QUATERNIONIC DISCRETE SERIES FOR $Sp(1, 1)$

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Abstract

In this paper we study the analytic realization of the discrete series representations for the group $G = Sp(1, 1)$ as a subspace of the space of square integrable sections in a homogeneous vector bundle over the symmetric space $G/K := Sp(1, 1)/(Sp(1) \times Sp(1))$. We use the Szegő map to give expressions for the restrictions of the K-types occurring in the representation spaces to the submanifold AK/K.

1. Introduction

In [1], Gross and Wallach considered representations of simple Lie groups G with maximal compact subgroup K such that the associated symmetric space G/K has a G-equivariant quaternionic structure (cf. [10]). This amounts to the group K containing a normal subgroup isomorphic to $SU(2)$. In fact, there is an isomorphism $K \cong SU(2) \times M$ for a subgroup $M \subseteq K$, and by setting $L := U(1) \times M$, the associated homogeneous space G/L is fibred over G/K with fibres diffeomorphic to $P^1(\mathsf{C})$. The quaternionic discrete series representations are then realised on the Dolbeault cohomology groups $H^1(G/L, \mathcal{L})$, where $\mathcal{L} \to G/L$ is a holomorphic line bundle. In this model they are able to classify all the K-types occurring in each of the obtained discrete series representations. Moreover, they consider the continuation of the discrete series and characterise the unitarizability of the underlying $(\mathfrak{g}, K)$-modules.

In this paper we consider another model of the quaternionic discrete series. If π is a quaternionic discrete series representation realised on the cohomology group $H^1(G/L, \mathcal{L})$, and τ is its minimal K-type, then the *Schmid D-operator* acts on the sections of the homogeneous vector bundle $G \times_K V_\tau \to G/K$ where V_τ is some vector space on which the K-type is unitarily realized. The Hilbert space $\ker D \cap L^2(G, \tau)$ then furnishes another realization of the representation π. We consider the special case when $G = Sp(1, 1)$. In this case the symmetric space G/K can be embedded into the bounded symmetric domain $SU(2, 2)/S(U(2) \times U(2))$ consisting of complex 2×2-matrices of norm less than one. The restriction of the Harish-Chandra embedding to G/K then yields a global trivialization of the vector bundle $G \times_K V_\tau$. In this model we

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compute the restrictions to the submanifold $A \cdot 0$ of the highest weight vectors for the occurring K-types. (Here A is associated with a particular Iwasawa decomposition $G = NAK$.) These functions turn out to be fibrewise highest weight vectors with a hypergeometric function as a coefficient. Hypergeometric functions occur frequently in representation theory, not only for Lie groups. For example, in [7], they play a role in the context of Hecke algebras.

We compute the K-types by using the Szegő map defined by Knapp and Wallach in [5] which exhibits any discrete series representation as a quotient of a nonunitary principal series representations. The K-types are determined on the level of the principal series representation, and then the Szegő map is applied to compute the above mentioned restrictions.

The paper is organized as follows. In Section 2 we explicitly state some results from the structure theory of the Lie group $Sp(1, 1)$ that will be needed. In Section 3 we describe the models for the discrete series in the general context of induced representations, and also give an explicit global trivialization. Section 4 describes the Szegő map by Knapp and Wallach, and we also compute K-types on the level of a nonunitary principal series representation. In Section 5 we compute the images of the K-types under the Szegő map and trivialize them to yield vector valued functions. The main theorem of this paper is Theorem 8 of this section.

2. Preliminaries

2.1. The quaternion algebra

The quaternion algebra, H, is a four-dimensional associative algebra over R with generators i, j, k satisfying the relations

$$i^2 = j^2 = k^2 = -1$$
$$ij = k, \quad jk = i, \quad ki = j, \quad \text{and}$$
$$ji = -ij, \quad ik = -ki, \quad kj = -jk.$$

Moreover, H is equipped with an involution, $*$, given by

$$(a + bi + cj + dk)^* = a - bi - cj - dk, a, b, c, d \in \mathsf{R}.$$

The Euclidean norm on the vector space $\mathsf{R}^4 \simeq \mathsf{H}$ can be expressed in terms of this involution by

$$|(a, b, c, d)|^2 = a^2 + b^2 + c^2 + d^2 = (a + bi + cj + dk)^*(a + bi + cj + dk).$$

It follows immediately that the quaternions of norm one, $Sp(1)$, form a group. The algebra H can be embedded as a subalgebra of the algebra, $M_2(\mathsf{C})$, of 2×2

complex matrices by

(1) $$\iota : \mathsf{H} \to M_2(\mathsf{C}),$$

where

(2) $$\iota(a + bi + cj + dk) = \begin{pmatrix} a + bi & c + di \\ -(c - di) & a - bi \end{pmatrix}.$$

In particular, the generators $1, i, j, k$ are embedded as

$$\iota(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \iota(i) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \ \iota(j) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ \iota(k) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

The embedding ι also satisfies the relation

$$\iota\big((a + bi + cj + dk)^*\big) = \begin{pmatrix} a - bi & -c - di \\ -(-c + di) & a + bi \end{pmatrix}$$

$$= \begin{pmatrix} a + bi & c + di \\ -(c - di) & a - bi \end{pmatrix}^*,$$

so ι is a homomorphism of involutive algebras. We observe that, letting $z = a + bi$, $w = c + di$,

$$\iota(\mathsf{H}) = \left\{ \begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix} \mid z, w \in \mathsf{C} \right\}$$

and moreover, we have the identity

$$a^2 + b^2 + c^2 + d^2 = |z|^2 + |w|^2 = \det \begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix}.$$

In particular,

$$Sp(1) \simeq \left\{ \begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix} \mid |z|^2 + |w|^2 = 1 \right\} = SU(2).$$

2.2. The group $Sp(1,1)$

The real vector space $\mathsf{H}^2 \cong \mathsf{R}^8$ is also equipped with the structure of an H-module by

(3) $$(\alpha, (h_1, h_2)) \mapsto (\alpha h_1, \alpha h_2), \qquad \alpha, h_1, h_2 \in \mathsf{H}.$$

If we identify H^2 with the set of 2×1 matrices over H, there is a natural H-linear action of the matrix group $GL(2, \mathsf{H})$ on H^2 given by

$$(4) \qquad (h_1 \quad h_2) \mapsto (h_1 \quad h_2) \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Consider the real vector space H^2 equipped with the nondegenerate indefinite Hermitian form

$$(5) \qquad \langle \cdot, \cdot \rangle_{1,1} : ((h_1, h_2), (h_1', h_2')) \mapsto h_1(h_1')^* - h_2(h_2')^*.$$

Recall that the group $Sp(1, 1)$ is defined as

$$(6) \qquad Sp(1, 1) := \{g \in GL(2, \mathsf{H}) | \langle gh, gh' \rangle_{1,1} = \langle h, h' \rangle_{1,1}\},$$

where $h := (h_1, h_2), h' := (h_1', h_2')$. The condition that the form $\langle, \rangle_{1,1}$ be preserved can be reformulated as

$$(7) \qquad g^* J g = J,$$

where $g^* = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^* := \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}$, and $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

The embedding (1) induces an embedding (which we also denote by the same symbol)

$$(8) \qquad \iota : M_2(\mathsf{H}) \to M_4(\mathsf{C})$$

by

$$(9) \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} \iota(a) & \iota(b) \\ \iota(c) & \iota(d) \end{pmatrix}.$$

This embedding is a homomorphism of algebras with involution. Applying it to the identity (7) reveals that the image of $Sp(1, 1)$ is a subgroup of the group

$$(10) \qquad SU(2, 2) = \{g \in M_4(\mathsf{C}) \mid g^* \tilde{J} g = \tilde{J}, \det g = 1\},$$

$$\tilde{J} := \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}.$$

2.3. The symmetric space $B_1(\mathsf{H}) = Sp(1, 1)/(Sp(1) \times Sp(1))$

Let $B_1(\mathsf{H})$ denote the unit ball

$$(11) \qquad B_1(\mathsf{H}) := \{h \in \mathsf{H} \mid |h| < 1\}$$

in H. The group $G := Sp(1, 1)$ acts transitively on $B_1(\mathsf{H})$ by the fractional linear action

$$(12) \quad g(h) := (ah + b)(ch + d)^{-1}, \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G, \ \ h \in B_1(\mathsf{H}).$$

The isotropic subgroup for the origin is

$$(13) \qquad K := G_0 = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in G \right\} \cong Sp(1) \times Sp(1),$$

and hence we have the description

$$(14) \qquad B_1(\mathsf{H}) \cong G/K$$

of $B_1(\mathsf{H})$ as a homogeneous space. Moreover, from eq. (7) it follows immediately that the group G is invariant under the Cartan involution

$$(15) \qquad \theta(g) := (g^*)^{-1}$$

and hence the space G/K is equipped with the family of reflections $\{\sigma_{gK}\}_{gK \in G/K}$ given by

$$(16) \qquad \sigma_{gK}(xK) := g\theta(g^{-1}x)K$$

which furnish G/K with the structure of a Riemannian symmetric space of the noncompact type. In particular, for any $h \in B_1(\mathsf{H})$, there is a unique geodesic joining 0 and h. We let φ_h denote the reflection in the midpoint, m_h, of this geodesic. The isometry $\varphi_h \in G$ is uniquely characterized by the properties

$$(17) \qquad \varphi_h(m_h) = h,$$

$$(18) \qquad d\varphi_h(m_h) = -\operatorname{Id}_{T_{m_h}(B_1(\mathsf{H}))}.$$

We let $Sp(1)_1$ and $Sp(1)_2$ denote the "upper" and "lower" subgroups of K given by

$$Sp(1)_1 = \left\{ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \in K \right\},$$

$$Sp(1)_2 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} \in K \right\}.$$

For $k = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in K$, we will write $k = (a, d) := (k_1, k_2)$. The group $Sp(1)_1 \cong SU(2)$ is then a normal subgroup of K. We will write

$$(19) \qquad \pi : K \to K/Sp(1)_1 \cong SU(2)$$

for the natural projection onto the quotient group.

The group K acts on the tangent space $T_0(B_1(\mathsf{H}))$ by the differentials at 0 of the actions on $B_1(\mathsf{H})$. By the restriction to the subgroup $Sp(1)_1$ we have a representation of $SU(2)$ on $T_0(B_1(\mathsf{H}))$. We can define an $SU(2)$-representation, μ_h, on the tangent space $T_h(B_1(\mathsf{H}))$ for any h by the formula

$$(20) \quad \mu_h(l)v := d\varphi_h(0) \circ dl(0) \circ d\varphi_h^{-1}(h)v, \quad v \in T_h(B_1(\mathsf{H})), \ \ l \in Sp(1)_1.$$

The family $\{\mu_h\}_{h \in B_1(\mathsf{H})}$ of $SU(2)$-representations amounts to an action of $SU(2)$ as gauge transformations of the tangent bundle $T(B_1(\mathsf{H}))$. It is, however, not invariant under the action of G as automorphisms of the bundle. Indeed, if we define, for $h \in B_1(\mathsf{H})$, $g \in G$,

$$(21) \qquad \kappa_{g,h} := \varphi_{g(h)}^{-1} g \varphi_h \in K,$$

then

$$(22) \qquad \mu_{g(h)}(l)dg(h)v = dg(h)\mu_h(\kappa_{g,h}^{-1}l\kappa_{g,h})v,$$

where the element $\kappa_{g,h}^{-1}l\kappa_{g,h}$ belongs to the subgroup $Sp(1)_1$ since it is normalized by K. Hence the principal fibre bundle over $B_1(\mathsf{H})$ defined by the family $\{\mu_h\}_{h \in B_1(\mathsf{H})}$ is G-equivariant, though not elementwise. This shows that the symmetric space has a quaternionic structure and is a quaternionic symmetric space in the sense defined by Wolf (cf. [10]).

2.4. Harish-Chandra realization

We consider again the embedding ι defined in eq. (8). If we set

$G' = SU(2, 2),$
$K' = S(U(2) \times U(2))$
$$:= \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \in SU(2, 2) \mid A \in U(2), D \in U(2), \det(A)\det(D) = 1 \right\},$$

ι induces an embedding of pairs $(G, K) \hookrightarrow (G', K')$ and hence descends to an embedding

$$(23) \qquad G/K \hookrightarrow G'/K'$$

of the corresponding symmetric spaces. We will write $SU(2)_1$ and $SU(2)_2$ for the images $\iota(Sp(1)_1)$ and $\iota(Sp(1)_2)$ respectively.

The Hermitian symmetric space G'/K' is by the Harish-Chandra realization holomorphically, and G-equivariantly, equivalent to the bounded symmetric domain of type I

$$(24) \qquad G'/K' \cong \mathscr{D} := \{Z \in M_2(\mathsf{C}) \mid I_2 - Z^*Z > 0\}.$$

The action of G' on $\mathscr{D}$ is given by

$$(25) \qquad g(Z) = (AZ + B)(CZ + D)^{-1},$$

if $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is a block matrix with blocks of size 2×2. The symmetric space G/K is thus embedded into $\mathscr{D}$ as the subset

$$\mathcal{D} := \left\{ Z \in M_2(\mathsf{C}) \mid I_2 - Z^*Z > 0, Z = \begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix} z, w \in \mathsf{C} \right\},$$

and the action is given by

$$(26) \qquad \iota(g)(\iota(h)) = (\iota(a)\iota(h) + \iota(b))(\iota(c)\iota(h) + \iota(d))^{-1} = \iota(g(h)),$$

where $g(h)$ is the action defined in (12).

For any $Z \in \mathscr{D}$, the tangent space $T_Z(\mathscr{D})$ is identified with the complex vector space $M_2(\mathsf{C})$ and the differentials at 0 of the K' actions are given by

$$(27) \qquad dk(0)Z = AZD^{-1}, \qquad Z \in M_2(\mathsf{C}), k = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \in K'.$$

2.5. Cartan subalgebra and root system

Recall the Cartan involution θ on G (15). Its differential at the identity determines a decomposition of $\mathfrak{g}$ into the ± 1-eigenspaces $\mathfrak{k}$ and $\mathfrak{p}$ respectively,

$$(28) \qquad \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p},$$

where

$$\mathfrak{k} = \left\{ \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \mid X, Y \in \mathsf{H}, X^* = -X, Y^* = -Y, \operatorname{tr} X + \operatorname{tr} Y = 0 \right\},$$

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix} \mid X \in \mathsf{H} \right\}.$$

Let $\mathfrak{t} \subset \mathfrak{k}$ denote the subalgebra (realized as complex matrices)

$$(29) \qquad \mathfrak{t} = \left\{ \begin{pmatrix} si & 0 & 0 & 0 \\ 0 & -si & 0 & 0 \\ 0 & 0 & ti & 0 \\ 0 & 0 & 0 & -ti \end{pmatrix} \mid s, t \in \mathsf{R} \right\}.$$

It has a basis $\{H_1, H_2\}$, where

$$(30) \qquad H_1 = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$(31) \qquad H_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}.$$

Let $\mathfrak{g}^{\mathsf{C}}$ be the complexification of $\mathfrak{g}$, and $\mathfrak{k}^{\mathsf{C}}$ and $\mathfrak{p}^{\mathsf{C}}$ denote the complexifications of $\mathfrak{k}$ and $\mathfrak{p}$ respectively. The Cartan decomposition induces the decomposition

$$(32) \qquad \mathfrak{g}^{\mathsf{C}} = \mathfrak{k}^{\mathsf{C}} \oplus \mathfrak{p}^{\mathsf{C}}.$$

The complexification $\mathfrak{t}^{\mathsf{C}} \subset \mathfrak{k}^{\mathsf{C}}$ is a compact Cartan subalgebra of $\mathfrak{g}^{\mathsf{C}}$. Let Δ denote the set of roots, and for $\alpha \in \Delta$, we let $\mathfrak{g}^{\alpha}$ denote the corresponding root space. Then, for each $\alpha \in \Delta$ either the inclusion $\mathfrak{g}^{\alpha} \subseteq \mathfrak{k}^{\mathsf{C}}$ or the inclusion $\mathfrak{g}^{\alpha} \subseteq \mathfrak{p}^{\mathsf{C}}$ holds. In the first case, we call the root compact, and in the second case we call it non-compact. Let $\Delta_{\mathfrak{k}}$, and $\Delta_{\mathfrak{p}}$ denote the set of compact roots and the set of non-compact roots respectively. We order the roots by letting the ordered basis $\{-\sqrt{-1}H_1^*, -\sqrt{-1}H_2^*\}$ for the real vector space $\sqrt{-1}\mathfrak{t}^*$ define a lexicographic ordering. We let $\Delta_{\mathfrak{k}}^+$ denote the set of positive compact roots, and we let $\Delta_{\mathfrak{p}}^+$ denote the set of positive non-compact roots.

The roots are given by

$$(33) \qquad \Delta_{\mathfrak{k}} = \{\pm 2\sqrt{-1}H_1^*, \pm 2\sqrt{-1}H_2^*\},$$

$$(34) \qquad \Delta_{\mathfrak{p}} = \{\pm\sqrt{-1}(H_1^* + H_2^*), \pm\sqrt{-1}(H_1^* - H_2^*)\}.$$

In terms of quaternionic matrices, the corresponding root spaces are

$$(35) \qquad \mathfrak{g}_{\pm 2\sqrt{-1}H_1^*} = \mathsf{C} \begin{pmatrix} j \mp \sqrt{-1}k & 0 \\ 0 & 0 \end{pmatrix},$$

$$(36) \qquad \mathfrak{g}_{\pm 2\sqrt{-1}H_2^*} = \mathsf{C} \begin{pmatrix} 0 & 0 \\ 0 & j \mp \sqrt{-1}k \end{pmatrix},$$

and

$$\tag{37} \mathfrak{g}_{\pm\sqrt{-1}(H_1^* + H_2^*)} = \mathsf{C}\left(\begin{pmatrix} 0 & j \\ -j & 0 \end{pmatrix} \mp \begin{pmatrix} 0 & k \\ -k & 0 \end{pmatrix}\right),$$

$$\tag{38} \mathfrak{g}_{\pm\sqrt{-1}(H_1^* - H_2^*)} = \mathsf{C}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mp \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}\right)$$

respectively.

According to the lexicographic ordering on $\sqrt{-1}\mathfrak{t}^*$ determined by the ordered basis $\{-\sqrt{-1}H_1^*, -\sqrt{-1}H_2^*\}$, the positive noncompact roots are

$$\tag{39} \alpha_1 = -\sqrt{-1}H_1^* + \sqrt{-1}H_2^*,$$

$$\tag{40} \alpha_2 = -\sqrt{-1}H_1^* - \sqrt{-1}H_2^*,$$

and $\alpha_1 < \alpha_2$. Moreover, $\alpha_1 + \alpha_2 = -2\sqrt{-1}H_1^*$, i.e., the sum is a root. Hence $\{\alpha_1\}$ is a maximal sequence of strongly orthogonal positive noncompact roots. We let $B(\cdot, \cdot)$ denote the Killing form on $\mathfrak{g}$. We use it to identify $\mathfrak{g}^*$ with $\mathfrak{g}$ according to

$$\tag{41} \alpha(X) := B(X, H_\alpha), \qquad \alpha \in \mathfrak{g}^*, \quad X \in \mathfrak{g}.$$

Via this identification, the Killing form induces a bilinear form on $\mathfrak{g}^*$ by

$$\tag{42} \langle \alpha, \beta \rangle := B(H_\alpha, H_\beta).$$

For $\alpha \in \Delta$, we select a root vector $E_\alpha \in \mathfrak{g}_\alpha$ in such a way that

$$\tag{43} B(E_\alpha, E_{-\alpha}) = \frac{2}{\langle \alpha, \alpha \rangle}.$$

2.6. Iwasawa decomposition

Consider the maximal abelian subspace

$$\tag{44} \mathfrak{a} := \mathsf{R}(E_{\alpha_1} + E_{-\alpha_1}) = \left\{ \begin{pmatrix} 0 & tI_2 \\ tI_2 & 0 \end{pmatrix} \mid t \in \mathsf{R} \right\}$$

of $\mathfrak{p}$. The Iwasawa decomposition of $\mathfrak{g}$ with respect to $\mathfrak{a}$ is given by

$$\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{k}.$$

The corresponding global decomposition is

$$G = NAK,$$

where, written as quaternionic matrices

$$N = \left\{ \begin{pmatrix} 1+q & -q \\ q & 1-q \end{pmatrix} \mid q \in \mathsf{H}, q^* = -q \right\},$$

$$A = \left\{ \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \mid t \in \mathsf{R} \right\}.$$

REMARK 1. One can just as well use an Iwasawa decomposition $G = KAN$, the correpondence between these two decompositions being $(nak)^{-1} = k^{-1}a^{-1}n^{-1}$. In the sequel will see that it is sometimes convenient use this other decomposition as a means for finding the components in our decomposition.

In the sequel we will need the explicit formulas for the NAK-factorization

(45) $$g = n(g)a(g)\kappa(g)$$

of an element $g \in Sp(1, 1)$.

LEMMA 2. *For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and $\log a(g) = t(E_{\alpha_1} + E_{-\alpha_1})$, e^t and $\kappa(g)$ are given by*

$$e^t = \frac{(1 - |bd^{-1}|^2)^{1/2}}{|1 - bd^{-1}|}, \qquad \kappa(g) = e^t \begin{pmatrix} a - c & 0 \\ 0 & d - b \end{pmatrix}.$$

PROOF. The proof is by straightforward computation. We prove only the second statement.

The identity

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1+q & -q \\ q & 1-q \end{pmatrix} \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix}$$

is equivalent to

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
$$= \begin{pmatrix} (\cosh t + q(\cosh t - \sinh t))u_1 & (\sinh t + q(\sinh t - \cosh t))u_2 \\ (\sinh t + q(\cosh t - \sinh t))u_1 & (\cosh t + q(\sinh t - \cosh t))u_2 \end{pmatrix}.$$

Hence

$$a - c = e^{-t}u_1$$
$$d - b = e^{-t}u_2.$$

3. Quaternionic discrete series representations

3.1. Generalities

The Cartan decomposition (28) decomposes $\mathfrak{g}$ into two invariant subspaces for the adjoint action of K. Moreover, we have the isomorphism of K-representations

$$\text{(46)} \qquad \text{Ad}_{\mathfrak{g}/\mathfrak{k}} \cong \text{Ad}_{\mathfrak{p}} .$$

We extend $\text{Ad}_{\mathfrak{p}}$ to a complex linear representation of K on the space $(\mathfrak{g}/\mathfrak{k})^{\mathsf{C}} \cong \mathfrak{p}^{\mathsf{C}}$.

Consider now the surjective mapping

$$\text{(47)} \qquad p : G \to G/K, \qquad p(g) = gK.$$

The differential at the origin

$$\text{(48)} \qquad dp(e) : \mathfrak{g} \to T_{eK}(G/K)$$

intertwines the adjoint action of K on $\mathfrak{g}$ with the differential action on the tangent space $T_{eK}(G/K)$. The kernel of $dp(e)$ is $\mathfrak{k}$, and as K-representations we thus have the isomorphism

$$\text{(49)} \qquad \text{Ad}_{\mathfrak{p}}^* \cong (dK(o)^{\mathsf{C}})^* ,$$

where the right hand side denotes the complex linear dual to the representation given by the complexified actions of the tangent maps at the origin. Using the quotient mapping induced by (48) and the realization of the differential action of K at the tangent space $T_0(\mathscr{D})$, we obtain the formula

$$\text{(50)} \qquad \text{Ad}_{\mathfrak{p}}^*(k)Z = (A^{-1})^t Z D^t .$$

Here $Z \in M_2(\mathsf{C}) \cong T_0^*(\mathscr{D}) \cong (T_0(\mathscr{D})^{\mathsf{C}})^*$, and $k = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \in K \cong SU(2) \times SU(2)$. The restriction, $\text{Ad}_{\mathfrak{p}}^*|_{SU(2)_1}$, to the subgroup $SU(2)_1$ is then given by

$$\text{(51)} \qquad \text{Ad}_{\mathfrak{p}}^*|_{SU(2)_1}(k)Z = (A^{-1})^t Z .$$

If we let $\{E_{ij}\}$, $i, j = 1, 2$ denote the standard basis for the complex vector space (i.e., E_{ij} has 1 at the position on the ith row and jth column and zeros elsewhere), then clearly the subspace

$$\text{(52)} \qquad V := \mathsf{C}E_{11} \oplus \mathsf{C}E_{21} \cong \mathsf{C}^2$$

spanned by the basis elements in the first column is $SU(2)_1$-invariant. Likewise, the subspace spanned by the basis elements of the second column is

invariant. We now let τ denote the representation given by restricting the K-representation $\mathrm{Ad}^*_{\mathfrak{p}}|_{SU(2)_1}$ to the subspace V, and let τ_k denote the kth symmetric tensor power of the representation τ. Then clearly, the natural identification of τ_k with a representation of $SU(2)$ is equivalent to the standard representation of $SU(2)$ on the space of homogeneous polynomial functions $p(z, w)$ on C^2 of degree k, i.e., we have

$$(53) \qquad \tau_k(l_1, l_2)p(z, w) := p(l_1^{-1}(z, w)) := p(az + \overline{b}w, -bz + \overline{a}w),$$

where $l_1^{-1} = \begin{pmatrix} a & \overline{b} \\ -b & \overline{a} \end{pmatrix} \in SU(2)$. We let V_{τ_k} denote the representation space for τ_k. The (smoothly) induced representation $\mathrm{Ind}_K^G(\tau_k)$ is then defined on the space

$$C^\infty(G, \tau_k) := \left\{ f \in C^\infty(G, V_{\tau_k}) \mid f(gl^{-1}) = \tau_k(l)f(g) \; \forall g \in G \; \forall l \in K \right\},$$

i.e., on the space of smooth sections on the G-homogeneous vector bundle

$$(54) \qquad \mathcal{V}^k \to G/K := G \times_K V_{\tau_k} \to G/K.$$

We fix the K-invariant inner product on $\langle \, , \rangle_k$ on V_{τ_k} given by

$$(55) \qquad \langle p, q \rangle_k := p(\partial)(q^*)(0),$$

where $p(\partial)$ is the differential operator defined by substituting $\frac{\partial}{\partial z}$ for z, and $\frac{\partial}{\partial w}$ for w in the polynomial function $p(z, w)$, and

$$\left(\sum_{j=1}^{k} a_j z^j w^{k-j} \right)^* := \sum_{j=1}^{k} \overline{a_j} z^j w^{k-j}.$$

We use this inner product to define an Hermitian metric on $\mathcal{V}^k$ by

$$(56) \qquad h_Z(u, v) := \langle (g^{-1})_Z u, (g^{-1})_Z v \rangle_k, \qquad u, v \in \mathcal{V}_Z^k,$$

where $Z = gK$ and $(g^{-1})_Z$ denotes the fibre map $\mathcal{V}_Z^k \to \mathcal{V}_0^k \cong V_k$ associated with g^{-1}. For a fixed choice, ι, of G-invariant measure on G/K we define $L^2(\mathrm{Ind}_K^G(\tau_k))$ as the Hilbert space completion of the space

$$(57) \qquad \left\{ s \in \Gamma(G/K, \mathcal{V}^k) \mid \int_{G/K} h_Z(s, s) \, d\iota(Z) < \infty \right\}.$$

The tensor product representation $\tau_k \otimes \mathrm{Ad}(K)|_{\mathfrak{p}^c}$ decomposes into K-types according to

$$(58) \qquad \tau_k \otimes \mathrm{Ad}(K)|_{\mathfrak{p}^c} = \sum_{\beta \in \Delta_{\mathfrak{p}}} m_\beta \pi_{\beta - k\sqrt{-1}H_1^*},$$

where $m_\beta \in \{0, 1\}$, and $\pi_{\beta - k\sqrt{-1}H_1^*}$ is the irreducible representation of K with highest weight $\beta - k\sqrt{-1}H_1^*$. Let τ_k^- be the subrepresentation of the tensor product given by

$$(59) \qquad \tau_k^- = \sum_{\beta \in \Delta_{\mathfrak{p}}^-} m_\beta \pi_{\beta - k\sqrt{-1}H_1^*},$$

and let V_k^- be the subspace of $V_{\tau_k} \otimes \mathfrak{p}^{\mathsf{C}}$ on which τ_k^- operates. Let $P : V_{\tau_k} \otimes \mathfrak{p}^{\mathsf{C}} \to V_k^-$ be the orthogonal projection. Define the space $C^\infty(G, \tau_k^-)$ in analogy with (3.1). We recall that the *Schmid D operator* is a differential operator mapping the space $C^\infty(G, \tau_k)$ into $C^\infty(G, \tau_k^-)$ and is defined as

$$(60) \qquad Df(g) = \sum_i P(X_i f(g) \otimes X_i),$$

where $\{X_i\}$ is any orthonormal basis for $\mathfrak{p}^{\mathsf{C}}$, and $X_i f$ denotes left invariant differentiation, i.e.,

$$Xf(g) := \frac{d}{dt} f(g \exp(tX))|_{t=0}, \qquad X \in \mathfrak{p},$$

$$Zf(g) := Xf(g) + iY(g), \qquad Z = X + iY \in \mathfrak{p}^{\mathsf{C}}.$$

The subspace $\ker D \cap L^2(\mathrm{Ind}_K^G(\tau_k))$ is then invariant under the left action of G and defines an irreducible representation of G belonging to the quaternionic discrete series. We let $\mathcal{H}_k$ denote this representation space. By [1], it belongs to the discrete series for $k \geq 1$.

REMARK 3. The model we use to describe the Hilbert space $\mathcal{H}_k$ can be used to realize any discrete series representation by induction from K to G of the minimal K-type for any pair (G, K) where G is semisimple and K is maximal compact (cf. [4]). By [1], for $k \geq 1$, τ_k occurs as a minimal K-type for some discrete series representation of $G = Sp(1, 1)$.

3.2. Global trivialization

Let us for a while view the representation space $\mathcal{H}_k$ as a space of sections of the vector bundle $\mathcal{V}^k \to G/K$. We recall the diffeomorphism $G/K \cong \mathcal{D}$ given by $gK \mapsto g \cdot 0$. This lifts to a global trivialization, Φ, of the bundle $\mathcal{V}^k \to G/K$ given by

$$(61) \quad \Phi : G \times_K V^{\tau_k} \to \mathcal{D} \times V_{\tau_k}, \qquad \Phi([(g, v)]) := (g \cdot 0, \tau_k(J(g, 0))v),$$

where $J(g, Z)$ denotes the K^{C}-component of $g \exp Z$ – the *automorphic factor of g at Z* (cf. [8]).

If $F : G \to V^{\tau_k}$ is a function in $C^\infty(G, \tau_k)$, its trivialized counterpart is the function $f : \mathcal{D} \to V_{\tau_k}$ given by

$$(62) \qquad\qquad f(g \cdot 0) := \tau_k(J(g, 0))F(g).$$

In the trivialized picture, the group G acts on functions on $\mathcal{D}$ by

$$(63) \qquad\qquad gf(Z) := \tau_k(J(g^{-1}, Z))^{-1} f(g^{-1}Z).$$

More explicitly, if $g^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ (considered as a matrix in $SU(2, 2)$), then

$$J(g^{-1}Z) = \begin{pmatrix} A - (AZ + B)(CZ + D)^{-1}C & 0 \\ 0 & D \end{pmatrix} \in SL(4, \mathsf{C}),$$

and

$$gf(Z) = \overset{k}{\odot}\, (A - (AZ + B)(CZ + D)^{-1}C)\, f((AZ + B)(CZ + D)^{-1}).$$

The action of $SU(2)$ on the vector space V_{τ_k} is here naturally extended to an action of $SL(2, \mathsf{C})$ by the formula (53).

In the trivialized picture, the norm (57) can also be described explicitly.

PROPOSITION 4. *Let $k \geq 1$. In the realization of the Hilbert space $\mathcal{H}_k$ as a space of V_{τ_k}-valued functions on $\mathcal{D}$, the norm (57) is given by*

$$(64) \qquad \|f\|_k := \int_{B_1(\mathsf{H})} (1 - |q|^2)^k \langle f(q), f(q)\rangle_k (1 - |q|^2)^{-4}\, dm(q).$$

PROOF. For $Z = gK$, a fibre map $(g^{-1})_Z : V_{\tau_k} \to V_{\tau_k}$ is given by

$$(65) \qquad\qquad (g^{-1})_Z v = \tau_k(J(g, 0))^{-1} v.$$

If $g = \begin{pmatrix} \cosh t\, I_2 & \sinh t\, I_2 \\ \sinh t\, I_2 & \cosh t\, I_2 \end{pmatrix}$, the automorphic factor $J(g, 0)$ is given by (cf. [4])

$$(66)$$
$$J(g, 0) = \begin{pmatrix} \cosh t^{-1} I_2 & 0 \\ 0 & \cosh t\, I_2 \end{pmatrix} = \begin{pmatrix} (1 - \tanh^2 t)^{1/2} I_2 & 0 \\ 0 & \cosh t\, I_2 \end{pmatrix}.$$

A general point $Z \in \mathcal{D}$ can be described as $Z = kgK$ for g as above. The cocycle condition

$$(67) \qquad\qquad J(kg, 0) = J(k, g0)J(g, 0)$$

then implies that

$$(68) \qquad J(kg, 0) = \begin{pmatrix} k_1(1 - \tanh^2 t)^{1/2} I_2 & 0 \\ 0 & k_2 \cosh t I_2 \end{pmatrix},$$

if $k = (k_1, k_2) \in SU(2) \times SU(2)$. Hence, for $k = 1$

$$h_Z(u, v) = \mathrm{tr}\left((k_1(1 - \tanh^2 t)^{1/2} I_2)^t u((k_1(1 - \tanh^2 t)^{1/2} I_2)^t v)^* \right)$$
$$= \mathrm{tr}\left((I_2 - ZZ^*)^t uv^* \right).$$

For arbitrary k, we have

$$(69) \qquad h_Z(u, v) = \mathrm{tr}(\overset{k}{\odot} (I_2 - ZZ^*)^t uv^*).$$

By analogous considerations, it follows that the invariant measure is given by

$$d\iota(Z) = \det(I_2 - Z^* Z)^{-2} dm(Z),$$

where $dm(Z)$ denotes the Lebesgue measure. Hence, we obtain the formula

$$(70) \qquad \int_{\mathcal{D}} \langle \overset{k}{\odot} (I_2 - ZZ^*)^t f(Z), f(Z) \rangle_k \det(I_2 - Z^* Z)^{-2} \, dm(Z)$$

for the norm (57). In quaternionic notation, this translates into the statement of the proposition.

4. Principal series representations and the Szegő map

In this section we will consider a realization of the discrete series representation $\ker D \cap L^2(\mathrm{Ind}\, K^G(\tau_k))$ as a quotient of a certain nonunitary principal series representation. We first state the theorem, and then we investigate how the given principal series representation decomposes into K-types. From now on we fix the number k and simply write τ for τ_k.

Recall the maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ and consider the parabolic subgroup

$$P = MAN$$

of G, where

$$M = Z_K(A) = \left\{ \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} \mid u \in SU(2) \right\},$$

and A and N are the ones that occur in the Iwasawa decomposition. Let σ be the restriction of the representation τ to the subgroup M. Then, clearly, the subspace defined by the M-span of the τ-highest weight-vector equals

V_τ and the representation σ is also irreducible. We will hereafter denote this representation space by V_σ. Recall the identification of V_σ with a space of homogeneous polynomials. We thus adopt a somewhat abusive notation and write z^σ for the highest weight-vector. Let $\nu \in \mathfrak{a}^*$ be a real-valued linear functional and consider the representation

$$\sigma \otimes \exp(\nu) \otimes 1 \tag{71}$$

of P. The induced representation $\mathrm{Ind}_P^G(\sigma \otimes \exp(\nu) \otimes 1)$ is defined on the set of continuous functions $f : G \to V_\sigma$ having the P-equivariant property

$$f(gman) = e^{-\nu(\log a)} \sigma(m)^{-1} f(g). \tag{72}$$

The action of G on this space is given by left translation,

$$\mathrm{Ind}_P^G(\sigma \otimes \exp(\nu) \otimes 1)(f)(x) := L_{g^{-1}} f(x) = f(g^{-1} x).$$

Consider now the smoothly induced representation $\mathrm{Ind}_M^K(\sigma)$ which operates on the space, $C^\infty(K, \sigma)$, of all smooth functions $f : K \to V_\sigma$ having the M-equivariance property

$$f(km) = \sigma(m)^{-1} f(k) \tag{73}$$

with K-action given by left translation. The Iwasawa decomposition $G = KAN$ shows that, a fortiori, $G = KMAN$ (although this factorization is not unique). Given a linear functional $\nu \in \mathfrak{a}^*$, we can therefore extend any such function on K to a function on G by setting

$$f(kman) = e^{-\nu(\log a)} \sigma(m)^{-1} f(k), \qquad \text{for} \quad g = kman.$$

The equivariance property (73) of f guarantees that this is indeed well-defined even though the factorization of g is not. The extended function f has the P-equivariance property (72). In fact, this extension procedure defines a bijection between the representation spaces of the representations $\mathrm{Ind}_M^K(\sigma)$ and $\mathrm{Ind}_P^G(\sigma \otimes \exp(\nu) \otimes 1)$. There is a natural pre-Hilbert space structure on this representation space given by

$$\|f\|^2 = \int_K \|f(k)\|_\sigma^2 \, dk,$$

where $\| \cdot \|_\sigma$ denotes the inner product on V_σ and dk is the Haar measure on K. The completion of the space of M-equivariant smooth functions $K \to V_\sigma$ with respect to this sesquilinear form can be identified with the space of all square-integrable V_σ-valued functions having the property (73). We will denote the

K-representation on this space by $L^2(\mathrm{Ind}_M^K(\sigma))$. By the extension procedure using ν described above, this completion can be extended to the space of all P-equivariant V_σ-valued functions on G such that the restriction to K is square-integrable.

We now state the theorem by Knapp and Wallach.

THEOREM 5 ([5], Thm. 6.1). *The Szegő mapping with parameters τ and ν given by*

$$(74) \qquad S(f)(x) := \int_K e^{\nu \log a(lx)} \tau(\kappa(lx)^{-1}) f(l^{-1}) \, dl$$

carries the space $C^\infty(K,\sigma)$ into $C^\infty(G,\tau) \cap \ker D$, provided that ν and τ are related by the formula

$$(75) \qquad \nu(E_{\alpha_1} + E_{-\alpha_1}) = \frac{2\langle -k\sqrt{-1}H_1^* + n_1\alpha_1, \alpha_1\rangle}{\langle \alpha_1, \alpha_1\rangle},$$

where

$$n_1 = |\{\gamma \in \Delta_\mathfrak{p}^+ \mid \alpha(\gamma) = \alpha_1 \text{ and } \alpha_1 + \gamma \in \Delta\}|.$$

In this case $n_1 = 1$, since the root α_2 is the only one satisfying the above condition. Moreover, an easy calculation gives that

$$(76) \qquad E_{\alpha_1} = \frac{1}{2}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}\right).$$

Hence, the condition (75) takes the form

$$(77) \qquad \nu\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) = k + 2.$$

Hereafter, we will make the identification

$$(78) \qquad \nu = k + 2$$

of the functional with a natural number. We now proceed with a more detailed study of the representation $L^2(\mathrm{Ind}_M^K(\sigma))$.

LEMMA 6. *The representation $L^2(\mathrm{Ind}_M^K(\sigma))$ is K-equivalent to $L^2(K/M) \otimes V_\sigma$.*

PROOF. Let f be a continuous function from K to $V_\sigma(= V_\tau)$ having the property of M-equivariance

$$f(km) = \sigma(m)^{-1} f(k), \qquad k \in K, \ m \in M.$$

Then the function

$$\tilde{f}(k) := \tau(k) f(k)$$

is clearly right M-invariant and hence we can define the function $F : K/M \to V_\sigma$ by

$$F(kM) = \tilde{f}(k).$$

This is obviously well-defined. By choosing a basis $\{e_j\}$ for V_σ, we can write

$$F(kM) = \sum_j F_j(kM) e_j$$

for some complex-valued functions F_j. We now define a mapping

$$T : L^2(\mathrm{Ind}_M^K(\sigma)) \to L^2(K/M) \otimes V_\sigma$$

by

$$Tf := \sum_j F_j \otimes e_j.$$

To see that this mapping is a bijection, note that any vector in the Hilbert space $L^2(K/M) \otimes V_\sigma$ can be uniquely expressed in the form $\sum_j G_j \otimes e_j$. We can thus define a mapping

$$S : L^2(K/M) \otimes V_\sigma \to L^2(\mathrm{Ind}_M^K(\sigma))$$

by

$$S\left(\sum_j G_j \otimes e_j\right)(k) := \tau(k)^{-1} \sum_j G_j(kM) e_j$$

and it is easy to see that S is the inverse of T.

It remains now only to prove the K-equivariance. Pick therefore any element $\sum_j G_j \otimes e_j$ from the Hilbert space on the right hand side. We have

$$k\left(\sum_j G_j \otimes e_j\right) = \sum_j G_j \circ L_{k^{-1}} \otimes \tau(k) e_j.$$

If we denote the matrix coefficients of $\tau(k)$ with respect to the basis $\{e_j\}$ by $\tau(k)_{ij}$, we have

$$\tau(k) e_j = \sum_i \tau(k)_{ij} e_i$$

and hence

$$\sum_j G_j \circ L_{k^{-1}} \otimes \tau(k) e_j = \sum_{i,j} G_j \circ L_{k^{-1}} \otimes \tau(k)_{ij} e_i.$$

Applying S to the above expression yields

$$S\left(\sum_{i,j} G_j \circ L_{k^{-1}} \otimes \tau(k)_{ij} e_i\right)(k') = \sigma(k')^{-1} \sum_{i,j} G_j(k^{-1}k'M)\tau(k)_{ij} e_i$$

$$= \tau(k')^{-1}\tau(k) \sum_j G_j(k^{-1}k'M)e_j$$

$$= S\left(\sum_j G_j \otimes e_j\right) \circ L_{k^{-1}}(k').$$

We shall now examine the left action of K on the $L^2(K/M)$-factor in the tensor product more closely. In particular, we are interested in a certain K-invariant subspace defined by a subclass of the K-types occurring in $L^2(K/M)$. We recall the identification of the K-representation τ_j with a standard representation of $SU(2)$. We therefore let τ_j also denote the corresponding $SU(2)$-representation, and we let V_j denote the associated vector space of polynomials. Any irreducible representation of $K = SU(2) \times SU(2)$ is isomorphic to a tensor product of irreducible $SU(2)$-representations, i.e., it is realized on a space

$$(79) \qquad\qquad\qquad V_j^* \otimes V_i,$$

for some $i, j \in \mathsf{N}$. With the fixed ordering of the roots, the polynomial function $(z, w) \mapsto z^j$ is a highest weight vector in V_j, and the polynomial function $(z, w) \mapsto w^j$ is a lowest weight vector. We will use the abusive notation where they are denoted by z^j and w^j respectively.

PROPOSITION 7. *The algebraic sum*

$$(80) \qquad\qquad\qquad W := \bigoplus_{j \in \mathsf{N}} V_j^* \otimes V_{k+j}$$

(where $k \in \mathsf{N}$ corresponds to the representation σ) of K-types is a subspace of $L^2(\mathrm{Ind}_M^K(\sigma))$. The highest weight vector for the K-type $V_j^* \otimes V_{k+j}$ is given by the function

$$(81) \qquad\qquad f_j(k) := \langle \tau_j \circ \pi(k)z^j, w^j \rangle_j \, \tau(k)^{-1} z^\sigma,$$

where π denotes the projection onto the first factor in $SU(2) \times SU(2)$.

PROOF. For

$$k = \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix}, \qquad k' = \begin{pmatrix} u_1' & 0 \\ 0 & u_2' \end{pmatrix} \in K,$$

we have

$$k'kM = \begin{pmatrix} u'_1 u_1 & 0 \\ 0 & u'_2 u_2 \end{pmatrix} M = \begin{pmatrix} u'_1 u_1 u_2^{-1} (u'_2)^{-1} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} u'_2 u_2 & 0 \\ 0 & u'_2 u_2 \end{pmatrix} M$$

$$= \begin{pmatrix} u'_1 u_1 u_2^{-1} (u'_2)^{-1} & 0 \\ 0 & I \end{pmatrix} M$$

and this shows that the left action of $K = SU(2) \times SU(2)$ on $L^2(K/M)$-functions is equivalent to the action $L_{g^{-1}} \otimes R_h$ on $L^2(SU(2))$:

$$(L_{g^{-1}} \otimes R_h) f(l) := f(g^{-1} l h).$$

Then, by the Peter-Weyl Theorem, $L^2(K/M)$ decomposes into K-types according to

(82) $$L^2(K/M) \simeq \bigoplus_{j \in \widehat{SU(2)}} (V_j \otimes V_j^*).$$

Tensoring with V_k gives the sequence of K-isomorphisms

$$L^2(K/M) \otimes V_k \simeq \bigoplus_{j \in \widehat{SU(2)}} (V_j \otimes V_j^*) \otimes V_k \simeq \bigoplus_{j \in \widehat{SU(2)}} (V_j^* \otimes V_j) \otimes V_k$$

$$\simeq \bigoplus_{j \in \widehat{SU(2)}} V_j^* \otimes (V_j \otimes V_k).$$

Moreover, each term $(V_j \otimes V_k)$ has a *Clebsch-Gordan* decomposition

$$(V_j \otimes V_k) \simeq (V_{k+j} \oplus \cdots)$$

and therefore each term $V_j \otimes V_{k+j}$ will constitute a K-type in $L^2(K/M) \otimes V_k$. Such a K-type has a highest weight-vector $(w^j)^* \otimes z^{k+j}$. Using first the embedding into $V_j^* \otimes (V_j \otimes V_k)$ and then the K-isomorphism given by Lemma 6, we see that highest weight-vector corresponds to the M-equivariant function

(83) $$f_j(k) := \langle \tau_j \circ \pi(k) z^j, w^j \rangle_j \, \sigma(k)^{-1} z^\sigma.$$

5. Realization of K-types

By [1], the only K-types occurring in the quaternionic discrete series for $Sp(1, 1)$ are the ones that form the subspace W in Proposition 7. In this section we compute their realizations as V_τ-valued functions on $B_1(\mathsf{H})$ when restricted

to the submanifold

$$(84) \qquad\qquad A \cdot 0 = \{t \in \mathsf{H} \mid -1 < t < 1\}$$

of $B_1(\mathsf{H})$. For $s \in \mathsf{R}$, we let

$$(85) \qquad\qquad a_s = \begin{pmatrix} \cosh s & \sinh s \\ \sinh s & \cosh s \end{pmatrix} \in Sp(1,1).$$

Then $a_s \cdot 0 = \tanh s \in A \cdot 0$. We start by computing the Szegő images of the f_j when restricted to points a_s.

Each of the standard $SU(2)$-representations, V_N, can be naturally extended to a representation of $GL(2, \mathsf{C})$ by

$$(86) \qquad\qquad p \mapsto p \circ g^{-1}, \qquad p \in V_N, \ g \in GL(2, \mathsf{C}).$$

This action of $GL(2, \mathsf{C})$ will occur frequently in the sequel.

LEMMA 8. *The Szegő transform of the highest weight-vector f_j is given by*

$$Sf_j(a_s) = (\cosh s)^{-\nu}$$

$$\times \int_{SU(2)} (\det(1 - l \tanh s))^{-(\nu+\sigma)/2} \langle \tau_j(l^{-1}) z^j, w^j \rangle_j \, \sigma(1 - l \tanh s) z^\sigma \, dl$$

when restricted to the A-component in the decomposition $G = NAK$.

PROOF. Take

$$k = \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} \qquad \text{and} \qquad x = \begin{pmatrix} \cosh s & \sinh s \\ \sinh s & \cosh s \end{pmatrix}.$$

Then

$$kx = \begin{pmatrix} u_1 \cosh s & u_1 \sinh s \\ u_2 \sinh s & u_2 \cosh s \end{pmatrix},$$

and Lemma 2 gives that

$$e^{\nu(\log H(kx))} = \left(\frac{1 - |u_1 u_2^{-1} \tanh s|^2}{|1 - u_1 u_2^{-1} \tanh s|^2} \right)^{\nu/2},$$

$$\kappa(kx) = \left(\frac{1 - |u_1 u_2^{-1} \tanh s|^2}{|1 - u_1 u_2^{-1} \tanh s|^2} \right)^{1/2}$$

$$\times \begin{pmatrix} u_1 \cosh s - u_2 \sinh s & 0 \\ 0 & u_2 \cosh s - u_1 \sinh s \end{pmatrix}.$$

Hence

$$\tau(\kappa(lx))^{-1} = \left(\frac{1 - |u_1 u_2^{-1} \tanh s|^2}{|1 - u_1 u_2^{-1} \tanh s|^2}\right)^{-\sigma} \sigma(u_1 \cosh s - u_2 \sinh s)^{-1}$$

and we get

$$
\begin{aligned}
Sf_j(a_s) &= \int_K \left(\frac{1 - |u_1 u_2^{-1} \tanh s|^2}{|1 - u_1 u_2^{-1} \tanh s|^2}\right)^{(\nu-\sigma)/2} \langle \tau_j \circ \pi(k^{-1})z^j, w^j \rangle_j \\
&\quad \times \sigma(u_1 \cosh s - u_2 \sinh s)^{-1}\sigma(u_1)z^\sigma \, dk \\
&= \int_K \left(\frac{1 - |\tanh s|^2}{|1 - u_1 u_2^{-1} \tanh s|^2}\right)^{(\nu-\sigma)/2} \langle \tau_j \circ \pi(k^{-1})z^j, w^j \rangle_j \\
&\quad \times \sigma(\cosh s - u_1^{-1}u_2 \sinh s)^{-1}z^\sigma \, dk \\
&= (\cosh s)^{-\nu} \int_K |1 - u_1 u_2^{-1} \tanh s|^{\sigma-\nu} \langle \tau_j \circ \pi(k^{-1})z^j, w^j \rangle_j \\
&\quad \times \sigma(1 - u_1^{-1}u_2 \tanh s)^{-1}z^\sigma \, dk.
\end{aligned}
$$

Using the identities

$$(1 - u_1^{-1}u_2 \tanh s)^{-1} = \frac{1 - u_2^{-1}u_1 \tanh s}{|1 - u_2^{-1}u_1 \tanh s|^2}$$

and

$$|1 - u_1 u_2^{-1} \tanh s| = |u_1^{-1}(1 - u_1 u_2^{-1} \tanh s)u_1| = |1 - u_2^{-1}u_1 \tanh s|$$

in the above equality yields

$$Sf_j(a_s) = (\cosh s)^{-\nu}$$
$$\times \int_K |1 - u_2^{-1}u_1 \tanh s|^{-(\sigma+\nu)} \langle \tau_j \circ \pi(k^{-1})z^j, w^j \rangle_j \times \sigma(1 - u_2^{-1}u_1 \tanh s)z^\sigma \, dk.$$

We observe that the integrand is right M-invariant. In fact,

$$
\begin{aligned}
&(\cosh s)^{-\nu} \int_K |1 - u_1 u_2^{-1} \tanh s|^{-(\sigma+\nu)} \langle \tau_j \circ \pi(k^{-1})z^j, w^j \rangle_j \\
&\qquad \times \sigma(1 - u_2^{-1}u_1 \tanh s)z^\sigma \, dk \\
&= (\cosh s)^{-\nu} \int_K |1 - \pi(k^{-1})^{-1} \tanh s|^{-(\sigma+\nu)} \langle \tau_j \circ \pi(k^{-1})z^j, w^j \rangle_j \\
&\qquad \times \sigma(1 - \pi(k^{-1})^{-1} \tanh s)z^\sigma \, dk.
\end{aligned}
$$

Therefore, it can be written as an integral over the coset space $K/M \simeq SU(2)$, i.e.,

$$Sf_j(a_s) = (\cosh s)^{-\nu}$$

$$\times \int_{SU(2)} |1 - l^{-1} \tanh s|^{-(\sigma+\nu)} \langle \tau_j(l) z^j, w^j \rangle_j \, \sigma(1 - l^{-1} \tanh s) z^\sigma \, dl.$$

Making the change of variables $l \mapsto l^{-1}$, and using the invariance of the Haar measure on $SU(2)$ under this map, yields

$$Sf_j(a_s) = (\cosh s)^{-\nu}$$

$$\times \int_{SU(2)} |1 - l \tanh s|^{-(\sigma+\nu)} \langle \tau_j(l^{-1}) z^j, w^j \rangle_j \, \sigma(1 - l \tanh s) z^\sigma \, dl,$$

and this finishes the proof.

5.1. Highest weight-vectors for K-types

The polynomial functions p_{l_1,l_2} defined by

$$p_{l_1,l_2}(z, w) := z^{l_1} w^{l_2},$$

for which $l_1 + l_2 = N$ form a basis for V_N. Occasionally we will however use the somewhat ambiguous notation $z^{l_1} w^{l_2}$ when there is no risk for misinterpretation.

We write $\zeta = \tanh s$ and consider the action of $(1 - \zeta l)$ on the basis vector p_{l_1,l_2}. We have

$$(87) \quad (1 - \zeta l) p_{l_1,l_2}(z, w) = \left((1 - \zeta l)^{-1} \begin{pmatrix} z \\ w \end{pmatrix} \right)_1^{l_1} \left((1 - \zeta l)^{-1} \begin{pmatrix} z \\ w \end{pmatrix} \right)_2^{l_2},$$

where the subscripts denote the projection functions

$$(z, w)_1 = z, \qquad (z, w)_2 = w$$

onto the first and second coordinate respectively.

The binomial theorem gives the following expression for the first factor

above:

$$\left((1 - \zeta l)^{-1} \begin{pmatrix} z \\ w \end{pmatrix}\right)_1^{l_1}$$

$$= \det(1 - \zeta l)^{-l_1} \sum_{j_1=0}^{l_1} \binom{l_1}{j_1} z^{j_1} (-\zeta)^{l_1-j_1} \left(l^{-1} \begin{pmatrix} z \\ w \end{pmatrix}\right)_1^{l_1-j_1}$$

$$= \det(1 - \zeta l)^{-l_1} \sum_{j_1=0}^{l_1} z^{j_1} \binom{l_1}{j_1} (-\zeta)^{l_1-j_1} l p_{l_1-j_1,0}(z, w),$$

and the second factor has a similar expression. Substituting these into (87) yields the double sum

$$(88) \quad (1 - \zeta l) p_{l_1,l_2}(z, w) = \det(1 - \zeta l)^{-\sigma}$$

$$\times \sum_{j_1=0}^{l_1} \sum_{j_2=0}^{l_2} \binom{l_1}{j_1} \binom{l_2}{j_2} (-\zeta)^{(l_1+l_2-j_1-j_2)} z^{j_1} w^{j_2} l p_{l_1-j_1,l_2-j_2}(z, w).$$

Denote the normalization of the basis vector $p_{r,s}$ by $e_{r,s}$. Then

$$p_{r,s} = (r!s!)^{1/2} e_{r,s}$$

and the term $l p_{l_1-j_1,l_2-j_2}$ in (88) can be written as the sum

$$(89) \quad l p_{l_1-j_1,l_2-j_2} = (r!s!)^{1/2} \sum_{r+s=l_1+l_2-j_1-j_2} M(l; l_1 - j_1, l_2 - j_2; r, s) e_{r,s}.$$

In what follows, we will use an expression for the first factor in the integrand in Lemma 8 as a series of $SU(2)$-characters. The following result can be found in [3].

LEMMA 9. *The function* $l \mapsto (\det(1 - l \tanh s)^{-\lambda}$ *has the character expansion*

$$(\det(1 - l \tanh s)^{-\lambda}$$

$$= \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \frac{(\lambda - 1)_{i+j+1}}{(i + j + 1)!} \frac{(\lambda - 1)_i}{i!} (j + 1)(\tanh s)^{2i+j} \chi_j$$

$$= \sum_{j=0}^{\infty} \frac{(\lambda - 1)_{j+1}}{j!} (\tanh s)^j {}_2F_1(\lambda + j, \lambda - 1; j + 2; \tanh^2 s) \chi_j.$$

PROPOSITION 10. *The function Sf_j has the following expression when restricted to A.*

$$Sf_j(a_s) = (1 - \tanh^2 s)^{\frac{\sigma+2}{2}} \sum_{i=0}^{\sigma} \binom{\sigma}{i} (-1)^{j+i} \frac{(2\sigma)_{j+i+1}\, j!}{(j+i+1)!} (\tanh s)^{j+2i}$$
$$\times {}_2F_1(2\sigma + 1 + j + i, 2\sigma; j + i + 2; \tanh^2 s)z^\sigma.$$

PROOF. In the defining integral

$$Sf_j(a_s) = (\cosh s)^{-\nu} \int_{SU(2)} (\det(1 - \zeta l))^{-(\sigma+1)} \langle \tau_j(l^{-1})z^j, w^j \rangle_j \sigma(1 - \zeta l)z^\sigma \, dl$$

we already have a character expansion for the factor $(\det(1 - \zeta l))^{-\lambda}$. Therefore, it suffices to determine the expansion of

$$\langle \tau_j(l^{-1})z^j, w^j \rangle_j \, \sigma(1 - \zeta l)z^\sigma$$

into matrix coefficients. As a special case of (88) we have

$$\sigma(1 - \zeta l)p_{\sigma,0} = \det(1 - \zeta l)^{-\sigma} \sum_{i=0}^{\sigma} \binom{\sigma}{i}(-\zeta)^{\sigma-i}z^i l p_{\sigma-i,0},$$

and the special case of (89) is

$$l p_{\sigma-i,0} = ((\sigma - i)!)^{1/2} \sum_{r=0}^{\sigma-i} \langle \tau_{\sigma-i}(l)e_{\sigma-i,0}, e_{r,\sigma-i-r} \rangle_{\sigma-i} e_{r,\sigma-i-r}.$$

To sum up, we have

$$\sigma(1 - \zeta l)p_{\sigma,0}(z,w) = \det(1 - \zeta l)^{-\sigma}$$

$$\times \sum_{i=0}^{\sigma} \binom{\sigma}{i}(-\zeta)^{\sigma-i}z^i((\sigma-i)!)^{1/2} \sum_{r=0}^{\sigma-i} \langle \tau_{\sigma-i}(l)e_{\sigma-i,0}, e_{r,\sigma-i-r} \rangle_{\sigma-i}$$

$$\times e_{r,\sigma-i-r}(z,w).$$

The integrand is thus a linear combination of terms of the form

$$(90) \qquad \langle \tau_j(l^{-1})z^j, w^j \rangle_j \, \langle \tau_{\sigma-i}(l)e_{\sigma-i,0}, e_{r,\sigma-i-r} \rangle_{\sigma-i}.$$

It is easy to see (using the formula (53)) that the identity

$$(91) \qquad \langle \tau_j(l^{-1})z^j, w^j \rangle_j = (-1)^j \langle \tau_j(l)z^j, w^j \rangle_j$$

holds, and using this identity in (90), the resulting terms are matrix coefficients for the tensor product representation $\tau_j \otimes \tau_{\sigma - i}$. In fact,

$$(92) \quad \langle \tau_j(l)z^j, w^j \rangle_j \langle \tau_{\sigma-i}(l)e_{\sigma-i,0}, e_{r,\sigma-i-r} \rangle_{\sigma-i}$$

$$= \left(\frac{1}{(\sigma-i)!\, r!\, (\sigma-i-r)!} \right)^{1/2} \langle (\tau_j \otimes \tau_{\sigma-i})(l)(z^j \otimes z^{\sigma-i}), w^j \otimes z^r w^{\sigma-i-r} \rangle_{j\,\otimes(\sigma-i)}.$$

We recall the Clebsch-Gordan decomposition for the tensor product $V_j \otimes V_{\sigma-i}$. There is an isometric $\mathrm{diag}(SU(2) \times SU(2))$-intertwining operator

$$\phi : V_j \otimes V_{\sigma-i} \to V_{j+\sigma-i} \oplus \cdots$$

which is of the form

$$\phi = \phi_{j+\sigma-i} \oplus \cdots \oplus \phi_{(\pm(j-(\sigma-i)))}$$

where each term is an intertwining partial isometry and the sum is orthogonal, i.e., the terms have mutually orthogonal kernels. The vector $z^j \otimes z^{\sigma-i}$ is a weight vector of weight $-(j + \sigma - i)\sqrt{-1}H_1^*$ and hence it maps to a highest weight vector in the summand $V_{j+\sigma-i}$. Therefore only the term corresponding to this summand in the orthogonal expansion of the inner product (92) is nonzero. To be more precise, we use the isometry ϕ to write the matrix coefficient (92) as the sum

$$(93) \quad \langle \phi(\tau_j \otimes \tau_{\sigma-i})(l)(z^j \otimes z^{\sigma-i}), \phi(w^j \otimes z^r w^{\sigma-i-r}) \rangle$$

$$= \sum_s \langle \phi_{j+\sigma-i-2s}(\tau_j \otimes \tau_{\sigma-i})(l)(z^j \otimes z^{\sigma-i}), \phi_{j+\sigma-i-2s}(w^j \otimes z^r w^{\sigma-i-r}) \rangle_{j+\sigma-i-2s}.$$

Since

$$\phi_{j+\sigma-i}((\tau_j \otimes \tau_{\sigma-i})(l)(z^j \otimes z^{\sigma-i})) = \left(\frac{j!(\sigma-i)!}{(j+\sigma-i)!} \right)^{1/2} \tau_{j+\sigma-i}(l)z^{j+\sigma-i},$$

the sum (93) is equal to its first term

$$\frac{j!(\sigma-i)!}{(j+\sigma-i)!} \langle \tau_{j+\sigma-i}(l)z^{j+\sigma-i}, z^r w^{j+\sigma-i-r} \rangle_{j+\sigma-i}.$$

Moreover, since we are integrating against characters, only the term corres-

ponding to $r = j + \sigma - i$ will contribute. Hence we have the equality

$$(94) \quad Sf_j(a_s) = (1 - \tanh^2 s)^{\frac{\sigma+2}{2}} \sum_{i=0}^{\sigma} \binom{\sigma}{i} (-\zeta)^{\sigma-i} (-1)^j \frac{j!(\sigma-i)!}{(j+\sigma-i)!} z^{\sigma}$$

$$\times \int_{SU(2)} (\det(1 - \zeta l))^{-(2\sigma+1)} \langle \tau_{j+\sigma-i}(l) z^{j+\sigma-i}, z^{j+\sigma-i} \rangle_{j+\sigma-i} \, dl.$$

So, by using the character expansion in Lemma 9 and the Schur orthogonality relations for matrix coefficients, we get the following expression for the above integral with the index i fixed.

$$\int_{SU(2)} (\det(1 - \zeta l))^{-(2\sigma+1)} \langle \tau_{j+\sigma-i}(l) z^{j+\sigma-i}, z^{j+\sigma-i} \rangle_{j+\sigma-i} \, dl$$

$$= \frac{(2\sigma)_{j+\sigma-i+1}}{(j+\sigma-i)!} (\tanh s)^{j+\sigma-i}$$

$$\times {}_2F_1(2\sigma + 1 + j + \sigma - i, 2\sigma; j + \sigma - i + 2; \tanh^2 s)$$

$$\times \frac{(j+\sigma-i)!}{j+\sigma-i+1}.$$

So, substitution of this into the sum (94) and reversing the order of summation yields

$$Sf_j(a_s) = (1 - \tanh^2 s)^{\frac{\sigma+2}{2}} \sum_{i=0}^{\sigma} \binom{\sigma}{i} (-1)^{j+i} \frac{(2\sigma)_{j+i+1} \, j!}{(j+i+1)!} (\tanh s)^{j+2i}$$

$$\times {}_2F_1(2\sigma + 1 + j + i, 2\sigma; j + i + 2; \tanh^2 s) z^{\sigma}.$$

We now return to the language of Section 3, so that σ corresponds to the natural number k. We can now state the main theorem on the K-types.

THEOREM 11. *For $k \geq 1$, the highest weight vector for the K-type $V_j^* \otimes V_{k+j}$ is the function $F_j : \mathcal{D} \to V_\tau$, whose restriction to $A \cdot 0$ is given by*

$$F_j(t) = (1 - t^2) \sum_{i=0}^{k} \binom{\sigma}{i} (-1)^{j+i} \frac{(2k)_{j+i+1} \, j!}{(j+i+1)!} t^{j+2i}$$

$$\times {}_2F_1(2k + 1 + j + i, 2k; j + i + 2; t^2) z^k.$$

PROOF. Letting $t = \tanh s = \begin{pmatrix} \cosh s & \sinh s \\ \sinh s & \cosh s \end{pmatrix} \cdot 0$, and applying the trivialization mapping (61), together with (66), to the functions Sf_j in Proposition 10 immediately gives the result.

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