline{A_h}(w, x, \ldots, x)$ and the linear mapping $\overline{A_h}(x, \ldots, x, -)$ is weak-star continuous on X^{**} for all $h = 1, \ldots, m$. As

$$C_{\sigma\mathbf{j}}(x, \ldots, x, w) = \overline{B}(P_0^{j_0}, P_1^{j_1}(x), \ldots, P_{h-1}^{j_{h-1}}(x), \overline{A_h}(x, \ldots, x, w), P_{h+1}^{j_{h+1}}(x), \ldots, P_m^{j_m}(x)),$$

the proof that property (a) holds for every $T_{\mathbf{j}}$ can be obtained in a similar way.

COROLLARY 2.2. *Suppose that Y is symmetrically regular. Then $\overline{g \circ f} = \overline{g} \circ \overline{f}$ for $f \in \mathscr{H}_b(X, Y)$ and $g \in \mathscr{H}_b(Y, Z)$.*

PROOF. We first note that the Taylor series $\sum_{n=0}^{\infty} Q_n$ of g at 0 converges to g in the Fréchet space $\mathscr{H}_b(Y, Z)$. Since the Aron-Berner extension induces a Fréchet isomorphism from $\mathscr{H}_b(Y, Z)$ into $\mathscr{H}_b(Y^{**}, Z^{**})$, it is enough to consider only the case where $g = Q \in \mathscr{P}(^kY, Z)$, for all $k \geq 1$.

For $R > 0$ we consider on $\mathscr{H}_b(X, Y)$ the norm $\|f\|_R = \sup\{|f(x)| : \|x\| \leq R\}$. We fix $Q \in \mathscr{P}(^kY, Z)$ and $f \in \mathscr{H}_b(X, Y)$. There exists $S > 0$ such that $f(RB_X) \subset SB_Y$. Since Q is uniformly continuous on the ball $(S+1)B_Y$ and since $\overline{Q}$ is also uniformly continuous on $(S+1)B_{Y^{**}}$, given $\varepsilon > 0$ we can find $0 < \delta < 1$ such that $\|Q(y_1) - Q(y_2)\| < \varepsilon$ for all $y_1, y_2 \in (S+1)B_Y$ with $\|y_1 - y_2\| < \delta$ and $\|\overline{Q}(v_1) - \overline{Q}(v_2)\| < \varepsilon$ for all $v_1, v_2 \in (S+1)B_{Y^{**}}$ with $\|v_1 - v_2\| < \delta$.

The Taylor series expansion $\sum_{m=0}^{\infty} P_m$ of f at zero converges absolutely and uniformly to f on any bounded set of X, and hence there exists m_0 such that

$$\left\| f - \sum_{m=0}^{m_0} P_m \right\|_R < \delta. \tag{1}$$

Thus, $\left\| Q \circ f - Q \circ \left(\sum_{m=0}^{m_0} P_m\right)\right\|_R < \varepsilon$. Hence, by [10, Theorem 8],

$$\left\| \overline{Q \circ f} - \overline{Q \circ \left(\sum_{m=0}^{m_0} P_m\right)} \right\|_R = \left\| Q \circ f - Q \circ \left(\sum_{m=0}^{m_0} P_m\right) \right\|_R < \varepsilon,$$

which, by Theorem 2.1, implies

$$\left\| \overline{Q \circ f} - \overline{Q} \circ \left(\sum_{m=0}^{m_0} \overline{P_m}\right) \right\|_R = \left\| \overline{Q \circ f} - \overline{Q \circ \left(\sum_{m=0}^{m_0} P_m\right)} \right\|_R < \varepsilon \tag{2}$$

On the other hand, by (1) and [10, Theorem 8] we have $\left\| \overline{f} - \sum_{m=0}^{m_0} \overline{P_m} \right\|_R = \left\| f - \sum_{m=0}^{m_0} P_m \right\|_R < \delta$, from which

$$\left\| \overline{Q} \circ \overline{f} - \overline{Q} \circ \left(\sum_{m=0}^{m_0} \overline{P_m}\right) \right\|_R < \varepsilon. \tag{3}$$

Now the conclusion is clear from (2) and (3).

An $f \in \mathscr{H}_b(X, Y)$ is called *weakly compact* if $f(rB_X)$ is a relatively weakly compact set for all $r > 0$. Let $\sum_{m=0}^{\infty} P_m$ be the Taylor series expansion of f at zero. An obvious modification of [4, Proposition 3.4] shows that f is weakly compact if and only if $P_m(B_X)$ is a relatively weakly compact set for all $m = 1, 2, \ldots$.

PROPOSITION 2.3. *Let X, Y and Z be complex Banach spaces and $m \geq 1$. If $P_h \in \mathscr{P}({}^hX, Y)$ is a weakly compact polynomial for all $h = 1, \ldots, m$ and $P = \sum_{h=0}^{m} P_h$, then $\overline{Q \circ P} = \overline{Q} \circ \overline{P}$ for every $Q \in \mathscr{P}({}^kY, Z)$ and $k \geq 1$.*

PROOF. Let B be the k-linear symmetric mapping associated to Q and $\overline{B}$ be its Aron-Berner extension. An inspection of the proof of Theorem 2.1 shows that the symmetry of $\overline{B}$ on $\left(\mathrm{span}(\overline{P}(X^{**})\right)^k$ is a sufficient condition for the equality $\overline{Q \circ P} = \overline{Q} \circ \overline{P}$. Since $\overline{P}(X^{**}) = P(X) \subset Y$, the conclusion follows.

It is well-known that the Banach space l_1 is not symmetrically regular ([3]). In the following we construct a 2-homogeneous polynomial $P : l_1 \to l_1$ such that $\overline{P \circ P} \neq \overline{P} \circ \overline{P}$.

EXAMPLE 2.4. Define the bounded symmetric bilinear mappings $A_1,\ A_2 : l_1 \times l_1 \to l_1$ by

$$A_1(x,y) = \sum_{n=1}^{\infty}\big[(x_1e_1 + x_3e_3 + \cdots + x_{2n-1}e_{2n-1})y_{2n} \\ + (y_1e_1 + y_3e_3 + \cdots + y_{2n-1}e_{2n-1})x_{2n}\big],$$

$$A_2(x,y) = \sum_{n=1}^{\infty}\big[(x_1 + x_3 + \cdots + x_{2n-1})y_{2n} \\ + (y_1 + y_3 + \cdots + y_{2n-1})x_{2n}\big]e_{2n},$$

where $x = (x_i), y = (y_i) \in l_1$ and $\{e_n\}$ is the canonical basis of l_1. Let $A = A_1 + A_2$.

Let P be the 2-homogeneous polynomial from l_1 to l_1 associated to A. Then $\overline{P \circ P} \neq \overline{P} \circ \overline{P}$.

PROOF. We can see easily that

$$A_1(e_{2p}, e_{2q}) = 0, \quad A_1(e_{2p-1}, e_{2q-1}) = 0,$$
$$A_2(e_{2p}, e_{2q}) = 0, \quad A_2(e_{2p-1}, e_{2q-1}) = 0$$

for every positive integers p, q. Further, we obtain that

$$A_1(e_{2p}, e_{2q-1}) = \begin{cases} e_{2q-1} & \text{if } p \geq q, \\ 0 & \text{if } p < q \end{cases}$$

and

$$A_2(e_{2p}, e_{2q-1}) = \begin{cases} e_{2p} & \text{if } p \geq q, \\ 0 & \text{if } p < q. \end{cases}$$

Let α and β be weak-star limit points in $\ell_1^{**} \setminus \ell_1$ of the sets $\{e_{2k-1} : k \in \mathsf{N}\}$ and $\{e_{2k} : k \in \mathsf{N}\}$, respectively. It follows immediately from the above that

$$\overline{A_1}(e_{2q-1}, \alpha) = \overline{A_1}(e_{2p}, \alpha) = \overline{A_1}(e_{2p}, \beta) = 0,$$
$$\overline{A_2}(e_{2q-1}, \alpha) = \overline{A_2}(e_{2p}, \alpha) = \overline{A_2}(e_{2p}, \beta) = 0,$$
$$\overline{A_1}(e_{2q-1}, \beta) = e_{2q-1},$$
$$\overline{A_2}(e_{2q-1}, \beta) = \beta$$

for every positive integers p and q. By taking limits we have that

$$\overline{A_1}(\alpha,\alpha) = \overline{A_1}(\beta,\alpha) = \overline{A_1}(\beta,\beta) = 0,$$
$$\overline{A_2}(\alpha,\alpha) = \overline{A_2}(\beta,\alpha) = \overline{A_2}(\beta,\beta) = 0,$$
$$\overline{A_1}(\alpha,\beta) = \alpha,$$
$$\overline{A_2}(\alpha,\beta) = \beta,$$

which implies that

$$\overline{A}(\alpha,\alpha) = \overline{A}(\beta,\beta) = \overline{A}(\beta,\alpha) = 0, \qquad \overline{A}(\alpha,\beta) = \alpha+\beta.$$

A simple computation shows that

$$\begin{aligned} &\overline{P}(\alpha+\beta) = \overline{A}(\alpha+\beta,\alpha+\beta) = \alpha+\beta, \\ &\overline{A}(e_{2q-1}+e_{2p},\alpha+\beta) = e_{2q-1}+\beta, \end{aligned} \tag{4}$$

for every positive integers p and q. Therefore, it is clear that $(\overline{P}\circ\overline{P})(\alpha+\beta) = \overline{P}(\alpha+\beta) = \alpha+\beta$. However, it can be computed that $\overline{P\circ P}(\alpha+\beta) = \frac{5}{3}(\alpha+\beta)$. Indeed, let (x_μ) be a net in X converging weak-star to $(\alpha+\beta)$ such that each x_μ is of the form $(e_{2q-1}+e_{2p})$. Let C be the bounded symmetric 4-linear mapping associated to $Q\circ P$. Then

$$\begin{aligned} C(x_1,x_2,x_3,x_4) = \frac{1}{3}\big[&A\big(A(x_1,x_2),A(x_3,x_4)\big) \\ &+ A\big(A(x_1,x_3),A(x_2,x_4)\big) + A\big(A(x_1,x_4),A(x_2,x_3)\big)\big]. \end{aligned}$$

Let $x_\mu^j = x_\mu$ for $j=1,2,3,4$. We also write each form of x_μ^j as $(e^j_{2q-1}+e^j_{2p})$ if necessary. Since (x_μ) converges weak-star to $\alpha+\beta$, we have

$$\overline{P\circ P}(\alpha+\beta) = (w^*-\lim)_{x_\mu^1}\cdots(w^*-\lim)_{x_\mu^4} C(x_\mu^1,x_\mu^2,x_\mu^3,x_\mu^4).$$

The computation of the limit is as follows:

$$\begin{aligned} &(w^*-\lim)_{x_\mu^1}\cdots(w^*-\lim)_{x_\mu^4} A\big(A(x_\mu^1,x_\mu^2),A(x_\mu^3,x_\mu^4)\big) \\ &\quad = (\overline{P}\circ\overline{P})(\alpha+\beta) \\ &\quad = \alpha+\beta, \end{aligned} \tag{1}$$

$$
\begin{aligned}
&(w^* - \lim)_{x_\mu^1} \cdots (w^* - \lim)_{x_\mu^4} A\big(A(x_\mu^1, x_\mu^3), A(x_\mu^2, x_\mu^4)\big) \\
&\quad = (w^* - \lim)_{x_\mu^1} (w^* - \lim)_{x_\mu^2} \overline{A}\big(\overline{A}(x_\mu^1, \alpha + \beta), \overline{A}(x_\mu^2, \alpha + \beta)\big) \\
&\quad = (w^* - \lim) x_\mu^1 (w^* - \lim)_{x_\mu^2} \overline{A}(e_{2q-1}^1 + \beta, e_{2q-1}^2 + \beta) \\
&\quad = 2(\alpha + \beta),
\end{aligned} \tag{2}
$$

and

$$
\begin{aligned}
&(w^* - \lim)_{x_\mu^1} \cdots (w^* - \lim)_{x_\mu^4} A\big(A(x_\mu^1, x_\mu^4), A(x_\mu^2, x_\mu^3)\big) \\
&\quad = (w^* - \lim)_{x_\mu^1} \cdots (w^* - \lim)_{x_\mu^4} A\big(A(x_\mu^2, x_\mu^3), A(x_\mu^1, x_\mu^4)\big) \\
&\quad = (w^* - \lim)_{x_\mu^1} \overline{A}\big(\overline{P}(\alpha + \beta), \overline{A}(x_\mu^1, \alpha + \beta)\big) \\
&\quad = (w^* - \lim)_{x_\mu^1} \overline{A}(\alpha + \beta, e_{2q-1}^1 + \beta) \\
&\quad = 2(\alpha + \beta).
\end{aligned} \tag{3}
$$

Therefore, $\overline{P \circ P}(\alpha + \beta) = \frac{5}{3}(\alpha + \beta)$.

The above example solves our main question in the negative, but the presentation given here is not our original point of view. Actually we found it by a more general mathematical tool, that is, the next lemma.

LEMMA 2.5. *Given two bounded 2-homogeneous polynomials $P \in \mathcal{P}(^2X, Y)$ and $Q \in \mathcal{P}(^2Y, Z)$, let A and B be the bounded symmetric bilinear mappings associated to P and Q, respectively. Then*

$$\overline{Q \circ P} = \overline{Q} \circ \overline{P}$$

if and only if $\overline{B}(\overline{P}(z), \overline{A}(x_\mu, z))$ converges weak-star to $\overline{Q} \circ \overline{P}(z)$ for every net $(x_\mu) \subset X$ converging weak-star to $z \in X^{**}$.

PROOF. By [7, Proposition 1.1], $\overline{Q \circ P} = \overline{Q} \circ \overline{P}$ holds if and only if the properties (a) and (b) stated at the beginning of the proof of Theorem 2.1 hold. We have that $\overline{A}(x, z) = \overline{A}(z, x)$ for all $x \in X$ and $z \in X^{**}$ and that $\overline{B}(y, u) = \overline{B}(u, y)$ for all $y \in Y$ and $u \in Y^{**}$. Hence it is easily checked that the property (a) holds always.

The bilinear mapping $S\overline{A} : X^{**} \times X^{**} \longrightarrow Y^{**}$ defined by $S\overline{A}(z_1, z_2) = \frac{1}{2}(\overline{A}(z_1, z_2) + \overline{A}(z_2, z_1))$ is the symmetrization of $\overline{A}$. If we consider $C : (X^{**})^4 \longrightarrow Z^{**}$ defined by $C(z_1, z_2, z_3, z_4) = \overline{B}(S(\overline{A})(z_1, z_2), S(\overline{A})(z_3, z_4))$ satisfies that $C(z, z, z, z) = \overline{Q} \circ \overline{P}(z)$ for all $z \in X^{**}$. Hence the 4-linear symmetric mapping associated to $\overline{Q} \circ \overline{P}$ is SC, the symmetrization of C. A straightforward calculation gives

$$
\begin{aligned}
&SC(z_1, z_2, z_3, z_4)\\
&\quad = \frac{1}{6}\Big(\overline{B}\big(S\overline{A}(z_1, z_2), S\overline{A}(z_3, z_4)\big) + \overline{B}\big(S\overline{A}(z_1, z_3), S\overline{A}(z_2, z_4)\big)\\
&\qquad + \overline{B}\big(S\overline{A}(z_1, z_4), S\overline{A}(z_2, z_3)\big) + \overline{B}\big(S\overline{A}(z_2, z_3), S\overline{A}(z_1, z_4)\big)\\
&\qquad + \overline{B}\big(S\overline{A}(z_2, z_4), S\overline{A}(z_1, z_3)\big) + \overline{B}\big(S\overline{A}(z_3, z_4), S\overline{A}(z_1, z_2)\big)\Big).
\end{aligned}
$$

Hence

$$
\begin{aligned}
D(\overline{Q} \circ \overline{P})(z)(x) &= 4SC(z, z, z, x)\\
&= 2\overline{B}(\overline{P}(z), S\overline{A}(z, x)) + 2\overline{B}(S\overline{A}(x, z), \overline{P}(z)),
\end{aligned}
$$

for all $x \in X$ and $z \in X^{**}$. As $S\overline{A}(z, x) = S\overline{A}(x, z) = \overline{A}(x, z)$ for all $x \in X$ and $z \in X^{**}$ we obtain that

$$
D(\overline{Q} \circ \overline{P})(z)(x) = 2(\overline{B}(\overline{P}(z), \overline{A}(x, z)) + \overline{B}(\overline{A}(x, z), \overline{P}(z))), \tag{5}
$$

for all $x \in X$ and $z \in X^{**}$. The linear mappings $\overline{B}(-, \overline{P}(z))$ and $\overline{A}(-, z)$ are $(w^\star, w^\star)$-continuous. Hence, given a net $(x_\mu) \subset X$ converging weak-star to $z \in X^{**}$ we have that the net $\overline{B}(\overline{A}(x_\mu, z), \overline{P}(z)))$ converges to $\overline{Q} \circ \overline{P}(z)$. Thus, by (5), the property (b) holds for $\overline{Q} \circ \overline{P}$ if and only if $\overline{B}(\overline{P}(z), \overline{A}(x_\mu, z))$ converges weak-star to $\overline{Q} \circ \overline{P}(z)$ for every net $(x_\mu) \subset X$ converging weak-star to $z \in X^{**}$.

In Proposition 2.3 we have shown, roughly speaking, that if the "size" of the image of $\overline{P}$ is "small", then the equality $\overline{Q \circ P} = \overline{Q} \circ \overline{P}$ holds even if the middle space Y is not symmetrically regular. The next example shows that even in the case $Z = \mathbb{C}$ we can find P and Q such that $\overline{Q \circ P} \neq \overline{Q} \circ \overline{P}$.

EXAMPLE 2.6. Define the bounded symmetric bilinear mappings $A : l_1 \times l_1 \to l_1$ by

$$
\begin{aligned}
A(x, y) = \sum_{n=1}^{\infty} &\big[(x_1 e_1 + x_3 e_3 + \cdots + x_{2n-1} e_{2n-1}) y_{2n}\\
&\quad + (y_1 e_1 + y_3 e_3 + \cdots + y_{2n-1} e_{2n-1}) x_{2n}\big]\\
&+ \sum_{n=1}^{\infty} \big[(x_1 + x_3 + \cdots + x_{2n-1}) y_{2n}\\
&\qquad + (y_1 + y_3 + \cdots + y_{2n-1}) x_{2n}\big] e_{2n},
\end{aligned}
$$

and $B : l_1 \times l_1 \to \mathsf{C}$

$$B(x, y) = \sum_{n=1}^{\infty} (x_1 + x_3 + \cdots + x_{2n-1}) y_{2n} + (y_1 + y_3 + \cdots + y_{2n-1}) x_{2n},$$

where $x = (x_i)$, $y = (y_i) \in l_1$ and $\{e_n\}$ is the canonical basis of l_1. Let P and Q be the 2-homogeneous polynomials from l_1 to l_1 associated to A and B, respectively. Then $\overline{Q \circ P} \neq \overline{Q} \circ \overline{P}$.

Proof. Clearly

$$B(e_{2p}, e_{2q}) = 0, \qquad B(e_{2p-1}, e_{2q-1}) = 0$$

for every positive integers p, q. Further, we obtain that

$$B(e_{2p}, e_{2q-1}) = \begin{cases} 1 & \text{if } p \geq q, \\ 0 & \text{if } p < q. \end{cases}$$

Let α and β be weak-star limit points in $\ell_l^{**} \setminus \ell_l$ of the sets $\{e_{2k-1} : k \in \mathsf{N}\}$ and $\{e_{2k} : k \in \mathsf{N}\}$, respectively. It follows immediately from the above that

$$\overline{B}(e_{2q-1}, \alpha) = \overline{B}(e_{2p}, \alpha) = \overline{B}(e_{2p}, \beta) = 0, \qquad \overline{B}(e_{2q-1}, \beta) = 1$$

for every positive integers p and q. By taking limits we have that

$$\overline{B}(\alpha, \alpha) = \overline{B}(\beta, \beta) = \overline{B}(\beta, \alpha) = 0, \qquad \overline{B}(\alpha, \beta) = 1.$$

Hence

$$\overline{Q}(\alpha + \beta) = \overline{B}(\alpha + \beta, \alpha + \beta) = 1, \qquad \overline{B}(\alpha + \beta, e_{2q-1} + \beta) = 2, \tag{6}$$

for every positive integer q.

Therefore, combining (4) and (6) we have that

$$(\overline{Q} \circ \overline{P})(\alpha + \beta) = \overline{Q}(\alpha + \beta) = 1$$

and

$$\overline{B}(\overline{P}(\alpha + \beta), \overline{A}(e_{2q-1} + e_{2p}, \alpha + \beta)) = 2,$$

for every positive integers p and q. Hence if (x_μ) is a net in X converging weak-star to $(\alpha + \beta)$ such that each x_μ is of the form $e_{2q-1} + e_{2p}$ we have that $\overline{B}(\overline{P}(\alpha + \beta), \overline{A}(x_\mu, \alpha + \beta))$ does not converge to $(\overline{Q} \circ \overline{P})(\alpha + \beta)$. By Lemma 2.5 we obtain that $\overline{Q \circ P} \neq \overline{Q} \circ \overline{P}$.

It is possible in the above example to proceed as in Example 2.4 to obtain that $\overline{Q \circ P}(\alpha + \beta) = 1$ but $\overline{Q} \circ \overline{P}(\alpha + \beta) = \frac{5}{3}$.

3. Numerical range of a holomorphic mapping

Let T be a bounded linear operator from a complex Banach space X into X. The numerical range of T is defined as

$$V(T) = \{\phi(Tx) : x \in S_X, \phi \in S_{X^*}, \phi(x) = 1\},$$

where S_X denotes the unit sphere of X ([6]). The numerical range for a holomorphic mapping was introduced by L. Harris [13]. We define the numerical range of $f \in \mathcal{H}_b(X, X)$ to be the set

$$V(f) = \{\phi(f(x)) : x \in S_X, \phi \in S_{X^*}, \phi(x) = 1\}.$$

The numerical ranges of multilinear mappings and polynomials have also been studied since 1996 ([1], [9]).

Bollobás [5] showed that $\mathrm{cl}(V(T)) = \mathrm{cl}(V(T^*))$, where T^* is the adjoint of T and $\mathrm{cl}(S)$ is the norm closure of the subset S of X. In the following we will prove that $\mathrm{cl}(V(f)) = \mathrm{cl}(V(\overline{f}))$ for $f \in \mathcal{H}_b(X, X)$.

THEOREM 3.1. *$\mathrm{cl}(V(f)) = \mathrm{cl}(V(\overline{f}))$ for $f \in \mathcal{H}_b(X, X)$.*

PROOF. Without loss of generality, we may assume that $\sup_{x\in B_X} \|f(x)\| \leq 1$. It is obvious that $\mathrm{cl}(V(f)) \subset \mathrm{cl}(V(\overline{f}))$. Thus it suffices to show that $V(\overline{f}) \subset \mathrm{cl}(V(f))$.

Suppose that $z \in S_{X^{**}}$, $\Psi \in S_{X^{***}}$ and $\Psi(z) = 1$. Hence $\Psi(\overline{f}(z)) \in V(\overline{f})$. By [10, Theorem 1], there is a net $(x_\alpha) \subset B_X$ such that (x_α) converges polynomial-star to z (i.e., $(P(x_\alpha))$ converges to $\overline{P}(z)$ for all scalar valued bounded polynomial P on X). Since

$$\liminf \|x_\alpha\| \geq \lim_\alpha |\phi(x_\alpha)| = |\overline{\phi}(z)| = |z(\phi)|$$

for all $\phi \in S_{X^*}$, we have that $\lim_\alpha \|x_\alpha\| = 1$. Set $y_\alpha = \frac{x_\alpha}{\|x_\alpha\|}$. Since

$$\lim_\alpha Q(y_\alpha) = \lim_\alpha \frac{1}{\|x_\alpha\|^k} Q(x_\alpha) = \overline{Q}(z)$$

for every $Q \in \mathcal{P}(^kX)$ and every positive integer k, the net (y_α) converges polynomial-star to z.

Let $\varepsilon > 0$ be given. Since f is uniformly continuous on B_X, there exists $\delta > 0$ such that $\|f(x) - f(y)\| \leq \frac{\varepsilon}{3}$ if $\|x - y\| \leq \delta$ and $x, y \in B_X$. Choose $0 < \varepsilon_0 < \frac{1}{2}$ so that $\varepsilon_0 + {\varepsilon_0}^2 < \delta$, and $3\varepsilon_0 \leq \epsilon$. As B_{X^*} is $w(X^{***}, X^{**})$-dense in $B_{X^{***}}$, considering two elements z and $\overline{f}(z)$ in X^{**} there exists $\varphi \in B_{X^*}$ such that

$$|\overline{\varphi}(z) - \Psi(z)| = |\overline{\varphi}(z) - 1| < \frac{\varepsilon_0^2}{4}$$

and

$$\left|\overline{\varphi}(\overline{f}(z)) - \Psi(\overline{f}(z))\right| < \frac{\varepsilon_0^2}{12},$$

which implies that $1 - \frac{\varepsilon_0^2}{4} < \|\varphi\| \leq 1$. Set $\psi = \frac{\varphi}{\|\varphi\|}$. We have

$$\begin{aligned}\left|\overline{\psi}(z) - 1\right| = \left|\frac{\overline{\varphi}}{\|\varphi\|}(z) - 1\right| &\leq \left|\frac{\overline{\varphi}}{\|\varphi\|}(z) - \overline{\varphi}(z)\right| + |\overline{\varphi}(z) - 1| \\ &\leq (1 - \|\varphi\|) + \frac{\varepsilon_0^2}{4} < \frac{\varepsilon_0^2}{2},\end{aligned}$$

and similarly,

$$\left|\Psi(\overline{f}(z)) - \overline{\psi}(\overline{f}(z))\right| < \frac{\varepsilon_0}{3}.$$

As (y_α) converges polynomial-star to z, we have that

$$1 - \psi(y_\alpha) \to 1 - \overline{\psi}(z) \qquad \text{and} \qquad \psi \circ f(y_\alpha) \to \overline{\psi \circ f}(z).$$

Hence we can choose $y_0 := y_{\alpha_0}$ such that

$$\left|\overline{\psi \circ f}(z) - \psi(f(y_0))\right| < \varepsilon_0/3 \qquad \text{and} \qquad |1 - \psi(y_0)| < {\varepsilon_0}^2/2.$$

By [5, Theorem 1], there exist $y \in S_X$ and $\phi \in S_{X^*}$ such that $\phi(y) = 1$, $\|\psi - \phi\| < \varepsilon_0$ and $\|y - y_0\| < \varepsilon_0 + {\varepsilon_0}^2$. By the construction of the Aron-Berner extension $\overline{f}$ it is easily checked that $\overline{\psi \circ f} = \overline{\psi} \circ \overline{f}$, and it follows that

$$\begin{aligned}&\left|\Psi(\overline{f}(z)) - \phi(f(y))\right| \\ &\quad \leq \left|\Psi(\overline{f}(z)) - \overline{\psi}(\overline{f}(z))\right| + \left|\overline{\psi}(\overline{f}(z)) - \psi(f(y_0))\right| \\ &\qquad + \left|\psi(f(y_0)) - \phi(f(y_0))\right| + \left|\phi(f(y_0)) - \phi(f(y))\right| \\ &\quad \leq \frac{\varepsilon_0}{3} + \frac{\varepsilon_0}{3} + \|\psi - \phi\| \|f(y_0)\| + \|\phi\| \|f(y_0) - f(y)\| \\ &\quad \leq \frac{2}{3}\varepsilon_0 + \varepsilon_0 + \frac{\varepsilon}{3} < \varepsilon,\end{aligned}$$

which implies that $\Psi(\overline{f}(z)) \in \mathrm{cl}(V(f))$, because $\phi(f(y)) \in V(f)$.

COROLLARY 3.2 ([8, Corollary 2.14]). *Let* $P \in \mathscr{P}(^m X, X)$. *Then* $\mathrm{cl}(V(\overline{P})) = \mathrm{cl}(V(P))$, *where* $\overline{P}$ *denotes the Aron-Berner extension of* P.

During the preparation of an earlier draft of this paper we became aware that in [1, Lemma 3] the above corollary had been proved for the case $P(x) = x_1^*(x) \dots x_m^*(x)$, where $x_j^* \in X^*$, $j = 1, \dots, m$. We also want to thank María

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