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ON BANACH IDEALS SATISFYING $c_0(\mathscr{A}(X,Y)) = \mathscr{A}(X, c_0(Y))$

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Abstract

We characterize Banach ideals $[\mathscr{A}, a]$ satisfying the equality $c_0(\mathscr{A}(X,Y)) = \mathscr{A}(X, c_0(Y))$ for all Banach spaces X and Y. Among other results we have proved that $\mathscr{K}$ (the normed operator ideal of all compact operators with the operator norm) is the only injective Banach ideal satisfying the equality.

1. Introduction

Let X and Y be Banach spaces. If I is an arbitrary index set, we denote by $\ell^\infty(I,Y)$ the Banach space of all bounded Y-valued functions defined on I, endowed with the supremum norm $\big(\|f\| = \sup\{\|f_i\| : i \in I\}$ for each $f = (f_i)_{i\in I} \in \ell^\infty(I,Y)\big)$. By $\ell^\infty_c(I,Y)$ (respectively, $c_0(I,Y)$) we mean the subspace of $\ell^\infty(I,Y)$ consisting of the functions with relatively compact range (respectively, converging to zero). As usual, we write $\ell^\infty(Y)$ (respectively, $c_0(Y)$) instead of $\ell^\infty(\mathsf{N},Y)$ (respectively, $c_0(\mathsf{N},Y)$) and $\ell^\infty(I)$ instead of $\ell^\infty(I,\mathsf{R})$.

If $\mathscr{L}(X,Y)$ is the space of bounded linear maps $T\colon X \longrightarrow Y$ and M is a bounded subset of $\mathscr{L}(X,Y)$, we can consider the operator $V\colon X \longrightarrow \ell^\infty(M,Y)$ defined by $Vx = (Tx)_{T\in M}$. When the set $M = (T_n)$ is a null sequence for the strong operator topology (in short, SOT), the values of the operator V lie in $c_0(Y)$, so we can define $U\colon X \longrightarrow c_0(Y)$ by $Ux = (T_n x)$.

There are many areas in which it is useful to elucidate whether the operators U or V belong to a certain Banach operator ideal. For example, the weak compactness of the operator U has turned out to be very helpful in the theory of multilinear operators (see [1, proposition 1.6], [7, theorem 5] and [8, theorem 3.5]); the well known Ryan's lemma [5, p. 375] provides a characterization in that case. Moreover, several authors have made use of the map V when it acts as a completely continuous operator. For instance, G. Emmanuele has obtained a characterization of Banach spaces not containing a copy of ℓ^1

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(see [2, theorem 2]) and T. Leavelle has deduced when a Banach space has the Reciprocal Dunford-Pettis property (see [3]).

Let us recall that a subset M of $\mathscr{K}(X, Y)$ (the space of all compact operators from X into Y) is *collectively compact* if the set $\{Tx : \|x\| \leq 1, T \in M\}$ is relatively compact. In [6], the notion of equicompactness for a subset M of $\mathscr{K}(X, Y)$ is introduced and it is proved that $M \subset \mathscr{K}(X, Y)$ is equicompact (that is, there exists a null sequence (x_n^*) in X^* such that $\|Tx\| \leq \sup_n |\langle x_n^*, x\rangle|$ for all $x \in X$ and all $T \in M$) iff the operator V is compact ([6, proposition 2.2]). Since M is equicompact iff $M^* = \{T^* : T \in M\}$ is collectively compact ([6, p. 689]), we have been able to obtain the following result using Palmer's criteria for relatively compact subsets of $\mathscr{K}(X, Y)$ ([4, theorem 2.2]):

PROPOSITION 1.1. *The following properties hold:*

(1) *If X and Y are Banach spaces and $M \subset \mathscr{K}(X, Y)$ is a bounded set, then M is relatively compact in $\mathscr{K}(X, Y)$ if and only if the operator*

$$V: x \in X \longmapsto (Tx)_{T \in M} \in \ell_c^\infty(M, Y)$$

is well defined and compact.

(2) *$\mathscr{K}(X, c_0(Y)) = c_0(\mathscr{K}(X, Y))$ (isometrically), for all Banach spaces X and Y.*

The aim of this paper is to study what happens when we replace the Banach ideal $[\mathscr{K}, \|\cdot\|]$ by an arbitrary Banach ideal in proposition 1.1. Thus, we consider the following definitions:

DEFINITION 1.2. Let $[\mathscr{A}, a]$ be a Banach ideal. We say that $[\mathscr{A}, a]$ has the *property* (P) if, for all Banach spaces X and Y, the relatively compact subsets of $\mathscr{A}(X, Y)$ are those bounded subsets M for which the operator

$$V: x \in X \longmapsto (Tx)_{T \in M} \in \ell_c^\infty(M, Y)$$

is well defined and belongs to $\mathscr{A}(X, \ell_c^\infty(M, Y))$.

DEFINITION 1.3. We say that $[\mathscr{A}, a]$ has the *property* (P_0) if the equality

$$\mathscr{A}(X, c_0(Y)) = c_0(\mathscr{A}(X, Y))$$

holds for all Banach spaces X and Y.

In section 2, we study a characterization of the property (P_0). Section 3 is devoted to making clear the relationship between the properties (P_0) and (P), as well as to describing arbitrary Banach ideals enjoying those properties. We show that the injectivity of the ideal plays a crucial role in this theory. In fact, theorem 3.4 proves that, among the injective Banach ideals, $[\mathscr{K}, \|\cdot\|]$ is the only one enjoying both properties (P) and (P_0). We conclude with some remarks and open problems.

Our notation is standard. B_X denotes the closed unit ball of X and X^* its topological dual. For a natural number n, $\ell_n^\infty(Y)$ is the space $(Y \times \overset{(n)}{\cdots} \times Y, \|\cdot\|_\infty)$. If $\{T_1, \dots, T_n\}$ is a finite subset of $\mathscr{L}(X, Y)$, we denote by $(T_k)_{k=1}^n$ the operator from X into $\ell_n^\infty(Y)$ defined by $(T_k)_{k=1}^n x = (T_1 x, \dots, T_n x)$ for all $x \in X$. Given a set $J \subset \mathbb{N}$, χ_J is the characteristic function of J. Now, if U is an operator defined as above, U_J is the operator from X into $c_0(Y)$ such that $U_J x = (\chi_J(n) \cdot T_n x)_n$ and $U_n = U_{\{1,\dots,n\}}$ for a fixed $n \in \mathbb{N}$.

Let us recall that an *operator ideal* $\mathscr{A}$ is a subclass of $\mathscr{L}$ (the class of all operators between arbitrary Banach spaces) such that, for each pair (X, Y) of Banach spaces, the component $\mathscr{A}(X, Y) = \mathscr{A} \cap \mathscr{L}(X, Y)$ is a linear space satisfying the following conditions:

(1) $x^* \otimes y \in \mathscr{A}(X, Y)$ for all $x^* \in X^*$ and $y \in Y$ ($x^* \otimes y : X \longrightarrow Y$ is defined by $(x^* \otimes y)(x) = \langle x^*, x \rangle y$).

(2) If $T \in \mathscr{L}(X_0, X)$, $S \in \mathscr{A}(X, Y)$ and $R \in \mathscr{L}(Y, Y_0)$ then $R \circ S \circ T \in \mathscr{A}(X_0, Y_0)$.

Let $\mathscr{A}$ be an operator ideal. The pair $[\mathscr{A}, a]$ is called a *Banach (operator) ideal* if a is a map from $\mathscr{A}(X, Y)$ into $\mathbb{R}^+$ and the following conditions are satisfied:

(1) $a(x^* \otimes y) = \|x^*\| \cdot \|y\|$ for all $x^* \in X^*$ and $y \in Y$.

(2) If $T \in \mathscr{L}(X_0, X)$, $S \in \mathscr{A}(X, Y)$ and $R \in \mathscr{L}(Y, Y_0)$ then

$$a(R \circ S \circ T) \leq \|R\| \cdot a(S) \cdot \|T\|.$$

(3) $[\mathscr{A}(X, Y), a]$ is a Banach space.

A Banach ideal $[\mathscr{A}, a]$ is *injective* if $a(i \circ T) = a(T)$ whenever X, Y and Z are Banach spaces, $i \in \mathscr{L}(Y, Z)$ is an isometry and $T \in \mathscr{A}(X, Y)$. Familiar examples of Banach ideals are $[\mathscr{L}, \|\cdot\|]$, the ideal of all bounded linear operators, or $[\mathscr{K}, \|\cdot\|]$, the ideal of all compact operators (here, $\|\cdot\|$ denotes the operator norm). These are examples of classical Banach ideals, that is, operator ideals supplied with the operator norm. We also work with the Banach ideal $[\mathscr{N}_\infty, \nu_\infty]$ of the ∞-nuclear operators.

2. The property (P_0)

We have studied the property (P_0) considering separately the inclusions $c_0(\mathscr{A}(X, Y)) \subset \mathscr{A}(X, c_0(Y))$ and $\mathscr{A}(X, c_0(Y)) \subset c_0(\mathscr{A}(X, Y))$. Then, we say that the ideal $[\mathscr{A}, a]$ has the *property* (P_{0r}) if the inclusion $c_0(\mathscr{A}(X, Y)) \subset$

$\mathscr{A}(X, c_0(Y))$ holds for all Banach spaces X and Y. In the same way, $[\mathscr{A}, a]$ has the *property* (P_{0l}) if the inclusion $\mathscr{A}(X, c_0(Y)) \subset c_0(\mathscr{A}(X,Y))$ holds for all Banach spaces X and Y.

THEOREM 2.1. *The Banach ideal $[\mathscr{A}, a]$ has the property (P_{0r}) if and only if there exists a positive constant C such that*

$$a((T_k)_{k=1}^n) \leq C \cdot \sup\{a(T_k) : 1 \leq k \leq n\}, \tag{1}$$

regardless of the Banach spaces X and Y and the finite set $\{T_1, \dots, T_n\} \subset \mathscr{A}(X,Y)$.

PROOF. Given a null sequence (T_n) in $\mathscr{A}(X,Y)$, we prove that the operator $U: x \in X \longmapsto (T_n x) \in c_0(Y)$ belongs to $\mathscr{A}(X, c_0(Y))$. Notice that

$$a(U_n - U_m) = a\big((T_k)_{k=m+1}^n\big)$$

when $n > m$. Then, by hypothesis, we have

$$a(U_n - U_m) \leq C \cdot \sup\{a(T_k) : m < k \leq n\},$$

so (U_n) is a Cauchy sequence in the Banach space $[\mathscr{A}(X, c_0(Y)), a]$ and, therefore, (U_n) is a-convergent. As $U = \lim_{n\to\infty} U_n$ for the operator norm, it must be $U = a\text{-}\lim_{n\to\infty}(U_n)$ and then $U \in \mathscr{A}(X, c_0(Y))$.

Conversely, suppose that $[\mathscr{A}, a]$ is a Banach ideal satisfying the property (P_{0r}). We have to find a positive constant C for which the inequality (1) holds, regardless of the Banach spaces X and Y and the finite subset $\{T_1, \dots, T_n\}$ of $\mathscr{A}(X,Y)$. By contradiction, for each $n \in \mathsf{N}$ there exist Banach spaces X_n and Y_n and operators $T_1^n, \dots, T_{p(n)}^n \in \mathscr{A}(X_n, Y_n)$ such that $\sup\{a(T_k^n) : 1 \leq k \leq p(n)\} = 1$ and

$$a\big((T_k^n)_{k=1}^{p(n)}\big) \geq n^2. \tag{2}$$

Put $X = \big(\sum_{n=1}^\infty X_n\big)_\infty$ and $Y = \big(\sum_{n=1}^\infty Y_n\big)_\infty$. Given $n, k \in \mathsf{N}$ so that $1 \leq k \leq p(n)$, consider the operator $S_k^n : X \longrightarrow Y$ defined by

$$S_k^n(x_m) = n^{-1}\left(\chi_{\{n\}}(m) T_k^n x_n\right)_m,$$

for all $(x_m) \in X$. It is a standard argument to show that $a(S_k^n) = n^{-1} a(T_k^n)$. So the sequence $(S_1^1, \dots, S_{p(1)}^1, S_1^2, \dots, S_{p(2)}^2, \dots)$ is null in $[\mathscr{A}(X,Y), a]$ and, by hypothesis, the operator

$$U: (x_n) \in X \longmapsto (S_1^1(x_m), \dots, S_{p(1)}^1(x_m), S_1^2(x_m), \dots, S_{p(2)}^2(x_m), \dots) \in c_0(Y)$$

belongs to $\mathscr{A}(X, c_0(Y))$. In particular, we have

$$a\big((S_k^n)_{k=1}^{p(n)}\big) \le a(U) \tag{3}$$

for all $n \in \mathsf{N}$. Since $a\big((S_k^n)_{k=1}^{p(n)}\big) = n^{-1}a((T_k^n)_{k=1}^{p(n)})$, the inequalities (3) and (2) lead to

$$a(U) \ge n$$

for all $n \in \mathsf{N}$, in contradiction to $U \in \mathscr{A}(X, c_0(Y))$.

The operator norm satisfies inequality (1). This yields the following result:

COROLLARY 2.2. *Every classical Banach ideal has the property* (P_{0r}).

COROLLARY 2.3. $[\mathscr{N}_\infty, \nu_\infty]$ *has the property* (P_{0r}).

PROOF. Given $T_1, \dots, T_N \in \mathscr{N}_\infty(X, Y)$ and $\varepsilon > 0$, for each $n \le N$ we can choose operators $A_n \in \mathscr{K}(X, c_0)$ and $B_n \in \mathscr{L}(c_0, Y)$ such that $T_n = B_n \circ A_n$, $\|A_n\| = 1$ and $\|B_n\| < \varepsilon + \nu_\infty(T_n)$. Then, consider the operators $A \in \mathscr{K}(X, c_0)$ and $B \in \mathscr{L}(c_0, \ell_N^\infty(Y))$ defined by

$$Ax = \big(\langle A_1x, e_1^*\rangle, \dots, \langle A_Nx, e_1^*\rangle, \langle A_1x, e_2^*\rangle, \dots \langle A_Nx, e_2^*\rangle, \dots\big)$$
$$B(\alpha_n)_n = \big(B_1(\alpha_{(n-1)\cdot N+1})_n, B_2(\alpha_{(n-1)\cdot N+2})_n, \dots, B_N(\alpha_{n\cdot N})_n\big)$$

for all $x \in X$ and $\alpha = (\alpha_n)_n \in c_0$ (here, $(e_k^*)_k$ is the unit vector basis of ℓ^1). Obviously, we have $T = (T_n)_{n=1}^N = B \circ A$ and

$$\nu_\infty(T) \le \|B\| \le \max_{n\le N} \|B_n\| < \max_{n\le N}(\nu_\infty(T_n) + \varepsilon)$$

for all $\varepsilon > 0$.

THEOREM 2.4. *If* $[\mathscr{A}, a]$ *is a Banach ideal, the following statements are equivalent:*

(a) $[\mathscr{A}, a]$ *has the property* (P_{0l}).

(b) *For all Banach spaces* X *and* Y, *the sequence* $(U - U_n)_n$ *is null in* $\mathscr{A}(X, c_0(Y))$ *whenever* $U \in \mathscr{A}(X, c_0(Y))$.

(c) *For all Banach spaces* X, Y *and* Z, *all operators* $T \in \mathscr{A}(X, Y)$ *and all SOT-null sequences* (S_n) *in* $\mathscr{L}(X, Y)$, *the sequence* $(S_n \circ T)$ *is null in* $\mathscr{A}(X, Y)$.

PROOF. In order to prove (a) ⇒ (b), suppose, by contradiction, that $U : x \in X \longmapsto (T_nx) \in c_0(Y)$ belongs to $\mathscr{A}(X, c_0(Y))$ but $(U - U_n)_n$ is not a null sequence in $\mathscr{A}(X, c_0(Y))$. Then, there exists $\varepsilon > 0$ and finite sets $J_n \subset \mathsf{N}$ such that $\max(J_n) < \min(J_{n+1})$ and

$$a(U_{J_n}) > \varepsilon \tag{4}$$

for all $n \in \mathsf{N}$. For each finite set $J \subset \mathsf{N}$, let us consider the operator ϕ_J from $c_0(Y)$ into $c_0(Y)$ defined by $\phi_J(y_k)_k = (\chi_J(k) \cdot y_k)_k$ and define $\phi = (\phi_{J_n})_n$. Put $Y_0 = c_0(Y)$ and consider the operator $\widehat{U}: x \in X \longmapsto (U_{J_n}x) \in c_0(Y_0)$. It follows from the equality $\widehat{U} = \phi \circ U$ that the operator $\widehat{U}$ belongs to $\mathscr{A}(X, c_0(Y_0))$. Hence, $\lim_n a(U_{J_n}) = 0$, which is in contradiction to (4).

Now, let us show (b) $\Rightarrow$ (c). Given $T \in \mathscr{A}(X,Y)$ and a SOT-null sequence (S_n) in $\mathscr{L}(Y,Z)$, we consider the operator $U: x \in X \longmapsto (S_n(Tx))_n \in c_0(Z)$. Since $U \in \mathscr{A}(X, c_0(Z))$, it follows that $\lim_n a(U - U_n) = 0$, so

$$\lim_n a(S_n \circ T) = \lim_n a(U_n - U_{n-1}) = 0.$$

Finally, suppose that the operator $U: x \in X \longmapsto (T_n)_n \in c_0(Y)$ belongs to $\mathscr{A}(X, c_0(Y))$. For each $n \in \mathsf{N}$, we consider $S_n: (y_k)_k \in c_0(Y) \longmapsto y_n \in Y$. In view of (c), we can ensure that

$$\lim_n a(T_n) = \lim_n (S_n \circ U) = 0$$

and (a) is proved from (c).

Since $[\mathscr{K}, \|\cdot\|]$ enjoys the property (P_0) (proposition 1.1) we have:

COROLLARY 2.5. *If* $[\mathscr{A}, a]$ *is a classical normed ideal contained in* $[\mathscr{K}, \|\cdot\|]$, *then* $[\mathscr{A}, a]$ *has the property* (P_{0l}).

COROLLARY 2.6. $[\mathscr{N}_\infty, \nu_\infty]$ *has the property* (P_{0l}).

PROOF. Let us consider $T \in \mathscr{N}_\infty(X,Y)$ and (S_n) a SOT-null sequence in $\mathscr{L}(Y,Z)$. Take an ∞-nuclear representation $T = \sum_m x_m^* \otimes y_m$, where (y_m) is an unconditionally summable sequence. Then, $S_n \circ T = \sum_m x_m^* \otimes S_n y_m$ and

$$\nu_\infty(S_n \circ T) \leq \left(\sup_m \|x_m^*\|\right) \cdot \sup_{\|y^*\| \leq 1} \sum_m |\langle y^*, S_n y_m \rangle|.$$

Hence, it is easy to show that $\lim_n \nu_\infty(S_n \circ T) = 0$.

3. Relationship between the properties (P_0) and (P)

We say that a Banach ideal $[\mathscr{A}, a]$ has the *property* (P_r) if, for all Banach spaces X and Y, the operator $V: x \in X \longmapsto (Tx)_{T \in M} \in \ell_c^\infty(M, Y)$ is well defined and belongs to $\mathscr{A}(X, \ell_c^\infty(M, Y))$ whenever the set $M \subset \mathscr{A}(X,Y)$ is relatively compact. We say that $[\mathscr{A}, a]$ enjoys the *property* (P_l) if, whenever the operator V belongs to $\mathscr{A}(X, \ell_c^\infty(M, Y))$, the set M is relatively compact regardless of the Banach spaces X and Y. We have found the following relations:

PROPOSITION 3.1. *Let $[\mathscr{A}, a]$ be a Banach ideal. The following hold:*

(a) *If $[\mathscr{A}, a]$ has the property (P_l), then it has the property (P_{0l}).*

(b) *If $[\mathscr{A}, a]$ has the property (P), then it has the property (P_0).*

(c) *If $[\mathscr{A}, a]$ has the property (P_{0r}), then it has the property (P_r).*

PROOF. Suppose that the operator $U: x \in X \longmapsto (T_n x) \in c_0(Y)$ belongs to $\mathscr{A}(X, c_0(Y))$ and let us denote by i the inclusion map from $c_0(Y)$ into $\ell_c^\infty(Y)$. As $i \circ U \in \mathscr{A}(X, \ell_c^\infty(Y))$, the set $\{T_n : n \in \mathsf{N}\}$ is relatively compact in $\mathscr{A}(X, Y)$. Since $\lim_n \|T_n x\| = 0$ for all $x \in X$, it follows that $\lim_n a(T_n) = 0$. This proves (a).

To show (b), suppose $[\mathscr{A}, a]$ enjoys the property (P). In view of the statement (a), we only need to prove that $[\mathscr{A}, a]$ has the property (P_{0r}). By contradiction, consider Banach spaces X and Y and a null sequence (T_n) in $\mathscr{A}(X, Y)$ so that the operator $U: x \in X \longrightarrow (T_n x) \in c_0(Y)$ does not belong to $\mathscr{A}(X, c_0(Y))$. An appeal to theorem 2.4 tells us that the sequence (U_n) is not convergent in $\mathscr{A}(X, c_0(Y))$. So, there exist $\varepsilon > 0$ and finite sets $J_n \subset \mathsf{N}$ such that $\max(J_n) < \min(J_{n+1})$ and

$$a(U_{J_n}) > \varepsilon \tag{5}$$

for all $n \in \mathsf{N}$. As in the proof of theorem 2.4, for each finite set $J \subset \mathsf{N}$ we consider the operator ϕ_J from $\ell_c^\infty(Y)$ into $c_0(Y)$ and we put $\phi = (\phi_{J_n})_n$. Since $\mathscr{A}$ has the property (P_r), the operator $V: x \in X \longmapsto (T_n x) \in \ell_c^\infty(Y)$ belongs to $\mathscr{A}(X, \ell_c^\infty(Y))$, so does $\phi \circ V$. Now, $\mathscr{A}$ has the property (P_l), so the set $\{U_{J_n} : n \in \mathsf{N}\}$ is relatively compact in $\mathscr{A}(X, c_0(Y))$. Take a convergent subsequence $(U_{J_{k(n)}})_n$ for the norm of $\mathscr{A}(X, c_0(Y))$; as $(U_{J_n})_n$ is null for the operator norm, we must have

$$\lim_n a(U_{J_{k(n)}}) = 0,$$

and this is in contradiction to (5).

Finally, let M be a relatively compact subset of $\mathscr{A}(X, Y)$ and let us show that the operator $V: x \in X \longmapsto (Tx)_{T \in M} \in \ell_c^\infty(M, Y)$ belongs to $\mathscr{A}(X, \ell_c^\infty(M, Y))$. Take a null sequence (T_n) in $\mathscr{A}(X, Y)$ so that $M \subset \overline{\text{aco}}\,(T_n) = \left\{\sum_n \alpha_n T_n : (\alpha_n) \in B_{\ell^1}\right\}$. For each $T \in M$, choose $(\alpha_n^T) \in B_{\ell^1}$ such that $T = \sum_n \alpha_n^T T_n$ and define the operators $U: X \longrightarrow c_0(Y)$ and $\tilde{i}: c_0(Y) \longrightarrow \ell_c^\infty(M, Y)$ by $Ux = (T_n x)$ for all $x \in X$ and $\tilde{i}(y_n) = \left(\sum_n \alpha_n^T y_n\right)_{T \in M}$ for all $(y_n) \in c_0(Y)$. Since $\mathscr{A}$ has the property (P_{0r}), the operator U belongs to $\mathscr{A}(X, c_0(Y))$, so $V = \tilde{i} \circ U$ belongs to $\mathscr{A}(X, \ell_c^\infty(M, Y)$.

PROPOSITION 3.2. *Let $[\mathscr{A}, a]$ be a Banach ideal. The following hold:*

(a) *If $[\mathscr{A}, a]$ has the property (P_l), then $\mathscr{A} \subset \mathscr{K}$.*

(b) *If $[\mathscr{A}, a]$ has the property (P_{0r}), then $\mathscr{N}_\infty \subset \mathscr{A}$.*

PROOF. Given $T \in \mathscr{A}(X, Y)$, let us denote by $j_Y \colon Y \longrightarrow \ell^\infty(B_{Y^*})$ the canonical isometry $j_Y(y) = (\langle y^*, y\rangle)_{y^* \in Y^*}$. The map $j_Y \circ T$ belongs to $\mathscr{A}((X, \ell^\infty(B_{Y^*}))$, so the property (P_l) tells us that the set $\{T^* y^* : y^* \in B_{Y^*}\}$ is relatively compact in $\mathscr{A}(X, \mathsf{R}) = X^*$.

In order to show (b), first notice that an operator $T \in \mathscr{L}(X, c_0)$ is compact iff $\lim_n \|T^* e_n^*\| = 0$; this yields the equality $\mathscr{N}_\infty(X, c_0) = \mathscr{K}(X, c_0)$ regardless of the Banach space X. Since the Banach ideal $\mathscr{A}$ enjoys the property (P_{0r}), we have $\mathscr{K}(X, c_0) \subset \mathscr{A}(X, c_0)$, so the statement (b) is proved when $Y = c_0$. Now, for an arbitrary Banach space Y, it suffices to have in mind that every ∞-nuclear operator admits a factorization $T = A \circ B$, where A is a compact map from X into c_0 and B an operator from c_0 into Y. Hence, the ideal property produces the inclusion $\mathscr{N}_\infty(X, Y) \subset \mathscr{A}(X, Y)$.

When the Banach ideal is injective, a further result can be deduced:

PROPOSITION 3.3. *Let $[\mathscr{A}, a]$ be an injective Banach ideal. The following hold:*

(a) *If $[\mathscr{A}, a]$ has the property (P_r), then it has the property (P_{0r}).*

(b) *If $[\mathscr{A}, a]$ has the property (P_{0l}), then $\mathscr{A} \subset \mathscr{K}$.*

(c) *If $[\mathscr{A}, a]$ has the property (P_{0l}), then it has the property (P_l).*

PROOF. (a) is evident since $\mathscr{A}$ is injective.

Given an operator T belonging to $\mathscr{A}(X, Y)$, we show that $T \in \mathscr{K}(X, Y)$ in two steps. First, suppose that the Banach space Y is separable. According to theorem 2.4, if (y_n^*) is a weak* null sequence, then $(y_n^* \circ T) = (T^* y_n^*)$ is null in $\mathscr{A}(X, \mathsf{R}) = \mathscr{L}(X, \mathsf{R})$ (isometrically). As B_{Y^*} is weak* sequentially compact, it follows that T^* is compact. Now, for arbitrary Banach spaces X and Y, consider a sequence (x_n) in B_X and put $X_0 = \overline{\text{span}}\{x_n : n \in \mathsf{N}\}$ and $Y_0 = \overline{\text{span}}\{T x_n : n \in \mathsf{N}\}$. If i denotes the inclusion map from X_0 into X, then $T \circ i$ belongs to $\mathscr{A}(X_0, Y)$. Since $(T \circ i)(X_0) \subset Y_0$ and $\mathscr{A}$ is injective, it follows that $T \circ i$ belongs to $\mathscr{A}(X_0, Y_0)$ viewed as an operator from X_0 into Y_0. The first part of the proof allows to deduce that such an operator is compact and, therefore, T is compact too.

Finally, let us prove (c). If $M \subset \mathscr{A}(X, Y)$ is pointwise compact, we obtain that M is compact in $\mathscr{K}(X, Y)$ using (b) and the fact that the Banach ideal

$[\mathscr{K}, \|\cdot\|]$ enjoys the property (P_l). It is easy to deduce that M is compact in $\mathscr{A}(X, Y)$ thanks to the injectivity of $\mathscr{A}$.

In the following result, it is proved that $[\mathscr{K}, \|\cdot\|]$ is the only injective Banach ideal satisfying the property (P) (or (P_0)).

THEOREM 3.4. *Let $[\mathscr{A}, a]$ be an injective Banach ideal. The following statements are equivalent:*

(a) *$[\mathscr{A}, a]$ has the property (P).*

(b) *$[\mathscr{A}, a]$ has the property (P_0).*

(c) *$\mathscr{A} = \mathscr{K}$.*

PROOF. (a) $\Rightarrow$ (b) occurs even in the absence of injectivity of the ideal $\mathscr{A}$ (proposition 3.1). In order to show (b) $\Rightarrow$ (c), we invoke propositions 3.2 and 3.3 to obtain $\mathscr{N}_\infty \subset \mathscr{A} \subset \mathscr{K}$. Now, (c) is deduced since the injective hull of $\mathscr{N}_\infty$ is $\mathscr{K}$ and the Banach ideals $\mathscr{A}$ and $\mathscr{K}$ are injective. Finally, (c) $\Rightarrow$ (a) is contained in proposition 1.1 [6].

4. Final notes and open problems

We do not know if the property (P_0) implies the property (P) for arbitrary Banach ideals. Therefore the following question arises naturally:

QUESTION 1. If $[\mathscr{A}, a]$ has the property (P_0), has $[\mathscr{A}, a]$ necessarily the property (P)?

$[\mathscr{N}_\infty, \nu_\infty]$ is a noninjective Banach ideal having the property (P_0), as we have proved in section 2. Nevertheless, this ideal does not serve as a counter-example to give a negative answer to question 1:

PROPOSITION 4.1. *$[\mathscr{N}_\infty, \nu_\infty]$ has the property (P).*

PROOF. According to corollary 2.3 and proposition 3.1, we only have to prove that $[\mathscr{N}_\infty, \nu_\infty]$ has the property (P_l). Let M be a bounded subset of $\mathscr{N}_\infty(X, Y)$ such that the operator

$$V\colon x \in X \longmapsto (Tx)_{T\in M} \in \ell_c^\infty(M, Y)$$

belongs to $\mathscr{N}_\infty(X, Y)$. Then V admits a representation $V = \sum_m x_m^* \otimes \hat{y}^m$, where (x_m^*) is a null sequence in X^* and $(\hat{y}^m)$ is an unconditionally summable sequence in $\ell_c^\infty(M, Y)$. So, each operator $T \in M$ admits the representation $T = \sum_m x_m^* \otimes \hat{y}_T^m$. Hence, the set $H = \{(\hat{y}_T^m)_m : T \in M)\}$ is unconditionally summable uniformly for $T \in M$, that is to say, for every $\varepsilon > 0$ there exists $m_0 \in \mathbf{N}$ so that

$$\sum_{m\geq m_0} |\langle y^*, \hat{y}_T^m\rangle| < \varepsilon$$

for all $T \in M$ and all $y^* \in B_{Y^*}$. Now, consider the continuous linear map

$$\Phi\colon (y_m) \in \ell_u^1(Y) \longmapsto \sum_m x_m^* \otimes y_m \in \mathcal{N}_\infty(X, Y)$$

(here, $\ell_u^1(Y)$ denotes the space of the unconditionally summable sequences in Y) and notice that $M = \Phi(H)$, so M must be relatively compact.

By proposition 3.2, if $[\mathscr{A}, a]$ has the property (P), then we have $\mathcal{N}_\infty \subset \mathscr{A} \subset \mathscr{K}$. Both operators ideals $\mathcal{N}_\infty$ and $\mathscr{K}$ enjoy that property, but the following question has no answer yet:

QUESTION 2. If $[\mathscr{A}, a]$ is a Banach ideal satisfying $\mathcal{N}_\infty \subset \mathscr{A} \subset \mathscr{K}$, does $\mathscr{A}$ enjoy the property (P) or (P_0)?

Of course, the answer is affirmative for every classical Banach ideal contained in that interval. Indeed, if $[\mathscr{A}, a]$ is another Banach ideal satisfying $\mathcal{N}_\infty \subset \mathscr{A} \subset \mathscr{K}$ and $M \subset \mathscr{A}(X, Y)$ is such that the operator $V\colon x \in X \longmapsto (Tx)_{T\in M} \in \ell_c^\infty(M, Y)$ belongs to $\mathscr{A}(X, \ell_c^\infty(M, Y))$, then the operator

$$\widehat{V}\colon x \in X \longmapsto \big(j_Y(Tx)\big)_{T\in M} \in \ell_c^\infty(M, \ell^\infty(B_{Y^*}))$$

is compact, j_Y being the canonical isometry $j_Y(y) = (\langle y^*, y\rangle)_{y^*\in Y^*}$. Since $\mathscr{K}$ has the property (P), we deduce that the set $j_Y(M) = \{j_Y \circ T : T \in M\}$ is relatively compact in $\mathscr{K}(X, \ell^\infty(B_{Y^*}))$. The injectivity of the Banach space $\ell^\infty(B_{Y^*})$ leads to $\mathscr{K}(X, \ell^\infty(B_{Y^*})) = \mathscr{A}(X, \ell^\infty(B_{Y^*}))$. In other words, we have shown that $M \subset \mathscr{A}(X, Y)$ is relatively compact viewed as a subset of $\mathscr{A}(X, \ell^\infty(B_{Y^*}))$.

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