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APPROXIMATELY INNER DERIVATIONS

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(Dedicated to the memory of Gert K. Pedersen)

Abstract

Let α be an approximately inner flow on a C^*-algebra A with generator δ and let δ_n denote the bounded generators of the approximating flows $\alpha^{(n)}$. We analyze the structure of the set

$$\mathscr{D} = \{x \in D(\delta) : \lim_{n\to\infty} \delta_n(x) = \delta(x)\}$$

of pointwise convergence of the generators. In particular we examine the relationship of $\mathscr{D}$ and various cores related to spectral subspaces.

1. Introduction

The theory of flows on operator algebras has been largely motivated by models of quantum statistical mechanics in which the flow α is usually constructed as the limit of a sequence $(\alpha^{(n)})$ of local flows. The latter are typically given by inner one-parameter automorphism groups of the algebra A and the flow is correspondingly called approximately inner. In particular this is the situation for models of quantum spin systems in which the algebra of observables A is a UHF-algebra (see, for example, [9], Chapter 6). An early result of Sakai established [25] that if δ is the generator of a flow α on a UHF-algebra then there exists an increasing sequence (A_n) of finite dimensional C^*-subalgebras A_n of A such that $A_n \subset D(\delta)$ and $\bigcup_n A_n$ is dense in A. Furthermore, there exists a sequence (h_n) of elements h_n of the self-adjoint part A_{sa} of A such that $\delta|_{A_n} = \mathrm{ad}(ih_n)|_{A_n}$ (see [8], Example 3.2.25). If, in addition, $\bigcup_n A_n$ is a core for δ_α, i.e. $\bigcup_n A_n$ is dense in $D(\delta)$ in the graph norm, then it follows that α is approximately inner. In particular one has

$$\lim_{n\to\infty} \max_{|t|\le 1} \|\alpha_t(x) - \alpha_t^{(n)}(x)\| = 0 \tag{1}$$

for all $x \in A$ where $\alpha_t^{(n)} = e^{t\delta_n}$ with $\delta_n = \mathrm{ad}(ih_n)$. This is a consequence of two general results. First the strong convergence (1) of the semigroups is equivalent to strong convergence of the resolvents $(\iota \pm \delta_n)^{-1}$ to $(\iota \pm \delta)^{-1}$ by the

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Kato-Trotter theorem (see, for example, [14], Theorem IX.2.16). Explicitly (1) is equivalent to the condition

(2) $$\lim_{n\to\infty} \|(\iota \pm \delta_n)^{-1}(x) - (\iota \pm \delta)^{-1}(x)\| = 0$$

for all x in a norm-dense subspace of A. Secondly, since $\bigcup_n A_n \subset \mathscr{D}$, where

(3) $$\mathscr{D} = \{x \in D(\delta) : \lim_{n\to\infty} \delta_n(x) = \delta(x)\} ,$$

it follows that $\mathscr{D}$ is a core of δ. In particular the subspaces $(\iota \pm \delta)(\mathscr{D})$ are norm-dense in A. Then if $y \in \mathscr{D}$ and $x_\pm = (\iota \pm \delta)(y)$ one has

$$\lim_{n\to\infty} \|(\iota \pm \delta_n)^{-1}(x_\pm) - (\iota \pm \delta)^{-1}(x_\pm)\| \le \lim_{n\to\infty} \|(\delta_n - \delta)(y)\| = 0 .$$

Therefore the strong resolvent convergence (2) and the equivalent semigroup convergence (1) is established. Thus the crucial feature in this argument is the core property of $\mathscr{D}$. This follows automatically if $\bigcup_n A_n$ is a core.

Secondly, note that it does not follow in general from Sakai's construction that $\bigcup_n A_n$ can be taken to be a core for δ. This was an open problem for many years. The affirmative answer would imply that α is approximately inner, and this was known as the Powers-Sakai conjecture. The problem was finally resolved in the negative in 2000 (see [20], Theorem 1.1). The counterexample is, however, an AF-algebra which is not UHF and the problem still seems to be open for UHF-algebras.

The purpose of this note is to analyze the convergence (1) for flows $\alpha_t^{(n)} = e^{it\delta_n}$ and $\alpha_t = e^{it\delta}$ on a general C^*-algebra A by examining the structure of the set $\mathscr{D}$ defined by (3). In particular we consider the relation between $\mathscr{D}$ and various natural cores of δ.

2. Cores and spectral subspaces

Let α be a flow on a C^*-algebra A with generator δ_α. Define the spectral subspace $A^\alpha(K)$, for each closed subset K of R, as the Banach subspace spanned by the $x \in A$ for which the conditions $f \in L_1(\mathsf{R})$ and $\operatorname{supp} \hat{f} \cap K = \emptyset$ imply $\alpha_f(x) = 0$. (Here $\hat{f}$ denotes the Fourier transform of f.) The α-spectrum of $x \in A$, denoted by $\mathrm{Sp}_\alpha(x)$, is defined to be the smallest closed subset K of R such that $x \in A^\alpha(K)$. Elements with compact α-spectra are referred to as geometric elements of α and the subspace of geometric elements is denoted by A^α_G. Explicitly, $A^\alpha_G = \bigcup_{n=1}^\infty A^\alpha([-n, n])$. Note that each geometric element is automatically an entire analytic element of α. Moreover, A^α_G is a *-subalgebra of A and an α-invariant core of δ_α.

The spectrum $\mathrm{Sp}(\alpha)$ of α is defined to be the smallest closed subset K of R such that $A = A^\alpha(K)$. The Connes spectrum $\mathsf{R}(\alpha)$ of α is defined to be the intersection of $\mathrm{Sp}(\alpha|_B)$ with B all non-zero α-invariant hereditary C^*-algebras of A. It is known that $\mathsf{R}(\alpha)$ is a closed subgroup of R.

Our first result establishes under quite general conditions that the geometric elements of an approximately inner flow are not contained in the subspace $\mathscr{D}$ of convergence.

THEOREM 2.1. *Let A be a separable C^*-algebra and α an approximately inner flow with generator δ_α. Suppose there exists a faithful family of α-covariant irreducible representations of A and that $\mathsf{R}(\alpha) \neq \{0\}$. Let (h_n) be a sequence of self-adjoint elements of A such that*

$$\lim_{n\to\infty} \max_{|t|\leq 1} \|\alpha_t(x) - \mathrm{Ad}\, e^{ith_n}(x)\| = 0$$

for all $x \in A$ and let

$$\mathscr{D} = \left\{x \in D(\delta_\alpha) : \lim_{n\to\infty} \mathrm{ad}\, ih_n(x) = \delta_\alpha(x)\right\}. \tag{4}$$

Then

$$\mathscr{D} \not\supset A_G^\alpha.$$

PROOF. Under the assumptions of the theorem it follows that there is a faithful family of irreducible representations $(\pi, \mathscr{H}_\pi)$ of A such that the representation

$$\overline{\pi} = \int_{\mathsf{R}}^{\oplus} dt\, \pi \circ \alpha_t$$

of A on $L_2(\mathsf{R}; \mathscr{H}_\pi)$ is of type I with centre $L_\infty(\mathsf{R})$ if $\mathsf{R}(\alpha) = \mathsf{R}$ and $L_\infty(\mathsf{R}/((2\pi p)^{-1}\mathsf{Z})))$ (as a subalgebra of $L_\infty(\mathsf{R})$) if $\mathsf{R}(\alpha) = p\mathsf{Z}$. (This is Theorem 1.2 of [18] when A is prime. See Remark 2.3 when A is not prime.) Define a unitary flow U on $L_2(\mathsf{R}; \mathscr{H}_\pi)$ by $U_t\xi(s) = \xi(s+t)$. Then $U_t\overline{\pi}(x)U_t^* = \overline{\pi}(\alpha_t(x))$. We denote by $\overline{\alpha}_t$ the weakly continuous flow $t \mapsto \mathrm{Ad}\ U_t$ on $\overline{\pi}(A)''$.

Since A is separable, there is a countable faithful family of such irreducible representations. Let (π_i) be such a family. Let $q \in \mathsf{R}(\alpha)$ and define $z_i \in \overline{\pi}_i(A)'' \cap \overline{\pi}_i(A)' \subset L_\infty(\mathsf{R})$ by $z_i(t) = e^{2\pi iqt}$. Note that z_i satisfies $\overline{\alpha}_t(z_i) = e^{2\pi iqt}z_i$. Then there is a net (y_μ) in the unit ball of A such that $\overline{\pi}_i(y_\mu)$ converges to z_i in the $*$-strong topology for any i and $\pi(y_\mu)$ converges to 0 weakly for any representation π disjoint from all $\overline{\pi}_i$. We may further suppose that the α-spectrum of y_μ decreases to $\{q\}$. Since A is separable and the direct sum of $\overline{\pi}_i$ is a representation on a separable Hilbert space, one can choose a sequence (y_k) from the convex combinations of (y_μ) such that (y_k) is a central sequence, $\overline{\pi}_i(y_k)$ converges to z_i in the $*$-strong topology, and the α-spectrum of y_k

shrinks to $\{q\}$. Since $\overline{\pi}_i(y_k y_k^*)$ converges to the identity for any i and the direct sum of $\overline{\pi}_i$ is faithful, one may conclude that $\|xy_k\|$ converges to $\|x\|$ for any $x \in A$ as $k \to \infty$. We use this fact below.

Assume that $\mathscr{D} \supset A^\alpha([-\varepsilon, \varepsilon])$ for an $\varepsilon > 0$. Then it follows from the uniform boundedness theorem that there is constant $c > 0$ such that

$$\| \operatorname{ad} ih_m|_{A^\alpha([-\varepsilon,\varepsilon])}\| < c$$

for all m. Choose $x \in A^\alpha([q - \varepsilon/2, q + \varepsilon/2])$ such that $\|x\| = 1$ where $q \in \mathsf{R}(\alpha)$ satisfies $q > c + \varepsilon$. By the arguments in the previous paragraph one can find a central sequence (y_k) in A such that $\|y_k\| \le 1$, $\mathrm{Sp}_\alpha(y_k) \subset \langle -q - \varepsilon/2, -q+\varepsilon/2\rangle$ and $\overline{\pi}(y_k y_k^*)$ converges weakly to the identity as $k \to \infty$. Then $\|xy_k\| \le 1$, $\|xy_k\| \to 1$ and $\mathrm{Sp}_\alpha(xy_k) \subset \langle -\varepsilon, \varepsilon\rangle$. Hence $\| \operatorname{ad} ih_m(xy_k)\| < c$ and

$$\lim_{k\to\infty} \| \operatorname{ad} ih_m(xy_k) - \operatorname{ad} ih_m(x)y_k\| = 0 \quad .$$

Since $\lim_{m\to\infty} \| \operatorname{ad} ih_m(x) - \delta_\alpha(x)\| = 0$ and $\lim_{k\to\infty} \|\delta_\alpha(x)y_k\| = \|\delta_\alpha(x)\|$ it follows that $\| \operatorname{ad} ih_m(xy_k)\| > \|\delta_\alpha(x)\| - \varepsilon/2 \ge q - \varepsilon > c$ for all sufficiently large m and k. This contradicts the bound $\| \operatorname{ad} ih_m(xy_k)\| < c$. Therefore $\mathscr{D} \not\supset \bigcup_{n=1}^\infty A^\alpha([-n, n])$.

REMARK 2.2. The foregoing proof establishes a slightly stronger statement: If $\mathscr{D}$ contains $A^\alpha([-\varepsilon, \varepsilon])$ for some $\varepsilon > 0$, then $\mathscr{D} \cap A^\alpha([q-\varepsilon/2, q+\varepsilon/2]) = \{0\}$ for all large $q \in \mathsf{R}(\alpha)$.

REMARK 2.3. In Theorem 2.1 we assumed the condition

(i) there exists a faithful family of α-covariant irreducible representations of A.

The essential requirement is, however, a consequence of the assumption

(ii) there exists a faithful family $\{\pi_i\}$ of irreducible representations of A such that $\bar{\pi}_i$ is of type I and the spectrum of $\bar{\alpha}$ on the center $\bar{\pi}_i(A)'' \cap \bar{\pi}_i(A)'$, which we denote by $\Delta(\pi)$, is $\mathsf{R}(\alpha)$, where $\bar{\pi}_i$ and $\bar{\alpha}$ are defined in the above proof.

As we asserted above (i) implies (ii). To confirm this assertion let us define $\mathsf{R}_2(\alpha)$ to be the set of $p \in \mathsf{R}$ satisfying: for any non-zero $x \in A$ and any $\varepsilon > 0$ there exists an $a \in A^\alpha([p-\varepsilon, p+\varepsilon])$ such that $\|a\| = 1$ and $\|x(a+a^*)x^*\| \ge (2 - \varepsilon)\|x\|^2$. It is obvious that $\mathsf{R}_2(\alpha)$ is a closed subset of $\mathsf{R}(\alpha)$. Furthermore one can show that the inclusion $\mathsf{R}_2(\alpha) \supset \bigcap_{\pi\in\mathscr{F}} \Delta(\pi)$ holds for any faithful family $\mathscr{F}$ of irreducible representations of A and the equality holds for some (see [17], Proposition 1). Consider the conditions

(i′) $\mathsf{R}_2(\hat{\alpha}) = \mathsf{R}$

and

(ii′) $\mathsf{R}_2(\alpha) = \mathsf{R}$.

One can show that (i) $\Leftrightarrow$ (i′) $\Rightarrow$ (ii′) $\Leftrightarrow$ (ii). If A is prime all these conditions are equivalent [18]. The equivalences of (i) with (i′) and (ii) with (ii′) are a kind of duality and these equivalences are straightforward. The only implication which is not explicitly given in the non-prime case seems to be (i′) $\Rightarrow$ (ii′). The arguments we adopt here are given in the proof of Theorem 3.3 of [16]. Let $B = A \times_\alpha \mathsf{R}$, let H be a discrete subgroup of $\mathsf{R}(\alpha)$, and let $\beta = \hat{\alpha}|_H$. Then it follows from (i') that there is a faithful family of covariant irreducible representations for $(B \times_\beta H, \hat{H}, \hat{\beta})$ and it follows from $H \subset \mathsf{R}(\alpha)$ that $H(\hat{\beta}) = H$. Let $x \in B \times_\beta H$. For $p \in H^\perp$ and any compact neighborhood U of p in R one can show from (i′) that

$$\sup\{\|x(a + a^*)x^*\| : a \in B^{\hat{\alpha}}(U), \|a\| = 1\} = 2\|x\|^2.$$

Moreover, for $s \in H$ one can show by using Glimm's type of theorem for the compact dynamical system $(B \times_\beta H, \hat{H}, \hat{\beta})$ [6] that

$$\sup\{\|x(a + a^*)x^*\| : a \in B\lambda(s)\} = 2\|x\|^2,$$

where $H \ni s \to \lambda(s)$ is the canonical unitary group in the multiplier algebra of $B \times_\beta H$ implementing β. Using these two conditions one can construct a faithful family $\{\pi_i\}$ of irreducible representations of $B \times_\beta H$ such that π_i restricts to an irreducible representation ρ_i of B and $\bar{\rho}_i$ which is the direct integral of $\rho_i\hat{\alpha}_p$, $p \in \mathsf{R}$ is of type I with $\Delta(\rho_i) = H^\perp$. Then, by Lemma 5 of [17], the duality implies that $\mathsf{R}_2(\alpha) \supset H$. Since H is an arbitrary discrete subgroup of $\mathsf{R}(\alpha)$ one can conclude that $\mathsf{R}_2(\alpha) = \mathsf{R}(\alpha)$.

It follows automatically from the assumptions of Theorem 2.1 that one has $\mathscr{D} \not\supset D$ for any core D of δ_α which contains the geometric elements A^α_G. In particular $\mathscr{D}$ cannot contain the analytic elements, or the C^∞ elements, of α. In addition one cannot construct a core in $\mathscr{D}$ by regularization of the subspace of geometric elements since the following lemma establishes that the subspace is unchanged by regularization with respect to the flow.

LEMMA 2.4. *Let A^α_G denote the geometric elements of the flow α on the C^*-algebra A. Then*

$$\begin{aligned} A^\alpha_G &= \{\alpha_f(x) : x \in A^\alpha_G, f \in L_1(\mathsf{R})\} \\ &= \operatorname{span}\{\alpha_f(x) : x \in A^\alpha_G, f \in C^\infty_c(\mathsf{R})\}. \end{aligned}$$

PROOF. First $x \in A^\alpha(K)$ if and only if $g \in L_1(\mathsf{R})$ and $(\operatorname{supp}\hat{g}) \cap K = \emptyset$ imply $\alpha_g(x) = 0$. But if $x \in A^\alpha(K)$ and $f \in L_1(\mathsf{R})$ then $\alpha_g(\alpha_f(x)) = \alpha_{f*g}(x)$.

Moreover, $\operatorname{supp}(\widehat{f * g}) \subseteq \operatorname{supp} \hat{g}$. Therefore $\alpha_g(\alpha_f(x)) = 0$ and one deduces that $\alpha_f(x) \in A^\alpha(K)$. Hence

$$\{\alpha_f(x) : x \in A^\alpha_G, f \in C^\infty_c(\mathsf{R})\} \subseteq \{\alpha_f(x) : x \in A^\alpha_G, f \in L_1(\mathsf{R})\} \subseteq A^\alpha_G.$$

Next if $x \in A^\alpha(K)$ and $f, g \in L_1(\mathsf{R})$ with $\hat{f} = \hat{g} = 1$ on an open neighbourhood U of K then $\alpha_f(x) = \alpha_g(x)$ (see, for example, [8], Lemma 3.2.38). Replacing g by an approximate identity, with the Fourier transform equal to one on U, and taking the limit one deduces that $\alpha_f(x) = x$. Therefore

$$A^\alpha_G = \{\alpha_f(x) : x \in A^\alpha_G, f \in L_1(\mathsf{R})\}.$$

But the same argument also gives

$$A^\alpha_G = \{\alpha_f(x) : x \in A^\alpha_G, f \in \mathscr{S}(\mathsf{R})\}$$

where $\mathscr{S}(\mathsf{R})$ is the usual Schwartz space. Finally, if $f \in \mathscr{S}(\mathsf{R})$ then it follows from the Rubel-Squires-Taylor factorization theorem [24] (see also [13]) that there exists a finite set of $g_i \in C^\infty_c(\mathsf{R})$ and $h_i \in \mathscr{S}(\mathsf{R})$ such that $f = \sum g_i * h_i$. In particular $\alpha_f(x) = \sum \alpha_{g_i}(y_i)$ with $y_i = \alpha_{h_i}(x)$. Therefore

$$\begin{aligned} A^\alpha_G &= \{\alpha_f(x) : x \in A^\alpha_G, f \in \mathscr{S}(\mathsf{R})\} \\ &\subseteq \operatorname{span}\{\alpha_g(y) : y \in A^\alpha_G, g \in C^\infty_c(\mathsf{R})\} \\ &\subseteq \operatorname{span}\{\alpha_g(y) : y \in A^\alpha_G, g \in \mathscr{S}(\mathsf{R})\} = A^\alpha_G \end{aligned}$$

and the proof is complete.

Despite these observations Example 3.6 illustrates that many approximately inner flows of interest in mathematical physics are such that the subspace of convergence $\mathscr{D}$ contains a dense invariant set of analytic elements.

It is remarkable that under the conditions of Theorem 2.1 convergence of a sequence of bounded derivations on the subspace A^α_G automatically implies boundedness of the limit derivation, at least if A is prime:

COROLLARY 2.5. *Let A be a separable prime C^*-algebra and α a flow. Suppose there exists a faithful family of α-covariant irreducible representations of A and that* $\mathsf{R}(\alpha) \neq \{0\}$.

Suppose that there is a sequence (b_n) of self-adjoint elements of A such that the limits

$$\delta(x) = \lim_{n\to\infty} \operatorname{ad} i b_n(x)$$

exist for all $x \in A^\alpha_G$. Then $x \in A^\alpha_G \mapsto \delta(x) \in A$ extends to a bounded *-derivation on A.

PROOF. It follows from the uniform boundedness theorem that the norm of $\operatorname{ad} b_n|_{A^\alpha([-k,k])}$ is bounded as $n \to \infty$ for each k. Hence $\delta|_{A^\alpha([-k,k])}$ is bounded. Then, by [15], δ is closable and its closure $\overline{\delta}$ generates a flow β.

By [19] there is a faithful covariant representation (π, U) of A such that the flow $\overline{\alpha} : t \mapsto \operatorname{Ad} U_t$ on the factor $M = \pi(A)''$ has Connes spectrum $\mathsf{R}(\alpha)$, e.g. π may be a type II_∞ representation extending the tracial representation of a UHF algebra (with a non-trivial UHF flow) "embedded" in A; here we need A to be prime. Let $f \in L_1(\mathsf{R})$ be an integrable real-valued function such that the Fourier transform $\hat{f}$ has compact support and define

$$\delta_f = \int_{\mathsf{R}} dt\, f(t)\alpha_t \delta \alpha_{-t}.$$

The closure $\overline{\delta}_f$ of δ_f is also a generator and we denote by β_f the flow generated by $\overline{\delta}_f$. By [5] the α-covariant representation π is also β_f-covariant. If $f \in L_1(\mathsf{R})$ satisfies $\hat{f}(0) = 0$ then δ_f is bounded. Moreover, there is a constant $c > 0$ such that $\|\delta_f\| \le c\|f\|_1$ for such f.

We fix a positive function $f \in L_1(\mathsf{R})$ such that $\operatorname{supp}(\hat{f})$ is compact and $\hat{f}(0) = \int dt\, f(t) = 1$ and define δ_n as the closure of $1/n \int dt\, f(t/n)\alpha_t\delta\alpha_{-t}$. Define Δ_n to be the weak extension of $x \mapsto \pi\delta_n(x)$ on $\pi(\mathscr{D}(\delta_n))$, i.e. the generator of the weak extension of the flow on $\pi(A)$ induced by δ_n. Since $\|\Delta_n - \Delta_1\| \le 2c$, we take a limit point d of the sequence of derivations $\Delta_n - \Delta_1 : \pi(A)'' \to \pi(A)''$ with pointwise weak topology. Thus the limit d is a derivation on M. Then by general theory d is an inner derivation on M. Hence $\Delta_1 + d$ is a generator. Since $\Delta_1 + d$ is a limit point of $\Delta_n : \mathscr{D}(\Delta_1) \to M$ and $\|\overline{\alpha}_t\Delta_n(x) - \Delta_n\overline{\alpha}_t(x)\| \to 0,\ x \in M_G^{\overline{\alpha}}$ as $n \to \infty$, we conclude that $\Delta_1 + d$ commutes with $\overline{\alpha}$. By the arguments originating in [21] we argue that $\Delta_1 + d$ generates a flow which is a bounded perturbation of a scaled $\overline{\alpha}$ as follows: Let γ be the flow on M generated by $\Delta_1 + d$. We denote by $\overline{\alpha} \otimes \gamma$ the action of R^2 on M defined by $(s, t) \mapsto \overline{\alpha}_s\gamma_t$. Since the spectrum of $\overline{\alpha} \otimes \gamma$ is bounded on each $\{p\} \times \mathsf{R}$ and the Connes spectrum $\mathsf{R}(\overline{\alpha} \otimes \gamma)$ is included in $\mathsf{R}(\alpha) \times \mathsf{R}$, there is a constant $\lambda \in \mathsf{R}$ such that $\mathsf{R}(\overline{\alpha} \otimes \gamma) = \{(p, \lambda p) : p \in \mathsf{R}(\alpha)\}$. Since $\operatorname{Sp}(\overline{\alpha} \otimes \gamma) + \mathsf{R}(\overline{\alpha} \otimes \gamma) = \operatorname{Sp}(\overline{\alpha} \otimes \gamma)$, we conclude that γ is a bounded perturbation of the flow $t \mapsto \overline{\alpha}_{\lambda t}$ or the flow on A generated by δ_1 is a bounded perturbation of the flow $t \mapsto \alpha_{\lambda t}$.

On the other hand define δ'_n to be the closure of $n \int dt\, f(nt)\alpha_t\delta\alpha_{-t}$ and Δ'_n to be the weak extension of $x \mapsto \pi(\delta'_n(x))$. Then by the same token we have that $\|\Delta_1 - \Delta'_n\| \le 2c$. Note that $\mathscr{D}(\Delta_1) = \mathscr{D}(\Delta'_n)$ and $\|\pi(\delta_1(x) - \delta'_n(x))\| \le 2c\|\pi(x)\|$ for $x \in \mathscr{D}(\delta_1)$. We then conclude that there is a derivation d' of M such that $\pi(\delta_1(x)) - d'(\pi(x)) = \pi(\delta(x))$ for $x \in \pi(A_G^\alpha)$. Since A_G^α is dense in A, this implies that d' leaves $\pi(A)$ invariant. Hence, since π is faithful,

$x \in A_G^\alpha \mapsto \delta_1(x) - \delta(x)$ extends to a derivation of A. Combining this with the result in the previous paragraph, the flow β (generated by $\overline{\delta}$) is a bounded perturbation of $t \mapsto \alpha_{\lambda t}$. If $\lambda \neq 0$, this would imply that $\delta_\alpha|_{A_G^\alpha}$ is approximated by inner derivations, which contradicts Theorem 2.1. Hence $\lambda = 0$ and β is uniformly continuous, or $\overline{\delta}$ is bounded.

Theorem 2.1 can be reformulated in various ways. The assumption of the existence of a faithful family of α-covariant irreducible representations is likely to follow from the approximate innerness of α alone. For example, if each non-zero ideal of A has a non-zero projection, this follows because there exist ground states for a perturbed α restricted to an invariant unital hereditary C^*-subalgebra. With different arguments we can show this is also a consequence of a property of the ideal structure of the C^*-algebra.

PROPOSITION 2.6. *Let A be a separable C^*-algebra and α an approximately inner flow. Suppose that A has at most countably many ideals. Then there exists a faithful family of α-covariant irreducible representations of A.*

PROOF. Let (h_n) be a sequence in A_{sa} such that

$$\alpha_t(x) = \lim_{n\to\infty} \operatorname{Ad} e^{ith_n}(x)$$

uniformly in t on every bounded interval of R for all $x \in A$.

Let γ denote the flow on $C_0(\mathsf{R}, A)$, the C^*-algebra of continuous functions into A vanishing at infinity, induced by translation; $\gamma_t(x)(s) = x(s+t)$, $x \in C_0(\mathsf{R}, A)$ and let $\hat{\alpha}$ denote the dual action of R on the crossed product $A \times_\alpha \mathsf{R}$.

We denote by N^+ the one-point compactification of N; ∞ is the newly added point. We assign the dynamical system $(C_0(\mathsf{R}, A), \gamma)$ to each point $n \in \mathsf{N}$ and $(A \times_\alpha \mathsf{R}, \hat{\alpha})$ to $\infty \in \mathsf{N}^+$. We assert that they form a *continuous field of dynamical systems over* N^+.

We define a map ϕ_n of $C_c(\mathsf{R}, A)$, the space of continuous functions on R into A with compact support, into $C_0(\mathsf{R}, A)$ by

$$\phi_n(f)(p) = \int_{\mathsf{R}} dt\ f(t)e^{it(h_n+p)} = \hat{f}(p + h_n).$$

We note that $\phi_n(f)\phi_n(g) = \phi_n(f *_n g)$ and $\phi_n(f)^* = \phi_n(f^{*_n})$, where

$$f *_n g(t) = \int_{\mathsf{R}} ds\ f(s) \operatorname{Ad} e^{ish_n}(g(t-s))$$

and $f^{*_n}(t) = \operatorname{Ad} e^{ith_n}(f(-t))$. We denote by ϕ_∞ the natural embedding of $C_c(\mathsf{R}, A)$ into $A \times_\alpha \mathsf{R}$, which is given by

$$\phi_\infty(f) = \int_{\mathsf{R}} dt\ f(t)\lambda_t,$$

where $t \mapsto \lambda_t$ is the canonical unitary flow implementing α in the multiplier algebra of $A \times_\alpha \mathsf{R}$. We note that $\phi_\infty(f)\phi_\infty(g) = \phi_\infty(f * g)$ and $\phi_\infty(f)^* = \phi_\infty(f^*)$, where

$$f * g(t) = \int_{\mathsf{R}} dt\, f(s)\alpha_s(g(t-s))$$

and $f^*(t) = \alpha_t(f(-t))$. Note that $f *_n g$ converges to $f * g$ (resp. f^{*_n} to f^*) uniformly as continuous functions of support contained in $\mathrm{supp}(f) + \mathrm{supp}(g)$ (resp. $-\,\mathrm{supp}(f)$). In particular $\|\phi_n(f *_n g) - \phi_n(f * g)\| \to 0$ and $\|\phi_n(f^{*_n}) - \phi_n(f^*)\| \to 0$.

For $f \in C_c(\mathsf{R}, A)$ and $q \in \mathsf{R}$ we define $f_q \in C_c(\mathsf{R}, A)$ by $f_q(t) = f(t)e^{iqt}$. Then $\gamma_q(\phi_n(f)) = \phi_n(f_q)$ for $n \in \mathsf{N}$ and $\hat{\alpha}_q(\phi_\infty(f)) = \phi_\infty(f_q)$. The assertion made above comprises this fact and the following:

LEMMA 2.7. *If $f \in C_c(\mathsf{R}, A)$ then $n \in \mathsf{N}^+ \mapsto \|\phi_n(f)\|$ is continuous. The range of ϕ_n is dense in $C_0(\mathsf{R}, A)$ if $n < \infty$ or in $A \times_\alpha \mathsf{R}$ if $n = \infty$.*

The only non-trivial claim is that $\lim_{n\to\infty} \|\phi_n(f)\| = \|\phi_\infty(f)\|$ for $f \in C_c(\mathsf{R}, A)$. Let $\rho(f) = \limsup \|\phi_n(f)\|$. Then it follows that ρ defines a C^*-seminorm on $C_c(\mathsf{R}, A)$ as a $*$-subalgebra of $A \times_\alpha \mathsf{R}$. This fact follows from

$$\begin{aligned}\rho(f * g) &= \limsup \|\phi_n(f * g)\| = \limsup \|\phi_n(f *_n g)\| \\ &\le \limsup \|\phi_n(f)\|\|\phi_n(g)\| \le \rho(f)\rho(g),\end{aligned}$$

$$\rho(f^* * f) = \limsup \|\phi_n(f^{*_n} *_n f)\| = \limsup \|\phi_n(f)\|^2 = \rho(f)^2,$$

etc.

Since $\rho\hat{\alpha}_p = \rho$ and ρ is non-zero on a non-zero element ag with $a \in A$ and $g \in C_c(\mathsf{R})$, one concludes that ρ is a norm, i.e. $\rho(f) = \|\phi_\infty(f)\|$. Since the same statement holds for any subsequence of (ϕ_n), the claim follows.

Let $a \in A$ and $g \in C_c(\mathsf{R})$ and define $x \in C_c(\mathsf{R}, A)$ by $x(t) = ag(t)$. Then $\phi_n(x)(p) = a\int_{\mathsf{R}} dt\, g(t)e^{it(h_n+p)} = a\hat{g}(h_n + p)$. Note that $p \mapsto \|\phi_n(x)(p)\|$ is a continuous function on R vanishing at ∞.

If $a \neq 0$ and $g \neq 0$, then $\phi_\infty(x) \neq 0$. For any $\theta \in \langle 0, \|\phi_\infty(x)\|\rangle$, and for all large n, we find the smallest $p_n \in \mathsf{R}$ such that $\|\phi_n(x)(p_n)\| = \theta$. Define a seminorm ρ on $C_c(\mathsf{R}, A)$ by

$$\rho(f) = \limsup_{n\to\infty} \max_{p \le p_n} \|\phi_n(f)(p)\|,$$

which extends to a C^*-seminorm on $A \times_\alpha \mathsf{R}$. Note that $\rho(x) = \theta < \|\phi_\infty(x)\|$ and $q \mapsto \rho\hat{\alpha}_q(f)$ is increasing (because $\gamma_q\phi_n(f)(p) = \phi_n(f_q) = \phi_n(f)(p + q)$).

Let

$$\rho_{-\infty}(f) = \lim_{q\to-\infty} \rho\hat{\alpha}_q(f),$$

which defines a C^*-seminorm on $A \times_\alpha \mathsf{R}$.

Suppose that $\rho_{-\infty} \neq \rho$. Since $\operatorname{Ker}\rho \subsetneq \operatorname{Ker}\rho_{-\infty}$, we take an irreducible representation π of the quotient $\operatorname{Ker}\rho_{-\infty}/\operatorname{Ker}\rho$ and regard it as an irreducible representation of $A \times_\alpha \mathsf{R}$. Then $\operatorname{Ker}\pi\hat{\alpha}_p \neq \operatorname{Ker}\pi$ for $p \neq 0$; if $\operatorname{Ker}\pi\hat{\alpha}_p = \operatorname{Ker}\pi$ for some $p \neq 0$, then $\operatorname{Ker}\pi$ contains $\operatorname{Ker}\rho\hat{\alpha}_q$ for any q (as $\operatorname{Ker}\pi \supset \operatorname{Ker}\rho$), from which follows that $\pi|_{\operatorname{Ker}\rho_{-\infty}} = 0$, a contradiction. This implies that the center of $\overline{\pi}$ (as defined as the direct integral of $\pi\hat{\alpha}_p$ over $p \in \mathsf{R}$ as before) is $L_\infty(\mathsf{R})$, which in turn implies that the representation π of $A \times_\alpha \mathsf{R}$ is induced from an α-covariant irreducible representation of A.

Suppose that $\rho_{-\infty} = \rho$, which implies that $\rho\hat{\alpha}_q = \rho$ for all q. Then there is an α-invariant ideal I of A such that $\operatorname{Ker}\rho$ is described as $I \times_\alpha \mathsf{R}$. For each $\theta \in \langle 0, \|\phi_\infty(x)\|\rangle$ we have defined a seminorm $\rho = \rho_\theta$ and an ideal $I = I_\theta$ of A if ρ_θ is $\hat{\alpha}$-invariant. Note that if ρ_θ and ρ_μ are $\hat{\alpha}$-invariant for $\theta \neq \mu$, then $I_\theta \neq I_\mu$. If all ρ_θ are $\hat{\alpha}$-invariant, we thus obtain a continuous family of ideals of A, which contradicts the assumption. Thus there is a θ such that ρ_θ is not $\hat{\alpha}$-invariant.

Thus we obtain an α-covariant irreducible representation π of A. If $\operatorname{Ker}\pi$ is non-zero, we apply this argument to $\operatorname{Ker}\pi$ and $\alpha|_{\operatorname{Ker}\pi}$, which is an approximately inner flow on a C^*-algebra with at most countably many ideals. By induction one concludes that there is a faithful family of α-covariant irreducible representations.

3. AF and UHF algebras

In this section we examine some properties of cores of generators for AF and UHF algebras. First we note that if α is an approximately inner flow on an AF-algebra then one may choose the sequence (h_n) which defines the generators of the approximating flows in such a way that the subspace $\mathscr{D}$ defined by (3) is dense in A.

PROPOSITION 3.1. *Let A be an AF algebra and α an approximately inner flow on A with generator δ_α. Then there exists a sequence (h_n) in A_{sa} such that*

$$\lim_{n\to\infty} \max_{|t|\le 1} \|\alpha_t(x) - \operatorname{Ad} e^{ith_n}(x)\| = 0$$

for all $x \in A$ and

$$\mathscr{D} = \left\{x \in D(\delta_\alpha) : \lim_{n\to\infty} \operatorname{ad} ih_n(x) = \delta_\alpha(x)\right\}$$

is dense in A.

REMARK 3.2. Even without assuming the existence of (h_n) it follows from the result of Sakai mentioned in the introduction [25] that there exists an increasing sequence A_n of finite dimensional C^*-subalgebras of A such that $A_n \subset D(\delta_\alpha)$ and $\bigcup_n A_n$ is dense in A. This is true even if δ is not assumed to be a generator, but only a closed derivation ([7], Theorem 11). Furthermore, there exists a sequence h_n in A_{sa} such that $\delta|_{A_n} = \mathrm{ad}(ih_n)|_{A_n}$ (see [8], Example 3.2.25). If $\delta = \delta_\alpha$ is a generator and $\bigcup_n A_n$ is a core for δ_α then the assertion in Proposition 3.1 follows with (h_n) equal to this sequence since in this case $\bigcup_n A_n \subset \mathscr{D}$. In the absence of the assumption on the existence of h_n, however, it does not in general follow from the generator property of δ that $\bigcup_n A_n$ can be taken to be a core for δ or that h_n exists (see again [20], Theorem 1.1).

PROOF OF PROPOSITION 3.1. Since α is approximately inner there exists a sequence $d_n = d_n^*$ in A_{sa} such that δ_α is the graph limit of $\mathrm{ad}(id_n)$. This means that for each x in $D(\delta_\alpha)$ there exists a sequence (x_n) of $x_n \in A$ such that

$$\lim_{n\to\infty} \|x_n - x\| + \lim_{n\to\infty} \|\,\mathrm{ad}(id_n)(x_n) - \delta_\alpha(x)\| = 0 \tag{5}$$

This condition is actually equivalent to each of the equivalent conditions (1) and (2) in the introduction (see [8], Theorem 3.1.28).

To prove the proposition we next choose an increasing sequence B_n of finite dimensional sub-algebras by Sakai's result as in the preceding remark such that $\bigcup_n B_n \subset D(\delta_\alpha)$. Then we are going to modify d_n to h_n by passing to a subsequence of d_n such that the conclusion of Proposition 3.1 is valid.

First, fix an n and let $\mathbf{X}$ be a set of matrix units for B_n (see e.g. [12], Chapter III). Now for each matrix unit x in B_n there exists, as remarked above, a sequence x_m in A such that

$$\lim_{m\to\infty} \|x_m - x\| + \lim_{m\to\infty} \|\,\mathrm{ad}(id_m)(x_m) - \delta_\alpha(x)\| = 0 \tag{6}$$

Using Glimm's technique we may furthermore use spectral theory to modify the approximants x_m for each fixed m such that the set of these approximants form a set of matrix units isomorphic to $\mathbf{X}$ for each m. This is possible if we go so far out in the sequences indexed by m that the approximation to x is good enough (see, for example, [12], Section III.3, or [10], Section 2). If $\mathbf{X}_m$ denotes the corresponding sequences of matrix units, we then have

$$\lim_{m\to\infty} \|\mathbf{X}_m - \mathbf{X}\| = 0 \tag{7}$$

However by careful scrutiny of the proof in [12] or [10] one may choose the approximants to **X** such that also

$$\lim_{m\to\infty} \| \operatorname{ad}(id_m)(\mathbf{X}_m) - \delta_\alpha(\mathbf{X})\| = 0. \tag{8}$$

Hence

$$\lim_{m\to\infty} \|\mathbf{X}_m - \mathbf{X}\| + \lim_{m\to\infty} \| \operatorname{ad}(id_m)(\mathbf{X}_m) - \delta_\alpha(\mathbf{X})\| = 0 \tag{9}$$

(The norms in these relations are defined by taking the maximum of the finite number of norms obtained by replacing the sets **X** by the individual matrix elements x in **X**.)

The reason for the convergence of the derivatives is that the matrix elements are obtained by applying functional calculus by smooth functions in $D(\delta)$. We use [8], Theorem 3.2.32, (or alternatively [23], or [2] and [3]) to deduce that if δ is any closed derivation and $x = x^*$ is in $D(\delta)$ then $f(x)$ is in $D(\delta)$. Since $f(x)$ only depends on the definition of f on $\operatorname{Spec}(x)$ we may assume

$$|||f||| = (2\pi)^{-1/2} \int_{\mathsf{R}} dp \, |\hat{f}(p)||p| < \infty$$

where $\hat{f}$ is the Fourier transform of f. Then

$$f(x) = (2\pi)^{-1/2} \int_{\mathsf{R}} dp \, \hat{f}(p) e^{ipx}$$

and

$$\delta(f(x)) = i(2\pi)^{-1/2} \int_{\mathsf{R}} dp \, \hat{f}(p) p \int_0^1 dt \, e^{itpx} \delta(x) e^{i(1-t)px}.$$

Hence

$$\|\delta(f(x))\| \le |||f||| \, \|\delta(x)\|. \tag{10}$$

We need the following lemma:

Lemma 3.3. *Let δ_n be a sequence of closed derivations on a C^*-algebra A with graph limit δ and (x_n) a sequence of elements of A with $\lim_{n\to\infty} \|x_n - x\| + \|\delta_n(x_n) - \delta(x)\| = 0$. Further let f_n be a sequence of functions converging to f with respect to the semi-norm $|||\cdot|||$.*

It follows that

$$\lim_{n\to\infty} \|\delta_n(f_n(x_n)) - \delta(f(x))\| = 0 \tag{11}$$

PROOF. We have

$$
\begin{aligned}
&\delta_n(f_n(x_n)) - \delta(f(x)) \\
&\quad = i(2\pi)^{-1/2}\int_{\mathbf{R}} dp\, \hat{f}_n(p)p\int_0^1 dt\, e^{itpx}\delta_n(x_n)e^{i(1-t)px} \\
&\qquad - i(2\pi)^{-1/2}\int_{\mathbf{R}} dp\, \hat{f}(p)p\int_0^1 dt\, e^{itpx}\delta(x)e^{i(1-t)px} \\
&\quad = i(2\pi)^{-1/2}\int_{\mathbf{R}} dp\, (\hat{f}_n(p)-\hat{f}(p))p\int_0^1 dt\, e^{itpx}\delta_n(x_n)e^{i(1-t)px} \\
&\qquad + i(2\pi)^{-1/2}\int_{\mathbf{R}} dp\, \hat{f}(p)p\int_0^1 dt\, e^{itpx}(\delta_n(x_n)-\delta(x))e^{i(1-t)px}
\end{aligned}
$$

from which one deduces that

$$\|\delta_n(f_n(x_n)) - \delta(f(x))\| \le |||f_n - f|||\,\|\delta_n(x_n)\| + |||f|||\,\|\delta_n(x_n)-\delta(x)\|. \tag{12}$$

The conclusion of the lemma follows immediately.

REMARK 3.4. The semi-norm $|||f|||$ occurring in (10) and (12) can be estimated by noting that $|p|(1+p^2)^{1/2} \le 2\,(1+p^4)^{1/2}$ and using the Cauchy-Schwarz inequality. Hence

$$
\begin{aligned}
|||f|||^2 &\le 4\Big(\int_{\mathbf{R}} dp\,(1+p^2)^{-1}\Big)\Big(\int_{\mathbf{R}} dp\,|\hat{f}(p)|^2(1+p^4)\Big) \\
&= 2\pi\,(\|f''\|_2^2 + \|f\|_2^2).
\end{aligned} \tag{13}
$$

It then follows that the space of functions with $|||f||| < \infty$ contains the Sobolev space $W^{2,2}(\mathbf{R})$. But these estimates are not optimal (see [2], [3], [1], [4]).

PROOF OF PROPOSITION 3.1 CONTINUED. We fix a set $\mathbf{X}^n$ of matrix units for B_n and suppose that we have obtained a sequence $(\mathbf{X}^n_m)$ of matrix units for each $\mathbf{X}^n$ by the arguments illustrated above. We then find a sequence $(u_{n,m})$ of unitaries in A such that

$$\lim_{m\to\infty}\|u_{n,m} - \mathbb{1}\| = 0 \tag{14}$$

and

$$u_{n,m}\mathbf{X}^n u^*_{n,m} = \mathbf{X}^n_m \tag{15}$$

where we again interpret (15) as the set of relations obtained by replacing $\mathbf{X} = \mathbf{X}^n_m, \mathbf{X}^n$ by each the matrix elements x in $\mathbf{X}$. We can choose a subsequence

$(m(n))$ such that $\|u_{n,m(n)} - \mathbb{1}\| < 1/n$ and $\|(\mathrm{ad}(id_{m(n)})\,\mathrm{Ad}\,u_{n,m(n)} - \delta)|B_n\| < 1/n$. Then it follows that $\|\hat{\delta_n}(x) - \delta(x)\| \to 0$ for all $x \in \bigcup_k B_k$ where

$$\hat{\delta}_n(x) = u^*_{n,m(n)}\,\mathrm{ad}(id_{m(n)})(u_{n,m(n)} x u^*_{n,m(n)}) u_{n,m(n)}, \tag{16}$$

and hence

$$\hat{\delta}_n = \mathrm{ad}(i u^*_{n,m(n)} d_{m(n)} u_{n,m(n)}). \tag{17}$$

Since $\|u_{n,m(n)} - 1\| \to 1$, one concludes that $\hat{\delta}_n$ converges to δ in the graph norm. This concludes the proof.

Remark 3.5. As we have already said after Proposition 3.1, if the increasing family of finite dimensional sub-algebras constitute a core for δ, there is nothing more to prove. So Proposition 3.1 only tells something new when the increasing family is not a core, or $D(\delta)$ is not an AF Banach algebra in the graph norm. We do not know in general whether we can choose approximating inner derivations converging pointwise on a core for δ.

The method of constructing the modified sequence again goes back to Glimm, and is expanded in Section II.3 in [12] and in Section 2 in [10].

Although Theorem 2.1 established under quite general conditions that the convergence subspace $\mathscr{D}$ cannot contain the analytic elements of the flow α the next example shows that there are many examples in which $\mathscr{D}$ contains an α-invariant dense subspace of analytic elements. The following example is a flow constructed on a one-dimensional lattice. In mathematical physics terms A is the algebra of observables of a one-dimensional spin-1/2 system. Note that we consider the one-dimensional case for simplicity. One can construct similar examples on higher dimensional lattices by analogous arguments.

Example 3.6. Let A denote the UHF-algebra given by the C^*-closure of the infinite tensor product $\bigotimes_{n\in\mathsf{Z}} M_2$ of copies of the 2×2-matrices M_2.

The algebra A has a natural quasi-local structure. Let $A_I = \bigotimes_{i\in I} M_2$ denote the family of local matrix algebras indexed by finite subsets $I = \{i_1, \ldots, i_n\}$ with $i_m \in \mathsf{Z}$. Further let $A_{\mathrm{loc}} = \bigcup_I A_I$. If σ denotes the shift automorphism on A then Z acts on A as a group of shifts (space translations) $n \in \mathsf{Z} \mapsto \sigma_n = \sigma^n$ which leaves A_{loc} invariant. In particular $\sigma_n(A_I) = A_{I+n}$.

Next we construct a flow corresponding to a finite-range interaction between the spins, i.e. an interaction which links close by points of Z. Fix $\Phi = \Phi^* \in A_J$ for some finite subset J. Define $H_I = H_I^*$ by

$$H_I = \sum_{m\in I} \sigma_m(\Phi).$$

Then introduce the corresponding inner *-derivations δ_I by

$$\delta_I(x) = \operatorname{ad} i H_I(x)$$

for all $x \in A$. Further define the *-derivation δ by $D(\delta) = A_{\mathrm{loc}}$ and

$$\delta(x) = \lim_I \operatorname{ad} i H_I(x)$$

for $x \in A_{\mathrm{loc}}$ where the limit is over an increasing family of I whose union is Z. (For each $x \in A_{\mathrm{loc}}$ there is an $I \in \mathsf{Z}$ such that $\delta(x) = \operatorname{ad} i H_I(x)$ by locality.)

There is a flow α on A given by

$$\alpha_t(x) = \lim_I \operatorname{Ad} e^{itH_I}(x).$$

The norm limit exists by the estimates of [9], Theorem 6.2.4. Moreover the generator δ_α of the flow is the norm closure $\overline{\delta}$ of the derivation δ. Then δ_α is the graph limit of the derivations δ_I (see [8], Theorem 3.1.28). Therefore if we define

$$\mathscr{D} = \left\{x \in D(\delta_\alpha) : \lim_I \delta_I(x) = \delta_\alpha(x)\right\}$$

one has

$$A_{\mathrm{loc}} = D(\delta) \subseteq \mathscr{D} \subseteq D(\delta_\alpha).$$

Finally define $\mathscr{A}$ as the *-algebra generated by $\{\alpha_t(A_{\mathrm{loc}}) : t \in \mathsf{R}\}$. Then we argue that $\mathscr{A} \subseteq \mathscr{D}$. To this end it suffices to show that if $x \in A_J$ for some J and $t \in \mathsf{R}$ then $\lim_I \operatorname{ad} H_I(\alpha_t(x))$ exists. But if $I_1 \subset I_2$ then

$$\operatorname{ad} H_{I_1}(\alpha_t(x)) - \operatorname{ad} H_{I_2}(\alpha_t(x)) = \sum_{p \in I_2 \backslash I_1} [\sigma_p(\Phi), \alpha_t(x)].$$

Then, as a consequence of [9], Proposition 6.2.9, there are $a, b, c > 0$ such that

$$\|[\sigma_p(\Phi), \alpha_t(x)]\| \le a\, \|x\|\, e^{-bp+ct}$$

uniformly for $p \in \mathsf{Z}$ and $t \in \mathsf{R}$. Therefore

$$\|\operatorname{ad} H_{I_1}(\alpha_t(x)) - \operatorname{ad} H_{I_2}(\alpha_t(x))\| \le a\, \|x\|\, e^{ct} \sum_{p \in I_1^{\mathrm{c}}} e^{-bp}$$

for all $t \in \mathsf{R}$. It follows immediately that the limit exists and this suffices to establish the inclusion $\mathscr{A} \subseteq \mathscr{D}$.

One concludes that $\mathscr{A}$ is an α-invariant subspace of $D(\delta_\alpha)$. Therefore it is a core of δ_α. But it also follows from [9], Theorem 6.2.4, that each $x \in \mathscr{A}$ is an analytic element of δ_α. Thus $\mathscr{D}$ contains an α-invariant dense subalgebra of analytic elements.

Since the flow α commutes with the group of translations by Z it follows that the Connes' spectrum $\mathsf{R}(\alpha)$, which is a subgroup of R, must be $\mathrm{Sp}(\alpha)$. Therefore $\mathsf{R}(\alpha) \cong \mathsf{Z}$ or R if α is non-trivial. Both cases can occur with a suitable choice of Φ.

Although the latter conclusions rely on translation invariance one can construct similar examples on the half line and the same conclusions are valid. In particular one can add a bounded *-derivation δ_P to δ_α in such a way that α factors into a product of flows $\alpha^{(\pm)}$ on the left and right half lattice $\mathsf{Z}_\pm$, respectively. Then the algebra $\mathscr{A}^+$ generated by $\{\alpha_t^{(+)}(A_I) : t \in \mathsf{R}, I \subset \mathsf{Z}_+\}$ is contained in the set $\mathscr{D}(\alpha^{(+)})$ corresponding to $\alpha^{(+)}$ and consists of analytic elements for the latter flow. Finally $\mathsf{R}(\alpha^{(+)}) = \mathsf{R}(\alpha)$ because $\alpha^{(-)} \otimes \alpha^{(+)}$ arises by a bounded perturbation of the generator of α.

The next example shows that $\mathscr{D}$ can be much larger but then the Connes' spectrum is equal to $\{0\}$.

EXAMPLE 3.7. Let A denote the UHF-algebra given by the C^*-closure of the infinite tensor product $\bigotimes_{n\geq 1} M_2$ of copies of the 2×2-matrices M_2, $A_I = \bigotimes_{i\in I} M_2$ the local matrix algebras and $A_{\mathrm{loc}} = \bigcup_I A_I$.

Now let $(\lambda_i)_{i\geq 1}$ be a sequence of positive numbers and define $h_i \in A_{\{i\}}$ by

$$h_i = \begin{pmatrix} \lambda_i & 0 \\ 0 & 0 \end{pmatrix}.$$

Set $H_n = \sum_{i=1}^n h_i$ and $\delta_n(x) = \mathrm{ad}\, i H_n(x)$ for $x \in A$. Then define α on A by

$$\alpha_t(x) = \lim_I \mathrm{Ad}\, e^{itH_I}(x).$$

Set

$$\mathscr{D} = \left\{x \in D(\delta_\alpha) : \lim_I \delta_I(x) = \delta_\alpha(x)\right\}.$$

We next argue that if the λ_i are chosen to increase sufficiently fast as $i \to \infty$ then $D(\delta_\alpha^2) \subset \mathscr{D}$ but in this case $\mathsf{R}(\alpha) = \{0\}$.

Let $a_i, a_i^* \in A_{\{i\}}$ be given by

$$a_i = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad a_i^* = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and note that $H_n = \sum_{i=1}^n \lambda_i\, a_i^* a_i$. Set $a(I) = \bigotimes_{i\in I} a_i$ and $a^*(J) = \bigotimes_{j\in J} a_j^*$.

Let C_2 be the diagonal matrices of M_2 and C the C^*-subalgebra generated by C_2 at every point of N; $C = \bigotimes_{i\in \mathsf{N}} C_2 \subset A = \bigotimes_{i\in\mathsf{N}} M_2$. For a subset K of N let $C_K = \bigotimes_{i\in K} C_2$.

We define an action γ of $G = \prod_{n=1}^{\infty} \mathsf{T}$ on A by

$$\gamma_z = \bigotimes_{n=1}^{\infty} \mathrm{Ad}\begin{pmatrix} z_n & 0 \\ 0 & 1 \end{pmatrix}.$$

Then the fixed point algebra of γ is C while $a(I)$ and $a^*(J)$ are eigen-operators for finite subsets I, J: $\gamma_z(a(I)) = \prod_{n\in I} z_n a(I)$. The spectrum of γ is $\coprod\{-1, 0, 1\} \subset \hat{G} = \coprod_{n\in\mathsf{N}} \mathsf{Z}$, which we identify with $\mathscr{S} = \{(I, J) \in P_f(\mathsf{N}) \times P_f(\mathsf{N}) \mid I \cap J = \emptyset\}$, where $P_f(\mathsf{N})$ is the set of finite subsets of N and $p \in \coprod\{-1, 0, 1\}$ maps to (I, J) with $I = \{n \mid p_n = -1\}$ and $J = \{n \mid p_n = 1\}$. Note that for each $(I, J) \in \mathscr{S}$ the eigen-space is given by $Ca(I)a^*(J) = C_{I^c\cap J^c}a(I)a^*(J)$.

Then each $x \in A_{\mathrm{loc}}$ has a unique representation

$$x = \sum_{(I,J)\in\mathscr{S}} x(I;J)\, a^*(I)a(J)$$

with $x(I; J) \in C_{I^c\cap J^c}$, where the sum is finite. Let $z^I = \prod_{n\in I} z_n$ and $\bar{z}^J = \prod_{n\in J} \bar{z}_n$. Since

$$\int_G z^I \bar{z}^J \gamma_z(x)\, dz = x(I, J)a^*(I)a(J)$$

with dz is normalized Haar measure on G, one deduces that $\|x(I; J)\| \le \|x\|$.

Now $x \in D(\delta_\alpha^2)$ by locality,

$$\delta_\alpha(x) = i \sum_{(I,J)\in\mathscr{S}} (\lambda(I) - \lambda(J))\, x(I;J)\, a^*(I)a(J) \tag{18}$$

where $\lambda(I) = \sum_{i\in I} \lambda_i$ and

$$\delta_\alpha^2(x) = -\sum_{(I,J)\in\mathscr{S}} (\lambda(I) - \lambda(J))^2\, x(I;J)\, a^*(I)a(J).$$

In particular $(\lambda(I) - \lambda(J))^2\, \|x(I; J)\| \le \|\delta_\alpha^2(x)\|$.

Next suppose $\lambda_n \ge 2\,(\lambda_1 + \cdots + \lambda_{n-1}) + 6^n$ for all n. If $n = \max(I \cup J)$ then

$$|\lambda(I) - \lambda(J)| \ge \lambda_n - (\lambda_1 + \cdots + \lambda_{n-1}) \ge 6^n.$$

Therefore $6^n|\lambda(I)-\lambda(J)|\,\|x(I;J)\| \le \|\delta_\alpha^2(x)\|$ and

$$\|\delta_\alpha(x)\| \le \sum_{(I,J)\in\mathscr{S}} |\lambda(I)-\lambda(J)|\,\|x(I;J)\| \le \sum_{n\ge 1}\ \sum_{\max(I\cup J)=n} 6^{-n}\|\delta_\alpha^2(x)\|$$
$$\le \sum_{n\ge 1} 2^{-n}\|\delta_\alpha^2(x)\| = \|\delta_\alpha^2(x)\|$$

where we have used $\sum_{\max(I\cup J)=n} 1 < 3^n$. But A_{loc} is a core of δ_α^2. The foregoing estimates then establish that the representation (18) extends to all $x \in D(\delta_\alpha^2)$; the infinite sum in (18) is absolutely convergent.

Then if $k \le l$ one has $H_l - H_k = \sum_{i=k+1}^{l} \lambda_i\, a_i^* a_i$ and so

$$\delta_l(x) - \delta_k(x) = i\sum_{I,J}(\lambda_{k,l}(I)-\lambda_{k,l}(J))\,x(I;J)\,a^*(I)a(J) \tag{19}$$

for all $x \in D(\delta_\alpha^2)$ with $\lambda_{k,l}(I) = \lambda(I\cap\{k+1,\ldots,l\})$. In particular the summand is only non-zero if $\max(I\cup J) > k$. But $\lambda_{k,l}(I) = \lambda_l(I) - \lambda_k(I)$ with $\lambda_l(I) = \lambda(I\cap\{1,\ldots,l\})$. Now if $n+1 \in I\cup J$ then

$$|\lambda_{n+1}(I) - \lambda_{n+1}(J)| \ge \lambda_{n+1} - (\lambda_1 + \cdots + \lambda_n)$$

and

$$|\lambda_n(I) - \lambda_n(J)| \le \lambda_1 + \cdots + \lambda_n.$$

Hence

$$|\lambda_n(I) - \lambda_n(J)| \le |\lambda_{n+1}(I) - \lambda_{n+1}(J)|. \tag{20}$$

But if $n+1 \notin I\cup J$ then $\lambda_n(I) - \lambda_n(J) = \lambda_{n+1}(I) - \lambda_{n+1}(J)$ so (20) is valid for all n. Then by iteration

$$|\lambda_k(I) - \lambda_k(J)| \le |\lambda_l(I) - \lambda_l(J)| \le |\lambda(I) - \lambda(J)|.$$

Combining these observations one concludes from (19) that

$$\|\delta_l(x) - \delta_k(x)\| \le \sum_{\max(I\cup J)>k} |\lambda(I)-\lambda(J)|\,\|x(I;J)\| \le 2^{-k}\|\delta_\alpha^2(x)\|$$

for all $x \in D(\delta_\alpha^2)$. Therefore $\delta_l(x) \to \delta_\alpha(x)$ as $l\to\infty$ and $\mathscr{D} \supseteq D(\delta_\alpha^2)$.

Note that in this example $\mathsf{R}(\alpha) = \{0\}$ but δ_α is not bounded.

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