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NORM INEQUALITIES IN MULTIDIMENSIONAL LORENTZ SPACES

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Abstract

In this paper we obtain necessary and sufficient conditions for double trigonometric series to belong to generalized Lorentz spaces, not symmetric in general. Estimates for the norms are given in terms of coefficients.

1. Introduction

Let $f(x, y)$ be a measurable function on $[0, 2\pi]^2$. We define the rearrangement of $f(x, y)$ with respect to y to be $f_y^*(x, t_2) = \big(f(x, y)\big)_y^*(x, t_2)$, i.e., $f_y^*(x, t_2)$ is a non-increasing function on t_2 and the functions $f_y^*(x, t_2)$ and $|f(x, y)|$ are equimeasurable as functions of one variable for almost all x. We define the rearrangement of $f_y^*(x, t_2)$ with respect to x to be $f_{yx}^*(t_1, t_2)$. Therefore $f_{yx}^*(t_1, t_2)$ is non-increasing on t_1 and t_2 and equimeasurable with $|f(x, y)|$.

According to [1]–[2], if $0 < \alpha < \infty$, we say that a measurable function $f(x, y)$, which is 2π-periodic on each variable, belongs to the two-dimensional weighted Lorentz space $\Lambda_2^\alpha(\omega)$, if

$$\|f\|_{\Lambda_2^\alpha(\omega)} = \left(\int_0^{2\pi}\int_0^{2\pi} w(t_1, t_2)\big(f_{yx}^*(t_1, t_2)\big)^\alpha \, dt_1\, dt_2\right)^{\frac{1}{\alpha}} < \infty,$$

where the weighted function $w \in W$, i.e., $w(t_1, t_2)$ is an a.e. positive locally integrable function on $[0, 2\pi]^2$.

Similarly, we say that $f(x, y)$ belongs to $\bar\Lambda_2^\alpha(\omega)$ provided that

$$\|f\|_{\bar\Lambda_2^\alpha(\omega)} = \left(\int_0^{2\pi}\int_0^{2\pi} w(t_1, t_2)\big(f_{xy}^*(t_1, t_2)\big)^\alpha \, dt_1\, dt_2\right)^{\frac{1}{\alpha}} < \infty.$$

We consider the following series

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{mn}\cos mx \cos ny, \tag{1.1}$$

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$$\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} a_{mn} \cos mx \sin ny, \tag{1.2}$$

$$\sum_{m=1}^{\infty}\sum_{n=0}^{\infty} a_{mn} \sin mx \cos ny, \tag{1.3}$$

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn} \sin mx \sin ny, \tag{1.4}$$

where the coefficients a_{mn} are real numbers and $\cos 0x := \cos 0y := \frac{1}{2}$. We also suppose that the sequence $\{a_{mn}\}$ satisfies the condition

$$a_{mn} \to 0 \qquad \text{as} \quad m+n \to \infty. \tag{1.5}$$

For integers k_1 and k_2 we define

$$\Delta_{k_1k_2}a_{mn} = \sum_{i=0}^{k_1}(-1)^i C_{k_1}^i \sum_{j=0}^{k_2}(-1)^j C_{k_2}^j a_{m+i\,n+j} \qquad (m, n \geq 0),$$

where $C_k^l = \frac{k(k-1)\cdots(k-l+1)}{l!}$ for $l \geq 1$ and $C_k^l = 1$ for $l = 0$.

We note that if $\{a_{mn}\}$ satisfies (1.5) and $\Delta_{k_1k_2}a_{mn} \geq 0$ for integers $k_1, k_2 \geq 1$, then $\Delta_{s_1s_2}a_{mn} \geq 0$ for all $0 \leq s_1 \leq k_1$ and $0 \leq s_2 \leq k_2$.

By C, C_i we denote positive constants that may be different on different occasions. Also, $F \asymp G$ means that there exist constants C_1 and C_2 such that $C_1 F \leq G \leq C_2 F$.

Now we recall the definition of convergence of double series in the Pringsheim's sense. The partial sums of the series $\sum_{\mu,\nu=0}^{\infty} c_{\mu\nu}$ are defined to be $S_{mn} = \sum_{\nu=0}^{n}\sum_{\mu=0}^{m} c_{\mu\nu}$. If there exists a number S such that for any $\varepsilon > 0$ there exist integers k and l such that $|S_{mn} - S| < \varepsilon$ for all $n > k$ and $m > l$, then series $\sum_{\mu,\nu} c_{\mu\nu}$ is said to converge to S, in the sense of Pringsheim.

It is well-known (see e.g. [7]) that if a sequence $\{a_{mn}\}$ satisfies (1.5) and $\Delta_{k_1k_2}a_{mn} \geq 0$ for any integers k_1 and k_2, then series (1.1)–(1.4) converge in Pringsheim's sense except, possibly, on a set of measure zero. We define by $f_1(x, y)$, $f_2(x, y)$, $f_3(x, y)$, and $f_4(x, y)$ the sums of series (1.1)–(1.4), respectively.

The aim of the paper is to study necessary and sufficient conditions for the functions $f_1(x, y)$, $f_2(x, y)$, $f_3(x, y)$, and $f_4(x, y)$ to belong to the Lorentz spaces, not symmetric in general, in terms of coefficients $\{a_{mn}\}$.

This problem has a long history starting from the well-known theorem by Hardy and Littlewood (see [8], [17, V.2, XII, §6]): *A necessary and sufficient* *condition that the function*

$$f(x) \sim \sum_n (a_n \cos nx + b_n \sin nx), \qquad a_n, b_n \downarrow,$$

should belong to L^p, $1 < p < \infty$, *is that*

$$\sum_n (a_n^p + b_n^p) n^{p-2} < \infty.$$

This result was generalized by Sagher for the Lorentz spaces (see [14] and [15]): *Assume that* $f(x)$ *is either the Fourier sine or Fourier cosine series associated with* $\{a_n\}$. *If* $a_n \downarrow$, *then for* $1 < p < \infty$ *and* $0 < q \le \infty$,

$$\|f(x)\|_{L^{p,q}} \asymp \|\{a_n\}\|_{l^{p',q}}$$

where $L^{p,q}$ *and* $l^{p',q}$ *are continuous and discrete Lorentz spaces [3].*

For non-weighted multidimensional L^p-spaces, where $1 < p < \infty$, the Hardy-Littlewood-type result was obtained by Moricz [12]. The cases of different Lebesgue and Lorentz spaces were also investigated in [4], [6], [11], [13], [16], among others.

In section 2 we present our main results for the spaces $\Lambda_2^\alpha(\omega)$ and $\bar{\Lambda}_2^\alpha(\omega)$ in the case of $\alpha \in (0, \infty)$. First (Theorem 2.1), we find the estimate of $\|f_i\|_{\Lambda_2^\alpha(\omega)}$ from above. Next (Theorems 2.2 and 2.3), we give the necessary conditions for f_i to belong to $\Lambda_2^\alpha(\omega)$, depending on the monotonicity properties of the weight ω. Sections 3 and 4 contain the preliminary results and proofs, respectively.

Finally, we remark that our results, in particular, provide criteria for f_1, f_2, f_3, and f_4 to belong to classical Lorentz space $L^{p,q}$, $0 < p, q < \infty$, and to the Lorentz-Zygmund space $L^{p,q}\varphi(L)$ (here φ is a slowly varying function; see [5], [11]).

2. Main Results

For convenience purposes we set

$$w_{mn} := \int_{\pi/(n+1)}^{\pi/\bar{n}} \int_{\pi/(m+1)}^{\pi/\bar{m}} w(x, y)\, dx\, dy, \quad \text{where } \bar{n} := \begin{cases} \frac{1}{2} & \text{if } n = 0, \\ n & \text{if } n \in \mathsf{N}. \end{cases}$$

Then our results are read as follows.

THEOREM 2.1. *Let* $0 < \alpha < \infty$, $w \in W$, *and let* w *satisfy the following condition: for all* $\delta_1, \delta_2 \in (0, 2\pi)$

$$\int_0^{\delta_1} \int_0^{\delta_2} w(t_1, t_2)\, dt_1\, dt_2 \le C \int_{\frac{\delta_1}{2}}^{\delta_1} \int_{\frac{\delta_2}{2}}^{\delta_2} w(t_1, t_2)\, dt_1\, dt_2, \tag{2.1}$$

where C is independent of δ_1, δ_2. Suppose the sequence $\{a_{mn}\}$ satisfies condition (1.5) *and $\Delta_{k_1k_2}a_{mn} \geq 0$ for all $m, n \in \mathbb{N}$ and appropriate k_1, k_2 appearing in* (A)–(D).

(A) *If $k_1 = 2$ and $k_2 = 2$, then*

$$\|f_1(x, y)\|_{\Lambda_2^\alpha(\omega)} \leq C_1\left(\sum_{n=0}^{\infty}(n + 1)^{2\alpha} \sum_{m=0}^{\infty}(m + 1)^{2\alpha} w_{mn}(\Delta_{11}a_{mn})^\alpha\right)^{\frac{1}{\alpha}}; \tag{2.2}$$

(B) *If $k_1 = 2$ and $k_2 = 1$, then*

$$\|f_2(x, y)\|_{\Lambda_2^\alpha(\omega)} \leq C_2\left(\sum_{n=1}^{\infty}(n + 1)^{\alpha} \sum_{m=0}^{\infty}(m + 1)^{2\alpha} w_{mn}(\Delta_{10}a_{mn})^\alpha\right)^{\frac{1}{\alpha}}; \tag{2.3}$$

(C) *If $k_1 = 1$ and $k_2 = 2$, then*

$$\|f_3(x, y)\|_{\Lambda_2^\alpha(\omega)} \leq C_3\left(\sum_{n=0}^{\infty}(n + 1)^{2\alpha} \sum_{m=1}^{\infty}(m + 1)^{\alpha} w_{mn}(\Delta_{01}a_{mn})^\alpha\right)^{\frac{1}{\alpha}}; \tag{2.4}$$

(D) *If $k_1 = 1$ and $k_2 = 1$, then*

$$\|f_4(x, y)\|_{\Lambda_2^\alpha(\omega)} \leq C_4\left(\sum_{n=1}^{\infty}(n + 1)^{\alpha} \sum_{m=1}^{\infty}(m + 1)^{\alpha} w_{mn}\, a_{mn}^\alpha\right)^{\frac{1}{\alpha}}; \tag{2.5}$$

where C_1, C_2, C_3, C_4 are independent of $\{a_{mn}\}$.

REMARK 2.1. For any $\gamma_1, \gamma_2 > 0$ and for any slowly varying functions $\varphi_1(x), \varphi_2(y)$ on $(0, \infty)$, $w(t_1, t_2) = t_1^{\gamma_1 - 1}\varphi_1(1/t_1)t_2^{\gamma_2 - 1}\varphi_1(1/t_2)$ satisfies (2.1).

THEOREM 2.2. *Let $0 < \alpha < \infty$, $w(t_1, t_2) \in W$, and let $w(t_1, t_2)$ be non-increasing with respect to t_1 for almost all t_2 and non-increasing with respect to t_2 for almost all t_1. Suppose the sequence $\{a_{mn}\}$ satisfies condition* (1.5) *and $\Delta_{k_1k_2}a_{mn} \geq 0$ for all $m, n \in \mathbb{N}$ and appropriate k_1, k_2 appearing in* (A)–(D).

(A) *If $k_1 = 2$ and $k_2 = 2$, then*

$$\|f_1(x, y)\|_{\Lambda_2^\alpha(\omega)} \geq C_1\left(\sum_{n=0}^{\infty}(n + 1)^{2\alpha} \sum_{m=0}^{\infty}(m + 1)^{2\alpha} w_{mn}(\Delta_{11}a_{mn})^\alpha\right)^{\frac{1}{\alpha}}; \tag{2.6}$$

(B) *If $k_1 = 2$ and $k_2 = 1$, then*

$$\|f_2(x, y)\|_{\Lambda_2^\alpha(\omega)} \geq C_2\left(\sum_{n=1}^{\infty}(n + 1)^{\alpha} \sum_{m=0}^{\infty}(m + 1)^{2\alpha} w_{mn}\,(\Delta_{10}a_{mn})^\alpha\right)^{\frac{1}{\alpha}}; \tag{2.7}$$

(C) *If* $k_1 = 1$ *and* $k_2 = 2$, *then*

$$\|f_3(x,y)\|_{\Lambda_2^\alpha(\omega)} \geq C_3 \Bigg(\sum_{n=0}^{\infty} (n+1)^{2\alpha} \sum_{m=1}^{\infty} (m+1)^\alpha w_{mn} \left(\Delta_{01} a_{mn}\right)^\alpha \Bigg)^{\frac{1}{\alpha}}; \tag{2.8}$$

(D) *If* $k_1 = 1$ *and* $k_2 = 1$, *then*

$$\|f_4(x,y)\|_{\Lambda_2^\alpha(\omega)} \geq C_4 \Bigg(\sum_{n=1}^{\infty} (n+1)^{\alpha} \sum_{m=1}^{\infty} (m+1)^\alpha w_{mn}\, a_{mn}^\alpha \Bigg)^{\frac{1}{\alpha}}; \tag{2.9}$$

where C_1, C_2, C_3, C_4 *are independent of* $\{a_{mn}\}$.

THEOREM 2.3. *Let* $1 \leq \alpha < \infty$, $w(t_1,t_2) \in W$, *and let* $w(t_1,t_2)$ *satisfy the following two conditions:*

(1) *for almost all* $t_2 \in [0,2\pi]$ $w(t_1,t_2)$ *is non-decreasing and satisfies* Δ_2-*condition with respect to* t_1, *i.e.,* $w(2t_1,t_2) \leq Cw(t_1,t_2)$;

(2) *for almost all* $t_1 \in [0,2\pi]$, $w(t_1,t_2)$ *is non-decreasing and satisfies* Δ_2-*condition with respect to* t_2, *i.e.,* $w(t_1,2t_2) \leq Cw(t_1,t_2)$.

Suppose the sequence $\{a_{mn}\}$ *satisfies condition* (1.5) *and* $\Delta_{k_1k_2}a_{mn} \geq 0$ *for all* $m,n \in \mathsf{N}$ *and appropriate* k_1, k_2 *appearing in* (A)–(D).

(A) *If* $k_1 = 2$ *and* $k_2 = 2$, *then inequality* (2.6) *is satisfied;*

(B) *If* $k_1 = 2$ *and* $k_2 = 1$, *then inequality* (2.7) *is satisfied;*

(C) *If* $k_1 = 1$ *and* $k_2 = 2$, *then inequality* (2.8) *is satisfied;*

(D) *If* $k_1 = 1$ *and* $k_2 = 1$, *then inequality* (2.9) *is satisfied;*

where C_1, C_2, C_3, C_4 *are independent of* $\{a_{mn}\}$.

COROLLARY 2.1. *Under the conditions of Theorem 2.1 and Theorem 2.2 (or Theorem 2.3), we have*

$$\|f_j(x,y)\|_{\Lambda_2^\alpha(\omega)} \asymp \|f_j(x,y)\|_{\bar{\Lambda}_2^\alpha(\omega)}, \qquad j = 1,2,3,4. \tag{2.10}$$

We note that, in general, (2.10) does not hold (see [2] and [5]). We also remark that in [2, §3] the authors showed that if $\|\cdot\|_{\Lambda_2^p(\omega)}$ is a rearrangement invariant norm (see [3, Ch. 2]), then ω is a constant and $\Lambda_2^p(\omega) \equiv L^p([0,2\pi]^2)$. The following corollary follows from Theorems 2.1–2.3 (see also[16]).

COROLLARY 2.2. *Let* $0 < p < \infty$. *Suppose the sequence* $\{a_{mn}\}$ *satisfies condition* (1.5) *and* $\Delta_{k_1k_2}a_{mn} \geq 0$ *for all* $m,n \in \mathsf{N}$ *and appropriate* k_1, k_2 *appearing in* (A)–(D).

(A) *If* $k_1 = 2$ *and* $k_2 = 2$*, then*

$$\|f_1(x,y)\|_{L^p([0,2\pi]^2)} \asymp \left(\sum_{n=0}^{\infty}(n+1)^{2p-2}\sum_{m=0}^{\infty}(m+1)^{2p-2}\left(\Delta_{11}a_{mn}\right)^p\right)^{\frac{1}{p}};$$

(B) *If* $k_1 = 2$ *and* $k_2 = 1$*, then*

$$\|f_2(x,y)\|_{L^p([0,2\pi]^2)} \asymp \left(\sum_{n=1}^{\infty}(n+1)^{p-2}\sum_{m=0}^{\infty}(m+1)^{2p-2}\left(\Delta_{10}a_{mn}\right)^p\right)^{\frac{1}{p}};$$

(C) *If* $k_1 = 1$ *and* $k_2 = 2$*, then*

$$\|f_3(x,y)\|_{L^p([0,2\pi]^2)} \asymp \left(\sum_{n=0}^{\infty}(n+1)^{2p-2}\sum_{m=1}^{\infty}(m+1)^{p-2}\left(\Delta_{01}a_{mn}\right)^p\right)^{\frac{1}{p}};$$

(D) *If* $k_1 = 1$ *and* $k_2 = 1$*, then*

$$\|f_4(x,y)\|_{L^p([0,2\pi]^2)} \asymp \left(\sum_{n=1}^{\infty}(n+1)^{p-2}\sum_{m=1}^{\infty}(m+1)^{p-2}\,a_{mn}^p\right)^{\frac{1}{p}}.$$

This result is the two-dimensional version of the classical Hardy-Littlewood theorem. Note also that if $1 < p < \infty$ and $k_1 = k_2 = 1$, then (compare with [12])

$$\|f_j(x,y)\|_{L^p([0,2\pi]^2)} \asymp \left(\sum_{n=1}^{\infty}(n+1)^{p-2}\sum_{m=1}^{\infty}(m+1)^{p-2}\,a_{mn}^p\right)^{\frac{1}{p}}, \qquad j = 1,2,3,4.$$

3. Lemmas

Let

$$B_0^1(x) = \frac{1}{2};$$

$$B_n^1(x) = \frac{1}{2} + \cos x + \cdots + \cos nx \qquad \text{for} \quad n \geq 1;$$

$$B_n^k(x) = \sum_{m=0}^{n} B_m^{k-1}(x) \qquad \text{for} \quad k = 2,3,\ldots \ \text{ and } \ n = 0,1,2,\ldots;$$

$$\overline{B}_n^1(x) = \sin x + \cdots + \sin nx \qquad \text{for} \quad n = 1, 2, \ldots;$$

$$\overline{B}_n^k(x) = \sum_{m=1}^{n} \overline{B}_m^{k-1}(x) \qquad \text{for} \quad k = 2, 3, \ldots \ \text{ and } \ n = 1, 2, \ldots.$$

We consider series

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \Delta_{k_1k_2} a_{mn} B_m^{k_1}(x) B_n^{k_2}(y); \tag{3.1}$$

$$\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} \Delta_{k_1k_2} a_{mn} B_m^{k_1}(x) \overline{B}_n^{k_2}(y); \tag{3.2}$$

$$\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \Delta_{k_1k_2} a_{mn} \overline{B}_m^{k_1}(x) B_n^{k_2}(y); \tag{3.3}$$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \Delta_{k_1k_2} a_{mn} \overline{B}_m^{k_1}(x) \overline{B}_n^{k_2}(y). \tag{3.4}$$

LEMMA 3.1. *Suppose the sequence* $\{a_{mn}\}$ *satisfies condition* (1.5) *and* $\Delta_{k_1k_2} a_{mn} \geq 0$ *for integers* k_1 *and* k_2. *Then any series* (3.1)–(3.4) *converges in Pringsheim's sense except possibly the set of measure zero to* $f_1(x, y)$, $f_2(x, y)$, $f_3(x, y)$, *and* $f_4(x, y)$, *respectively.*

The proof follows easily from the Abel transform (see also [16]).

LEMMA 3.2 ([10]). *Let* $a_n > 0$, $b_n \geq 0$ $(n = 1, 2, \ldots)$; $1 \leq p < \infty$. *Then*

$$\sum_{k=1}^{\infty} a_k \left(\sum_{m=1}^{k} b_m\right)^p \leq p^p \sum_{m=1}^{\infty} a_m^{1-p} \left(b_m \sum_{n=m}^{\infty} a_n\right)^p; \tag{A}$$

$$\sum_{k=1}^{\infty} a_k \left(\sum_{m=k}^{\infty} b_m\right)^p \leq p^p \sum_{m=1}^{\infty} a_m^{1-p} \left(b_m \sum_{n=1}^{m} a_n\right)^p. \tag{B}$$

LEMMA 3.3 ([9, Ch. II, §2]). *Let* f *and* g *be measurable functions on* $[0, 2\pi]$. *Then*

$$(f + g)^*(t_1 + t_2) \leq f^*(t_1) + g^*(t_2),$$

$$(fg)^*(t_1 + t_2) \leq f^*(t_1) g^*(t_2).$$

LEMMA 3.4 ([9, Ch. II, §2]). *Let* f *and* g *be measurable functions on* $[0, 2\pi]$. *Suppose* $|f(t)| \leq |g(t)|$; *then* $f^*(t) \leq g^*(t)$.

LEMMA 3.5 ([3, Ch. 2, §2]). *Let f be a measurable function on* $[0, 2\pi]$. *Then for all* $0 < \alpha < \infty$

$$\left(f^*(t)\right)^\alpha = (|f|^\alpha)^*(t).$$

LEMMA 3.6 ([9]). *Let $h(t)$ be an increasing non-negative function. Then*

$$\int_0^\infty (f+g)^*(t)h(t)\,dt \geq \int_0^\infty f^*(t)h(t)\,dt + \int_0^\infty g^*(t)h(t)\,dt.$$

LEMMA 3.7 ([5]). (A) *Suppose* $|f(x,y)| \leq |g(x,y)|$; *then* $f^*_{yx}(t_1,t_2) \leq g^*_{yx}(t_1,t_2)$;

(B) $(f+g)^*_{yx}(s_1+s_2, t_1+t_2) \leq f^*_{yx}(s_1,t_1) + g^*_{yx}(s_2,t_2)$.

LEMMA 3.8 ([9, Ch. II, §2]). *Let $f(t)$ and $g(t)$ be non-negative summable functions on* $[0, 2\pi]$ *such that*

$$\int_0^x f(t)\,dt \leq \int_0^x g(t)\,dt$$

Suppose $h(t)$ is non-increasing non-negative function on $(0, 2\pi)$; *then*

$$\int_0^{2\pi} f(t)h(t)\,dt \leq \int_0^{2\pi} g(t)h(t)\,dt.$$

LEMMA 3.9 ([5]). *Let* $f^*(t_1,t_2) = f^*_{yx}(t_1,t_2)$, $\sigma > 0$ *and let*

$$\lambda_f(\sigma) = \mu\big\{(x,y) \in \Omega_1 \times \Omega_2 : |f(x,y)| > \sigma\big\},$$
$$\lambda_{f^*}(\sigma) = \mu\big\{(s,t) \in [0,\infty) \times [0,\infty) : f^*(s,t) > \sigma\big\}.$$

Then $\lambda_f(\sigma) = \lambda_{f^*}(\sigma)$.

LEMMA 3.10. *Let a measurable set* $E \subset [0,a] \times [0,b]$, *where* $0 < a, b < \infty$. *Suppose for some* $k \in (0,1)$,

$$\mu E > k\mu([0,a] \times [0,b]) = kab.$$

Then

$$U := \left\{(s,t) \in \left[0, \frac{ka}{2}\right] \times \left[0, \frac{kb}{2}\right]\right\}$$
$$\subset \left\{(s,t) \in [0,\infty) \times [0,\infty) : \left(\chi_E(x,y)\right)^*_{yx}(s,t) > \frac{1}{2}\right\} =: T.$$

PROOF. From the non-increase of $\left(\chi_E(x,y)\right)^*_{yx}(s,t)$ it is clear that the boundary of $\left\{(s,t) : \left(\chi_E(x,y)\right)^*_{yx}(s,t) > \frac{1}{2}\right\}$ is the non-decreasing function $t = d(s)$. Let us assume that $U \not\subseteq T$. Therefore, $\mu(T) \le \frac{kab}{2} + \frac{kab}{2} - \frac{k^2ab}{4} < kab$. By Lemma 3.9, we have $\mu E = \mu T$ and hence $\mu E < kab$ which contradicts our assumption. Thus, $U \subset T$, which completes the proof.

4. Proofs

PROOF OF THEOREM 2.1. By Lemma 3.1, the following identity holds a.e.

$$F(x,y) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \Delta_{l_1l_2}a_{mn}\widetilde{B}^{l_1}_m(x)\widetilde{B}^{l_2}_n(y),$$

where in the case of

(A) $F(x,y) = f_1(x,y)$, $l_1 = l_2 = 2$, $\widetilde{B}^{l_1}_m(x) = B^2_m(x)$, $\widetilde{B}^{l_2}_n(y) = B^2_n(y)$;

(B) $F(x,y) = f_2(x,y)$, $l_1 = 2$, $l_2 = 1$, $\widetilde{B}^{l_1}_m(x) = B^2_m(x)$, $\widetilde{B}^{l_2}_n(y) = \overline{B}^1_n(y)$, and $\Delta_{21}a_{m0} = \Delta_{10}a_{m0} = 0$, $m = 0, 1, 2, \ldots$;

(C) $F(x,y) = f_3(x,y)$, $l_1 = 1$, $l_2 = 2$, $\widetilde{B}^{l_1}_m(x) = \overline{B}^1_m(x)$, $\widetilde{B}^{l_2}_n(y) = B^2_n(y)$, and $\Delta_{12}a_{0n} = \Delta_{01}a_{0n} = 0$, $n = 0, 1, 2, \ldots$;

(D) $F(x,y) = f_4(x,y)$, $l_1 = l_2 = 1$, $\widetilde{B}^{l_1}_m(x) = \overline{B}^1_m(x)$, $\widetilde{B}^{l_2}_n(y) = \overline{B}^1_n(y)$, and $\Delta_{11}a_{m0} = a_{m0} = 0$, $m = 0, 1, 2, \ldots$, $\Delta_{11}a_{0n} = a_{0n} = 0$, $n = 0, 1, 2, \ldots$.

We estimate $\|F(x,y)\|_{\Lambda^p_2(\omega)}$ from above. Using Lemmas 3.3 and 3.4, we obtain

$$\begin{aligned}
&\left\|F(x,y)\right\|^{\alpha}_{\Lambda^{\alpha}_2(\omega)} \\
&= \int_0^{2\pi}\int_0^{2\pi}\omega(t_1,t_2)\Big(F(x,y)\big[\chi_{(0,\pi]}(x)\chi_{(0,\pi]}(y) + \chi_{(0,\pi]}(x)\chi_{(\pi,2\pi]}(y) \\
&\qquad + \chi_{(\pi,2\pi]}(x)\chi_{(0,\pi]}(y) + \chi_{(\pi,2\pi]}(x)\chi_{(\pi,2\pi]}(y)\big]^*_{yx}(t_1,t_2)\Big)^{\alpha}\,dt_1\,dt_2 \\
&\le C\int_0^{\pi/2}\int_0^{\pi/2}\omega(4t_1,4t_2)\Big(\big[F(x,y)\chi_{(0,\pi]}(x)\chi_{(0,\pi]}(y)\big]^*_{yx}(t_1,t_2)\Big)^{\alpha}\,dt_1\,dt_2 \\
&\le C\left(I_1 + I_2 + I_3 + I_4\right),
\end{aligned}$$

where

$$\begin{aligned}
I_1 = &\sum_{n=0}^{\infty}\int_{\pi/2^{n+4}}^{\pi/2^{n+3}}\sum_{m=0}^{\infty}\int_{\pi/2^{m+4}}^{\pi/2^{m+3}}\omega(16t_1,16t_2) \\
&\left(\left[\sum_{\nu=0}^{2^{n+1}}\sum_{\mu=0}^{2^{m+1}}\Delta_{l_1l_2}a_{\mu\nu}\widetilde{B}^{l_1}_{\mu}(x)\widetilde{B}^{l_2}_{\nu}(y)\chi_{(0,\pi]}(x)\chi_{(0,\pi]}(y)\right]^*_{yx}(t_1,t_2)\right)^{\alpha}dt_1\,dt_2;
\end{aligned}$$

$$I_2 = \sum_{n=0}^{\infty} \int_{\pi/2^{n+4}}^{\pi/2^{n+3}} \sum_{m=0}^{\infty} \int_{\pi/2^{m+4}}^{\pi/2^{m+3}} \omega(16t_1, 16t_2)$$
$$\left(\left[\sum_{\nu=2^{n+1}+1}^{\infty} \sum_{\mu=0}^{2^{m+1}} \Delta_{l_1 l_2} a_{\mu\nu} \widetilde{B}_{\mu}^{l_1}(x) \widetilde{B}_{\nu}^{l_2}(y) \chi_{(0,\pi]}(x) \chi_{(0,\pi]}(y) \right]_{yx}^{*} (t_1, t_2) \right)^{\alpha} dt_1\, dt_2;$$

$$I_3 = \sum_{n=0}^{\infty} \int_{\pi/2^{n+4}}^{\pi/2^{n+3}} \sum_{m=0}^{\infty} \int_{\pi/2^{m+4}}^{\pi/2^{m+3}} \omega(16t_1, 16t_2)$$
$$\left(\left[\sum_{\nu=0}^{2^{n+1}} \sum_{\mu=2^{m+1}+1}^{\infty} \Delta_{l_1 l_2} a_{\mu\nu} \widetilde{B}_{\mu}^{l_1}(x) \widetilde{B}_{\nu}^{l_2}(y) \chi_{(0,\pi]}(x) \chi_{(0,\pi]}(y) \right]_{yx}^{*} (t_1, t_2) \right)^{\alpha} dt_1\, dt_2;$$

$$I_4 = \sum_{n=0}^{\infty} \int_{\pi/2^{n+4}}^{\pi/2^{n+3}} \sum_{m=0}^{\infty} \int_{\pi/2^{m+4}}^{\pi/2^{m+3}} \omega(16t_1, 16t_2)$$
$$\left(\left[\sum_{\nu=2^{n+1}+1}^{\infty} \sum_{\mu=2^{m+1}+1}^{\infty} \Delta_{l_1 l_2} a_{\mu\nu} \widetilde{B}_{\mu}^{l_1}(x) \widetilde{B}_{\nu}^{l_2}(y) \chi_{(0,\pi]}(x) \chi_{(0,\pi]}(y) \right]_{yx}^{*} (t_1, t_2) \right)^{\alpha} dt_1\, dt_2.$$

First we estimate I_1. Since $|\widetilde{B}_{\mu}^{l_1}(x)| \le C_1(\mu+1)^{l_1}$, $|\widetilde{B}_{\nu}^{l_2}(y)| \le C_2(\nu+1)^{l_2}$, where C_1, C_2 are independent of x, y and μ, ν, we have

$$I_1 \le \sum_{n=0}^{\infty} \int_{\pi/2^{n+4}}^{\pi/2^{n+3}} \sum_{m=0}^{\infty} \int_{\pi/2^{m+4}}^{\pi/2^{m+3}} \omega(16t_1, 16t_2)$$
$$\left(\sum_{\nu=0}^{2^{n+1}} \sum_{\mu=0}^{2^{m+1}} \Delta_{l_1 l_2} a_{\mu\nu} (\mu+1)^{l_1} (\nu+1)^{l_2} \right)^{\alpha} dt_1\, dt_2.$$

Since $\Delta_{l_1-1\, l_2} a_{mn} \ge 0$, $\Delta_{l_1-1\, l_2-1} a_{mn} \ge 0$ and $\Delta_{l_1\, l_2} a_{mn} = \Delta_{l_1-1\, l_2} a_{mn} - \Delta_{l_1-1\, l_2} a_{m+1\, n}$, $\Delta_{l_1-1\, l_2} a_{mn} = \Delta_{l_1-1\, l_2-1} a_{mn} - \Delta_{l_1-1\, l_2-1} a_{m\, n+1}$, then

$$\text{(4.1)} \quad \sum_{n=0}^{N} \sum_{m=0}^{M} \Delta_{l_1 l_2} a_{mn} (m+1)^{l_1} (n+1)^{l_2} \le C \sum_{n=0}^{N} \sum_{m=0}^{M} \Delta_{l_1-1\, l_2-1} a_{mn} (m+1)^{l_1-1} (n+1)^{l_2-1},$$

where $C = C(l_1, l_2)$. Using this estimate, we get

$$I_1 \le C \sum_{n=0}^{\infty} \int_{\pi/2^{n+4}}^{\pi/2^{n+3}} \sum_{m=0}^{\infty} \int_{\pi/2^{m+4}}^{\pi/2^{m+3}} \omega(16t_1, 16t_2)\, dt_1\, dt_2 \left(\sum_{\nu=0}^{n} \sum_{\mu=0}^{m} \Delta_{l_1-1\, l_2-1} a_{2^\mu-1, 2^\nu-1} 2^{\mu l_1} 2^{\nu l_2} \right)^{\alpha}.$$

Now if $\alpha \ge 1$, we use Lemma 3.2 and if $\alpha < 1$, we apply Jensen's inequality. Then using (2.1), we write

$$\begin{aligned} I_1 &\le C \sum_{n=0}^{\infty} \int_{\pi/2^{n+4}}^{\pi/2^{n+3}} \sum_{m=0}^{\infty} \int_{\pi/2^{m+4}}^{\pi/2^{m+3}} \omega(16t_1, 16t_2)\, dt_1\, dt_2 \\ &\quad \left(\Delta_{l_1-1\, l_2-1} a_{2^m-1, 2^n-1} 2^{m l_1} 2^{n l_2} \right)^{\alpha} \\ &\le C \sum_{n=0}^{\infty} (n+1)^{2\alpha l_1} \sum_{m=0}^{\infty} (m+1)^{2\alpha l_2} w_{mn} \left(\Delta_{l_1-1\, l_2-1} a_{mn} \right)^{\alpha} =: C B(\alpha, l_1, l_2). \end{aligned}$$

Now we estimate I_2. We note that for $y \in (0, \pi)$, $|\widetilde{B}_n^{l_2}(y)| \le C y^{-l_2} (l_2 = 1, 2)$, where C is independent of n and y. Then by Lemma 3.4, we have

$$\begin{aligned} I_2 &\le C \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \int_{\pi/2^{n+4}}^{\pi/2^{n+3}} \int_{\pi/2^{m+4}}^{\pi/2^{m+3}} \omega(16t_1, 16t_2) \\ &\times \left(\left[\sum_{\nu=2^{n+1}+1}^{\infty} \sum_{\mu=0}^{2^{m+1}} \Delta_{l_1 l_2} a_{\mu\nu} (\mu+1)^{l_1} y^{-l_2} \chi_{(0,\pi]}(x) \chi_{(0,\pi]}(y) \right]_{yx}^{*} (t_1, t_2) \right)^{\alpha} dt_1\, dt_2 \\ &\le C \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \int_{\pi/2^{n+4}}^{\pi/2^{n+3}} \int_{\pi/2^{m+4}}^{\pi/2^{m+3}} \omega(16t_1, 16t_2)\, dt_1\, dt_2 \\ &\quad \left(2^{n l_2} \sum_{\nu=2^{n+1}+1}^{\infty} \sum_{\mu=0}^{2^{m+1}} \Delta_{l_1 l_2} a_{\mu\nu} (\mu+1)^{l_1} \right)^{\alpha} \\ &\le C \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \int_{\pi/2^{n+4}}^{\pi/2^{n+3}} \int_{\pi/2^{m+4}}^{\pi/2^{m+3}} \omega(16t_1, 16t_2)\, dt_1\, dt_2 \\ &\quad \left(2^{n l_2} \sum_{\mu=0}^{2^{m+1}} 2^{\mu l_1} \Delta_{l_1-1\, l_2-1} a_{\mu, 2^{n+1}+1} \right)^{\alpha}. \end{aligned}$$

To verify the last inequality, we use $\sum_{\nu=2^{n+1}+1}^{\infty} \Delta_{l_1 l_2} a_{\mu\nu} = \Delta_{l_1 l_2-1} a_{\mu, 2^{n+1}+1}$ and (4.1).

Finally, using Lemma 3.2 for $\alpha \geq 1$, and Jensen's inequality for $\alpha < 1$, we arrive at $I_2 \leq CB(\alpha, l_1, l_2)$.

In a similar way to the estimates of I_1 and I_2, one can also obtain the inequality $I_3 + I_4 \leq CB(\alpha, l_1, l_2)$, i.e., $I \leq CB(\alpha, l_1, l_2)$. Then this gives us inequality (2.2) for $l_1 = l_2 = 2$, inequality (2.3) for $l_1 = 2, l_2 = 1$, inequality (2.4) for $l_1 = 1, l_2 = 2$, and inequality (2.5) for $l_1 = l_2 = 1$.

PROOF OF THEOREM 2.2. By Lemma 3.1, the following identities hold a.e.

(A)

$$\varphi_1(x,y) = f_1(x,y) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{mn} \cos mx \cos ny = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \Delta_{22} b_{mn} \frac{\sin^2 \frac{(m+1)x}{2}}{4\sin^2 \frac{x}{2}} \frac{\sin^2 \frac{(n+1)y}{2}}{4\sin^2 \frac{y}{2}},$$

where $b_{mn} = a_{mn}$,

(B)

$$\varphi_2(x,y) = \frac{f_2(x,y) + f_2(x,\pi - y)}{2} = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} a_{m\,2n-1} \cos mx \sin(2n-1)y = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} \Delta_{21} b_{mn} \frac{\sin^2 \frac{(m+1)x}{2}}{4\sin^2 \frac{x}{2}} \frac{\sin^2 ny}{\sin y},$$

where $b_{mn} = a_{m\,2n-1}$,

(C)

$$\varphi_3(x,y) = \frac{f_3(x,y) + f_3(\pi - x, y)}{2} = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} a_{2m-1\,n} \sin(2m-1)x \cos ny = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \Delta_{12} b_{mn} \frac{\sin^2 mx}{\sin x} \frac{\sin^2 \frac{(n+1)y}{2}}{4\sin^2 \frac{y}{2}},$$

where $b_{mn} = a_{2m-1\,n}$,

(D)

$$\varphi_4(x,y) = \frac{f_4(x,y) + f_4(\pi - x, y) + f_4(x, \pi - y) + f_4(\pi - x, \pi - y)}{2}$$

$$= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{2m-1\,2n-1} \sin(2m-1)x \sin(2n-1)y$$

$$= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \Delta_{11} b_{mn} \frac{\sin^2 mx}{\sin x} \frac{\sin^2 ny}{\sin y},$$

where $b_{mn} = a_{2m-1\,2n-1}$.

Thus, we have a.e.

$$\Phi(x, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \Delta_{l_1 l_2} b_{mn} \widehat{B}_m^{l_1}(x) \widehat{B}_n^{l_2}(y),$$

where

(A) $\Phi = \varphi_1, l_1 = 2, l_2 = 2, \widehat{B}_m^{l_1}(x) = B_m^2(x), \widehat{B}_n^{l_2}(y) = B_n^2(y)$;

(B) $\Phi = \varphi_2, l_1 = 2, l_2 = 1, \widehat{B}_m^{l_1}(x) = B_m^2(x), \widehat{B}_n^{l_2}(y) = \frac{\sin^2 ny}{\sin y}$; and $\Delta_{21} b_{m0} = \Delta_{10} b_{m0} = b_{m0} = 0, m = 0, 1, 2, \ldots$;

(C) $\Phi = \varphi_3, l_1 = 1, l_2 = 2, \widehat{B}_m^{l_1}(x) = \frac{\sin^2 mx}{\sin x}, \widehat{B}_n^{l_2}(y) = B_n^2(y)$; and $\Delta_{12} b_{0n} = \Delta_{01} b_{0n} = b_{0n} = 0, n = 0, 1, 2, \ldots$;

(D) $\Phi = \varphi_4, l_1 = 1, l_2 = 1, \widehat{B}_m^{l_1}(x) = \frac{\sin^2 mx}{\sin x}, \widehat{B}_n^{l_2}(y) = \frac{\sin^2 ny}{\sin y}$ and $\Delta_{11} b_{m0} = b_{m0} = 0, m = 0, 1, 2, \ldots, \Delta_{11} b_{0n} = b_{0n} = 0, n = 0, 1, 2, \ldots$.

Using Lemmas 3.5 and 3.8, we obtain

$$\int_0^{2\pi} \int_0^{2\pi} w(t_1, t_2) \big(F_{yx}^*(t_1, t_2)\big)^\alpha \, dt_1 \, dt_2 \geq \int_0^{2\pi} \int_0^{2\pi} w(x, y) |F(x, y)|^\alpha \, dx \, dy,$$

because of

$$\big(F_{yx}^*(t_1, t_2)\big)^\alpha = \Big(\big((F(x, y))_y^*\big)^\alpha\Big)_x^*(t_1, t_2) = \big(|F(x, y)|^\alpha\big)_{yx}^*(t_1, t_2).$$

We also note the following:

$$\begin{aligned}
&\int_0^{\pi} \int_0^{\pi} w\Big(\frac{t_1}{2}, \frac{t_2}{2}\Big) w\Big(\frac{t_1}{2}, \frac{t_2}{2}\Big) \Big[\big(f(x, y) + f(\pi - x, y)\big)_{yx}^*\Big]^\alpha (t_1, t_2) \, dt_1 \, dt_2 \\
&\quad \leq C \Bigg(\int_0^{\pi} \int_0^{\pi} w\Big(\frac{t_1}{2}, \frac{t_2}{2}\Big) \Big(\big|f(x, y)\big|_{yx}^* \Big(\frac{t_1}{2}, \frac{t_2}{2}\Big)\Big)^\alpha dt_1 \, dt_2 \\
&\qquad + \int_0^{\pi} \int_0^{\pi} w\Big(\frac{t_1}{2}, \frac{t_2}{2}\Big) \Big(\big|f(\pi - x, y)\big|_{yx}^* \Big(\frac{t_1}{2}, \frac{t_2}{2}\Big)\Big)^\alpha dt_1 \, dt_2\Bigg) \\
&\quad \leq C \int_0^{2\pi} \int_0^{2\pi} w(t_1, t_2) \big(f_{yx}^*(t_1, t_2)\big)^\alpha \, dt_1 \, dt_2.
\end{aligned}$$

Using this, let us estimate

$$I := \int_0^{\pi} \int_0^{\pi} w\Big(\frac{x}{2^k}, \frac{y}{2^k}\Big) \big|\Phi(x, y)\big|^\alpha \, dx \, dy,$$

where

$$k := \begin{cases} 0 & \text{if } \Phi = \varphi_1, \\ 1 & \text{if } \Phi = \varphi_2 \text{ (or } \varphi_4), . \\ 2 & \text{if } \Phi = \varphi_3 \end{cases}$$

We have

$$I = \sum_{s_2=0}^{\infty} \int_{\pi/2^{s_2}}^{\pi/2^{s_2+1}} \sum_{s_1=0}^{\infty} \int_{\pi/2^{s_1}}^{\pi/2^{s_1+1}} w\left(\frac{x}{2^k}, \frac{y}{2^k}\right) |\Phi(x, y)|^{\alpha}\, dx\, dy,$$

$$\geq C \sum_{s_2=0}^{\infty} \sum_{s_1=0}^{\infty} w\left(\frac{\pi}{2^{k+s_1}}, \frac{y}{2^{k+s_2}}\right) \int_{\pi/2^{s_2}}^{\pi/2^{s_2+1}} \int_{\pi/2^{s_1}}^{\pi/2^{s_1+1}} |\Phi(x, y)|^{\alpha}\, dx\, dy$$

$$=: C \sum_{s_2=0}^{\infty} \sum_{s_1=0}^{\infty} w\left(\frac{\pi}{2^{k+s_1}}, \frac{y}{2^{k+s_2}}\right) A_{s_1 s_2} \geq C \sum_{s_2=j}^{\infty} \sum_{s_1=i}^{\infty} w\left(\frac{\pi}{2^{k+s_1}}, \frac{\pi}{2^{k+s_2}}\right) A_{s_1 s_2},$$

where $i = j = 0$ for φ_1, $i = 0, j = 1$ for φ_2, $i = 1, j = 0$ for φ_3, and $i = j = 1$ for φ_4.

Using the following estimate (see [16])

$$A_{s_1 s_2} \geq C 2^{s_1(l_1\alpha-1)+s_2(l_2\alpha-1)} \left(\Delta_{l_1-1\, l_2-1} b_{[2^{s_1-1}][2^{s_2-1}]}\right)^{\alpha},$$

we have

$$I \geq C \sum_{s_2=j}^{\infty} \sum_{s_1=i}^{\infty} w\left(\frac{\pi}{2^{k+s_1}}, \frac{y}{2^{k+s_2}}\right) 2^{s_1(l_1\alpha-1)+s_2(l_2\alpha-1)} \left(\Delta_{l_1-1\, l_2-1} b_{[2^{s_1-1}][2^{s_2-1}]}\right)^{\alpha}.$$

Thus, we get for $i = j = 0$

$$\| f_1(x, y)\|_{\Lambda_2^{\alpha}(\omega)} \geq C_1 \left(\sum_{n=0}^{\infty} (n+1)^{2\alpha} \sum_{m=0}^{\infty} (m+1)^{2\alpha} w_{mn} \, (\Delta_{11} a_{mn})^{\alpha} \right)^{\frac{1}{\alpha}}.$$

In a similar way, one can also obtain inequalities (2.7)–(2.9).

Proof of Theorem 2.3. We only consider in detail the case (A). By Lemma 3.1, one has

$$f_1(x, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \Delta_{22} a_{mn} B_m^2(x) B_n^2(y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \Delta_{11} b_{mn} B_m^2(x) B_n^2(y),$$

where

$$b_{mn} = \Delta_{11} a_{mn}, \qquad B_k^2(z) = \frac{\sin^2 \frac{(k+1)z}{2}}{4 \sin^2 \frac{z}{2}}.$$

Let $x \in \left[\frac{\pi}{2^{m+1}}, \frac{\pi}{2^m}\right]$, $y \in \left[\frac{\pi}{2^{n+1}}, \frac{\pi}{2^n}\right]$, $I_{mn} = \left[\frac{\pi}{2^{m+1}}, \frac{\pi}{2^m}\right] \times \left[\frac{\pi}{2^{n+1}}, \frac{\pi}{2^n}\right]$, and let

$$\Psi_{mn}(x, y) = \sum_{\nu=[2^{n-1}]}^{\infty} \sum_{\mu=[2^{m-1}]}^{\infty} \Delta_{11} b_{\mu\nu} B_\mu^2(x) B_\nu^2(y).$$

Therefore,

$$f_1(x, y) \geq \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \chi_{I_{mn}}(x, y) \Psi_{mn}(x, y).$$

Then by Lemmas 3.5, 3.6, 3.7, and Jensen's inequality, we have

$$\begin{aligned}
&\left\| f_1(x, y) \right\|_{\Lambda_2^\alpha(\omega)}^\alpha \\
&= \int_0^{2\pi}\int_0^{2\pi} \omega(t_1, t_2)\Big(\big([f_1(x, y)]_y^*\big)_x^*(t_1, t_2)\Big)^\alpha \, dt_1\, dt_2 \\
&\geq C \int_0^{2\pi}\int_0^{2\pi} \omega(t_1, t_2)\left(\left(\left[\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \chi_{I_{mn}}(x, y)\Psi_{mn}(x, y)\right]_y^*\right)_x^*(t_1, t_2)\right)^\alpha dt_1\, dt_2 \\
&= C \int_0^{2\pi}\int_0^{2\pi} \omega(t_1, t_2)\left(\left[\left(\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \chi_{I_{mn}}(x, y)\Psi_{mn}(x, y)\right)^\alpha\right]_y^*\right)_x^*(t_1, t_2)\, dt_1\, dt_2 \\
&\geq C \int_0^{2\pi}\int_0^{2\pi} \omega(t_1, t_2)\left(\left[\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \big(\chi_{I_{mn}}(x, y)\Psi_{mn}(x, y)\big)^\alpha\right]_y^*\right)_x^*(t_1, t_2)\, dt_1\, dt_2 \\
&\geq C \int_0^{2\pi}\int_0^{2\pi} \omega(t_1, t_2)\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \Big(\big[\chi_{I_{mn}}(x, y)\Psi_{mn}(x, y)\big]_{yx}^*(t_1, t_2)\Big)^\alpha \, dt_1\, dt_2.
\end{aligned}$$

We will use the following fact (see [16]): there exists a set $J_{mn} \subset I_{mn}$ such that

$$J_{mn} = \{(x, y) \in I_{mn} : \Psi_{mn}(x, y) \geq C_1 2^{2(n+m)} \Delta_{11} b_{[2^{m-1}][2^{n-1}]}\}$$

and $\mu(J_{mn}) \geq C_2 \mu(I_{mn})$, where C_1, C_2 do not depend on $n, m, \{b_{mn}\}$. Hence, we have

$$\begin{aligned}
&\left\| f_1(x, y) \right\|_{\Lambda_2^\alpha(\omega)}^\alpha \\
&\quad\geq C \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \int_0^{2\pi}\int_0^{2\pi} \omega(t_1, t_2) \\
&\qquad \Big(\big(\chi_{I_{mn}}(x, y) 2^{2(n+m)} \Delta_{11} b_{[2^{m-1}][2^{n-1}]}\big)_{yx}^*(t_1, t_2)\Big)^\alpha \, dt_1\, dt_2 \\
&\quad= C \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \int_0^{2\pi}\int_0^{2\pi} \omega(t_1, t_2)\big(\chi_T(t_1, t_2) 2^{2(n+m)} \Delta_{11} b_{[2^{m-1}][2^{n-1}]}\big)^\alpha \, dt_1\, dt_2,
\end{aligned}$$

where

$$T = \left\{ (t_1, t_2) : \left(\chi_{I_{mn}}(x, y)\right)^*_{yx} (t_1, t_2) > \frac{1}{2} \right\}.$$

By Lemma 3.10, we write

$$\begin{aligned}
&\| f_1(x, y)\|^{\alpha}_{\Lambda_2^{\alpha}(\omega)} \\
&\geq C \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left(2^{2(n+m)} \Delta_{11} b_{[2^{m-1}][2^{n-1}]}\right)^{\alpha} \int_{C_2\pi 2^{-(n+2)}}^{C_2\pi 2^{-(n+1)}} \int_{C_2\pi 2^{-(m+2)}}^{C_2\pi 2^{-(m+1)}} \omega(t_1, t_2)\, dt_1\, dt_2 \\
&\geq C \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left(2^{2(n+m)} \Delta_{11} b_{[2^{m-1}][2^{n-1}]}\right)^{\alpha} \omega\left(\frac{\pi}{2^m}, \frac{\pi}{2^n}\right) \\
&\geq C \sum_{n=0}^{\infty} (n+1)^{2\alpha} \sum_{m=0}^{\infty} (m+1)^{2\alpha} w_{mn} \left(\Delta_{11} a_{mn}\right)^{\alpha}.
\end{aligned}$$

In a similar way, one can also obtain inequalities (2.7)–(2.9).

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