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GENERALIZED WALLACE THEOREMS

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Abstract

We present a number of generalizations of a classical result of Wallace regarding countable extensions of totally projective primary abelian groups.

1. Introduction

By the term "group" we will mean an abelian p-group, for some fixed prime p. Our terminology and notations will generally follow [10] and [13]. For instance, if α is an ordinal and G is a group, $p^\alpha G$ denotes the subgroup consisting of elements of height at least α. In particular, $p^\omega G$ will be the first Ulm subgroup of G, i.e., the set of elements of infinite height. We will also use without comment standard terminology on valuated groups and vector spaces (see, for example, [24] and [12]).

In [25], K. D. Wallace proved the following interesting result:

Theorem 1.1. *Suppose the reduced group A has a totally projective subgroup G such that the quotient A/G is countable. Then A is totally projective.*

In [3]–[8] this result was generalized to several classes of groups which properly contain the totally projectives; the goal of the present effort is to advance this investigation. In general, if G is a subgroup of A such that A/G is countable and $\mathscr{P}$ is some property, one can ask whether G satisfies $\mathscr{P}$ implies that A satisfies $\mathscr{P}$, or visa versa. Similarly, one can consider the dual question, i.e., if K is a countable subgroup of G and $A = G/K$, does assuming that G satisfies $\mathscr{P}$ imply that A satisfies $\mathscr{P}$, or visa versa.

Using some terminology of [17], these two questions can be combined as follows: If G and A are groups, κ is an infinite cardinal and $f : G \to A$ is a homomorphism, then f is said to be *κ-injective* if $|K| < \kappa$, where K is the kernel of f; f is said to be *κ-surjective* if $|C| < \kappa$, where $C = A/f(G)$ is the cokernel of f; and f is said to be *κ-bijective* if it is both κ-injective and κ-surjective. This terminology, then, leads us to investigate the following type of question: If $\mathscr{P}$ is some property and $f : G \to A$ is an ω_1-bijective

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homomorphism, does G satisfying $\mathscr{P}$ imply that A satisfies $\mathscr{P}$, or visa versa. For example, when $\mathscr{P}$ is the property "X is a separable $p^{\omega+n}$-projective group" or the property "X is a Q-group," we show that if $f : G \to A$ is an ω_1-bijective homomorphism, then G has property $\mathscr{P}$ iff A has property $\mathscr{P}$ (see Theorem 4.2 and Corollary 5.2(b), respectively).

Note that if $f : G \to A$ is an ω_1-bijective homomorphism and I is the image of f, then $G \to I$ is an ω_1-bijection which is actually surjective, and $I \subseteq A$ is an ω_1-bijection which is actually injective. This observation often allows us to split our arguments into two cases; one where $A = G/K \cong I$ is a factor group with K countable and one where $G = I \subseteq A$ is a subgroup such that the cokernel $C = A/G$ is countable.

Naturally, not every property considered will allow us to generalize Wallace's Theorem in this way; it is often necessary to restrict our attention somewhat. For example, when $\mathscr{P}$ is the property "X is simply presented", or the property "$\alpha \leq \omega_1$ and X is a C_α-group", and $f : G \to A$ is an ω_1-bijective homomorphism, then if G satisfies $\mathscr{P}$, it follows that A satisfies $\mathscr{P}$, but the converse does not hold (Theorems 2.4 and 3.5 and Examples 2.2 and 2.3). Sometimes the natural proofs of our results use additional properties of either the kernel of f as a subgroup of G or the image of f as a subgroup of A, such as requiring that it be pure, isotype or nice. We construct a number of examples to verify that these statements can fail without these additional hypotheses.

2. Simply presented groups

We begin with the following strengthening of Wallace's Theorem:

PROPOSITION 2.1. *Suppose G and A are reduced groups and $f : G \to A$ is an ω_1-bijective homomorphism. If G is totally projective, then A is totally projective.*

PROOF. Let I be the image of f. Since I is a subgroup of A, it is also reduced. If we can show that I is totally projective, then since A/I is countable, it will follow from Wallace's Theorem that A is totally projective, as well. It therefore suffices to suppose that $I = A$, so that f is, in fact, surjective. Let K be the kernel of f, so that K is countable, and in fact, assume $A = G/K$. By Theorem 81.9(α) of [10], G has a *nice system*, i.e, a collection of nice subgroups $\mathscr{N}$ which is closed under group unions (i.e., about Σ) and has the property that if $X \subseteq G$ is countable, then there is a countable $N \in \mathscr{N}$ such that $X \subseteq N$.

Let $\mathscr{N}' = \{N/K : N \in \mathscr{N} \text{ and } K \subseteq N\}$. By Lemma 79.3(i) of [10] every element of $\mathscr{N}'$ is a nice subgroup of $A = G/K$. It is easy to see that since $\mathscr{N}$ is closed under group unions, so is $\mathscr{N}'$. Finally, if $X' \subseteq A$ is countable, then

$X = K \cup \{x \in G : x + K \in X'\} \subseteq G$ is also countable; if $N \in \mathcal{N}$ is countable and $X \subseteq N$, then it follows that $N/K \in \mathcal{N}'$ is countable and $X' \subseteq N/K$. Therefore, $\mathcal{N}'$ is a nice system for A, so that it is totally projective, as required.

It might be tempting to conjecture that the converse of this proposition holds, i.e., if G and A are reduced groups, $f : G \to A$ is an ω_1-bijection and A is totally projective, then G is also totally projective. In fact, this fails for at least two reasons. The following shows that it can fail when f is simply injective:

EXAMPLE 2.2. Suppose A is a totally projective group of length ω_1 – so A is a *dsc group*, i.e., a direct sum of countable groups – and suppose

$$0 \to G \to A \to \mathbf{Z}_{p^\infty} \to 0$$

is p^{ω_1}-pure exact (i.e., it represents an element of $p^{\omega_1}\operatorname{Ext}(\mathbf{Z}_{p^\infty}, G)$). Then A/G is certainly countable, but G is not totally projective; in fact, it is an *elementary S-group* (see [26]).

Recall that if X is a subgroup of a group Y, then $p^\omega(Y/X) = \cap_{i<\omega}(p^i Y + X)/X = \overline{X}/X$, where $\overline{X}$ is the closure of X in the p-adic topology on Y. In particular, X is closed iff Y/X is separable. The next example shows that the converse of Proposition 2.1 can also fail when f is actually surjective:

EXAMPLE 2.3. We show that there is a pure-exact (and hence isotype) sequence:

$$0 \to K \to G \to A \to 0$$

where K is a countable direct sum of cyclics, A is a dsc group of length $\omega + 1$, and G is a separable group which is not a direct sum of cyclics. To this end, suppose B is an unbounded countable direct sum of cyclics with torsion completion $\overline{B}$ and L is some group such that there is a subgroup $P \subseteq L[p]$ for which L/P is a direct sum of cyclics and there is an isometry $\phi : \overline{B}[p] \to P$ (i.e., an isomorphism that also preserves heights computed in $\overline{B}$ and L). Since $L[p]$ is not free (as a valuated vector space), L is not a direct sum of cyclics. Let $G = (B \oplus L)/X$, where $X = \{(x, \phi(x)) : x \in B[p]\}$, $K = [(B \oplus \{0\}) + X]/X$. Because X is closed in $B \oplus L$, we can conclude G is separable. Since L embeds in G, it also follows that G is not a direct sum of cyclics. It follows easily that $K \cong B$ is a pure, and hence isotype, but not nice subgroup of G, and that $G/K \cong L/\phi(B[p])$; we denote this last group by A. Note that $P/\phi(B[p]) = p^\omega(L/\phi(B[p])) = p^\omega A$ is p-bounded and $A/p^\omega A = (L/\phi(B[p]))/(P/\phi(B[p]) \cong L/P$ is a direct sum of cyclics. Therefore, $A \cong G/K$ is a dsc group, and hence totally projective, as required.

Following [10], a group is said to be *simply presented* if it is the direct sum of a divisible and a (reduced) totally projective group. The following gives a strengthening of Proposition 2.1 to this broader class:

THEOREM 2.4. *Suppose G and A are groups and $f : G \to A$ is an ω_1-bijective homomorphism. If G is simply presented, then A is simply presented.*

We want to reduce Theorem 2.4 to Proposition 2.1. To that end, we introduce the following terminology: If $\kappa > \aleph_0$ is a cardinal and G is reduced, then we say G is a *reduced κ-$*$-group* if whenever C is an infinite subgroup of G with $|C| < \kappa$, then $|C'| = |C|$, where C' is the subgroup of G containing C such that C'/C is the maximal divisible subgroup of G/C. Note that a reduced group G is an ω_1-$*$-group iff whenever C is a countably infinite subgroup of G, then there is a countably infinite subgroup C'' of G containing C such that G/C'' is reduced; or equivalently, for all surjective homomorphisms, $g : G \to X$, with a countably infinite kernel, the maximal divisible subgroup of X is also countable.

PROPOSITION 2.5. *If G is a totally projective group, then G is an ω_1-$*$-group.*

PROOF. Suppose $\mathcal{N}$ is a nice system for G and C is a countable subgroup of G. Then there is a countable subgroup $N \in \mathcal{N}$ such that $C \subseteq N$. However, since G is reduced and N is nice, it follows that G/N is reduced, as required.

An arbitrary (possibly non-reduced) group will be called a κ-$*$-group if it is the direct sum of a reduced κ-$*$-group and a divisible group. The following is the critical step in the reduction of Theorem 2.4 to Proposition 2.1.

PROPOSITION 2.6. *Suppose G is an ω_1-$*$-group and A is a group and $f : G \to A$ is an ω_1-bijective homomorphism. Then,*

(a) *A is an ω_1-$*$-group;*
(b) *If $G = G_0 \oplus D$ and $A = A_0 \oplus E$ where D and E are divisible and G_0 and A_0 are reduced, then there is an ω_1-bijective homomorphism $f_0 : G_0 \to A_0$.*

Before proving Proposition 2.6, observe how it gives the following:

PROOF OF THEOREM 2.4. If $G = G_0 \oplus E$ and $A = A_0 \oplus E$, where D and E are divisible and G_0 and A_0 are reduced, then G_0 is totally projective, and by Proposition 2.6(b), there is an ω_1-bijective homomorphism $f_0 : G_0 \to A_0$. By Proposition 2.1, A_0 is totally projective, so that A is simply presented.

PROOF OF PROPOSITION 2.6. If K is the kernel of f and I is the image of f, then the obvious maps, $G \to G/K$ and $I \subseteq A$, are ω_1-bijections with composition f. Therefore, if we can prove the result assuming $A =$

G/K where K is countable, and $G \subseteq A$ where A/G is countable, then (a) will follow immediately and (b) will be a consequence of the fact that the composition of ω_1-bijective homomorphisms retains that property (see, for example, Proposition 2.1(c) of [17]).

Assume first, therefore, that K is a countable subgroup of G and $A = G/K$. Let $D = D_1 \oplus D_2$, where D_1 is countable and $K \subseteq G_0 \oplus D_1$. Since $A = (G_0 \oplus D_1 \oplus D_2)/K \cong [(G_0 \oplus D_1)/K] \oplus D_2$, we may, without loss of generality, assume that $D = D_1$ is countable.

Note that there is an exact sequence:

$$0 \to [D + K]/K \to A \to G/[D + K] \to 0$$

whose left-hand group is divisible and hence a summand of E. If $E = ([D + K]/K) \oplus E_1$, it follows that $A_0 \oplus E_1 \cong G/[D + K]$. Note that $G/[D + K] = (G_0 \oplus D)/[D+K] \cong G_0/(G_0 \cap [D+K])$. Since $C = G_0 \cap [K+D]$ is countable and G_0 is an ω_1-$*$-group, there is a countable subgroup C' of G_0 containing C such that C'/C is the maximal divisible subgroup of G_0/C. It follows that $G_0/C' \cong (G_0/C)/(C'/C)$ is isomorphic to A_0, and the obvious map $G_0 \to G_0/C' \cong A_0$ is an ω_1-$*$-bijective homomorphism (which is actually surjective), so that (b) follows. As for (a), we need to show that this A_0 is an ω_1-$*$-group. To that end, suppose X is a group and $g : A_0 \to X$ is a surjective homomorphism with countable kernel. Then the composition $G_0 \to A_0 \to X$ is also a surjective homomorphism with countable kernel, so, since G_0 is an ω_1-$*$-group, we can conclude that the divisible part of X is countable, so that A_0 is an ω_1-$*$-group, as required.

Suppose next that $G \subseteq A$ with A/G countable. Note $A = A_1 \oplus D$, where $G_0 \subseteq A_1$ and A_1/G_0 is countable. It suffices, therefore, to let $A = A_1$ and $G = G_0$, so that G is reduced.

We next claim that the maximal divisible subgroup E of A must be countable and that A_0 is an ω_1-$*$-group; before establishing the claim, note it immediately gives (a), and in addition, since the homomorphism $f_0 : G_0 = G \subseteq A \to A/E \cong A_0$ is a composition of ω_1-bijective homomorphisms, as aforementioned it is also ω_1-bijective. This proves (b), and hence the entire result.

Turning, therefore, to the claim, suppose C_0 is an arbitrary countable subgroup of A_0, and C_1 is a countable subgroup of A such that $A = G + C_1$. If $C = G \cap [C_0 + C_1]$, then C is countable, and if C'/C is the maximal divisible subgroup of G/C, then since G is an ω_1-$*$-group, C' is countable, as well. Note that $G \cap [C' + C_0 + C_1] = C'$. It follows that

$$\begin{aligned} A/[C' + C_0 + C_1] &= [G + C_1]/[C' + C_0 + C_1] \cong G/(G \cap [C' + C_0 + C_1]) \\ &= G/C' \cong (G/C)/(C'/C) \end{aligned}$$

is reduced. This implies that $E \subseteq C' + C_0 + C_1$ is countable, giving the first part of the claim. Next, since $A_0/(A_0 \cap [C' + C_0 + C_1])$ embeds in $A/[C' + C_0 + C_1]$ and the latter is reduced, so is the former. Letting $C'' = A_0 \cap [C' + C_0 + C_1]$, we conclude that $C'' \subseteq A_0$ is countable, $C_0 \subseteq C''$ and A_0/C'' is reduced, showing that A_0 is an ω_1-$*$-group, establishing the claim, and hence the result.

3. n-Σ-groups and C_α-groups

If α is an ordinal, then a subgroup H of a group A is said to be *p^α-high* if it is maximal with respect to the property that $H \cap p^\alpha A = \{0\}$ (see, for example, [20] and [1]). A p^ω-high subgroup is usually referred to simply as a *high* subgroup. We summarize a few standard properties of this notion in the following:

LEMMA 3.1. *If α is an ordinal and H is a p^α-high subgroup of A, then:*

(a) *If $\alpha < \omega$ is finite, then H is a summand of A;*

(b) *If $\alpha \geq \omega$ is infinite, then H is a $p^{\alpha+1}$-pure subgroup of A and A/H is divisible. In particular, this means that H is an isotype subgroup of A;*

(c) *If H is a dsc group and H' is another p^α-high subgroup of A, then H' is also a dsc group;*

(d) *There is a decomposition, $A[p] = H[p] \oplus (p^\alpha A)[p]$;*

(e) *If $\alpha = \beta + \gamma$, then $p^\beta H$ is p^γ-high in $p^\beta A$.*

PROOF. (a) is Theorem 27.7 of [10]. (b) follows from (2°) and Proposition 1 of [23]. (c) is Corollary 5 of [23]. (d) and (e) are simple consequences of the maximality of H.

The following definition appeared in [16]: A group A is a *Σ-group* provided that some high subgroup of A is a direct sum of cyclic groups. It follows from Lemma 3.1(c) that all its high subgroups are direct sums of cyclic groups. The following generalization of this terminology was given in [18]: If $\alpha \leq \omega_1$, then A is said to be a *C_α-group* iff for every $\beta < \alpha$, some (and hence all) p^β-high subgroup of A is a dsc group. If α is isolated, then we only need that some (and hence all) $p^{\alpha-1}$-high subgroup of A is a dsc group (since if $\beta < \alpha - 1$, then a p^β-high subgroup will be an isotype subgroup of a $p^{\alpha-1}$-high subgroup, and a classical result of Hill's states an isotype subgroup of a dsc group of countable length also has that form – see, for example, Theorem 104 of [13]). The $C_{\omega+1}$-groups are precisely the Σ-groups.

Next, we review a concept from [7]. Imitating a criterion from [2], if $1 \leq n < \omega$, we shall say that A is an *n-Σ-group* if $A[p^n] = \cup_{i<\omega} A_i$, where for all $i < \omega$, $A_i \subseteq A_{i+1}$ and $A_i \cap p^i A = (p^\omega A)[p^n]$. With this terminology, the 1-Σ-groups are precisely the Σ-groups (this observation is generalized in our next result). Clearly, if $m \leq n$, then every n-Σ-group is an m-Σ-group. Thus each n-Σ-group is a Σ-group, while the converse implication is false (in [6], a Σ-group was constructed which is not a 2-Σ-group).

PROPOSITION 3.2. *If A is a group and $0 < n < \omega$, then A is an n-Σ-group iff A is a $C_{\omega+n}$-group.*

PROOF. Suppose H is a $p^{\omega+n-1}$-high subgroup of A. Then H is isotype in A, so that heights computed in H and A agree. By Lemma 3.1(e) and (a), $p^\omega H = H \cap p^\omega A$ is a p^{n-1}-high subgroup of $p^\omega A$, and so there is a subgroup $X \subseteq p^\omega A$ such that $p^\omega A = p^\omega H \oplus X$. By Lemma 3.1(d), we have $A[p] = H[p] \oplus (p^{\omega+n-1}A)[p] = H[p] \oplus X[p]$. Now, $X[p^n]$ is isomorphic to the direct sum of a collection of copies of $\mathbf{Z}_{p^n}$, and since H is pure in A, $H[p^n]$ is a summand of $A[p^n]$. It follows that there is a decomposition:

$$A[p^n] = H[p^n] \oplus X[p^n].$$

In fact, we claim that the above decomposition is valuated, i.e., if $z \in H[p^n]$ and $x \in X[p^n]$, then $\mathrm{ht}(z + x) = \min\{\mathrm{ht}(z), \mathrm{ht}(x)\}$ (where all heights are computed in A): Note that if z has infinite height in A (and H), then this follows because $p^\omega A = p^\omega H \oplus X$, and if z has finite height in A (and H), then this follow from $\mathrm{ht}(z) < \omega \leq \mathrm{ht}(x)$.

Suppose first that H is some $p^{\omega+n-1}$-high subgroup that is a dsc group. Then $H/p^\omega H$ is a direct sum of cyclics, and so $H/p^\omega H \cong \oplus_{j<\omega} C_j$, where each C_j is a direct sum of copies of $\mathbf{Z}_{p^{j+1}}$. Considering the composition:

$$\phi : A[p^n] \cong H[p^n] \oplus X[p^n] \to H[p^n] \to H[p^n]/p^\omega H \subseteq H/p^\omega H \cong \oplus_{j<\omega} C_j,$$

we let $A_i = \phi^{-1}(\oplus_{j<i} C_j)$. Clearly $A_i \subseteq A_{i+1} \subseteq A[p^n]$, and since

$$\oplus_{j<\omega} C_j = \cup_{i<\omega}(\oplus_{j<i} C_j),$$

it follows that $A[p^n] = \cup_{i<\omega} A_i$. Next, note that all of the maps used to construct ϕ preserve the heights of elements whenever they are finite, and therefore, the kernel of ϕ is $(p^\omega A)[p^n]$. It follows that for every $i < \omega$ we have

$$\phi(p^i A \cap A_i) \subseteq p^i(\oplus_{j<i} C_j) = \{0\}.$$

From this we can conclude that $p^i A \cap A_i = (p^\omega A)[p^n]$, which means that A is an n-Σ-group.

Conversely, suppose A is an n-Σ-group and H is any $p^{\omega+n-1}$-high subgroup of A. To show that A is a $C_{\omega+n}$-group, we need to show that H is a dsc group, or, since $p^\omega H$ is bounded, that the Ulm factor $H/p^\omega H$ is a direct sum of cyclics. Let the $A_i \subseteq A[p^n]$ be as in the definition of an n-Σ-group. Note that $p^\omega H \subseteq H[p^{n-1}]$, so that $(H/p^\omega H)[p] \subseteq H[p^n]/p^\omega H$. In addition, for any $i < \omega$, $p^\omega H \subseteq (p^\omega A)[p^n] \subseteq A_i$, so we let

$$S_i = (H/p^\omega H)[p] \cap ((A_i \cap H)/p^\omega H).$$

Note that if $x + p^\omega H \in S_i$, where $x \in A_i \cap H$, then $x \in p^i A$ implies that $x \in p^i A \cap A_i \cap H = (p^\omega A)[p^n] \cap H = p^\omega H$. Therefore, the heights (in $H/p^\omega H$) of the non-zero elements of S_i are bounded by i. Since

$$\begin{aligned}(H/p^\omega H)[p] \subseteq H[p^n]/p^\omega H &= (A[p^n] \cap H)/p^\omega H \\ &= \cup_{i<\omega}[(A_i \cap H)/p^\omega H],\end{aligned}$$

we can conclude that

$$(H/p^\omega H)[p] = \cup_{i<\omega} S_i.$$

However, this implies that $H/p^\omega H$ is a direct sum of cyclic groups, which implies that H is a dsc group, as required.

Note that the above provides a non-homological proof of the fact that if one $p^{\omega+n-1}$-high subgroup of A is a dsc group, then all $p^{\omega+n-1}$-high subgroups of A are dsc groups.

COROLLARY 3.3. *If $0 < n < \omega$ and A is a group of length at most $\omega+n-1$, then A is an n-Σ-group iff it is a dsc group.*

PROOF. In this case, A is a $p^{\omega+n-1}$-high subgroup of itself.

A homological approach to these definitions can be given as follows: We let $A \bigtriangledown B$ denote the torsion product of the groups A and B. This admittedly non-standard notation better reflects the multiplicative nature of the operation. If $\alpha \leq \omega_1$ is an ordinal, let H_α denote the *generalized Prüfer group* of length α. It follows from Theorem 2 of [18] that A is a C_α-group iff $A \bigtriangledown H_\alpha$ is a dsc group (this latter characterization may be, in fact, a more natural definition of the term). The following, therefore, follows directly from Proposition 3.2:

COROLLARY 3.4. *If $0 < n < \omega$, then A is an n-Σ-group iff $A \bigtriangledown H_{\omega+n}$ is a dsc group.*

These considerations lead to the following result:

THEOREM 3.5. *Suppose $\alpha \leq \omega_1$ is an ordinal, G and A are groups and $f : G \to A$ is an ω_1-bijective homomorphism. If G is a C_α-group, then A is also a C_α-group.*

PROOF. Suppose first that $\alpha < \omega_1$ is countable, so that H_α is countable, as well. Let K and I be the kernel and image of f, respectively. There is a long-exact sequence:

$$0 \to K \triangledown H_\alpha \to G \triangledown H_\alpha \to I \triangledown H_\alpha \to K \otimes H_\alpha,$$

and since the outer two groups are countable, it follows that $G \triangledown H_\alpha \to I \triangledown H_\alpha$ is an ω_1-bijection. Since these both have length at most α, they are reduced (in fact, by Lemma 64.2 of [10], $p^\alpha(G \triangledown H_\alpha) = (p^\alpha G) \triangledown (p^\alpha H_\alpha) = \{0\}$). In addition, since G is a C_α-group, $G \triangledown H_\alpha$ is necessarily a dsc group, and hence $I \triangledown H_\alpha$ is also a dsc group by Proposition 2.1, so that I is a C_α-group. Now, if $C = A/I$, then C is countable and there is a left-exact sequence:

$$0 \to I \triangledown H_\alpha \to A \triangledown H_\alpha \to C \triangledown H_\alpha,$$

Since $I \triangledown H_\alpha$ is a dsc group, $A \triangledown H_\alpha$ is reduced and $C \triangledown H_\alpha$ is countable, it once again follows via Proposition 2.1 that $A \triangledown H_\alpha$ is a dsc group, showing that A is a C_α-group, as required.

Finally, if $\alpha = \omega_1$, then $H_\alpha = \oplus_{\beta<\alpha} H_\beta$, so if G is a C_α-group, then $G \triangledown H_\alpha = G \triangledown (\oplus_{\beta<\alpha} H_\beta) \cong \oplus_{\beta<\alpha}(G \triangledown H_\beta)$ is a dsc group, which implies that $G \triangledown H_\beta$ is a dsc group for all $\beta < \alpha$, which implies that $A \triangledown H_\beta$ is a dsc group for all $\beta < \alpha$, which implies that $A \triangledown H_\alpha = A \triangledown (\oplus_{\beta<\alpha} H_\beta) \cong \oplus_{\beta<\alpha}(A \triangledown H_\beta)$ is a dsc group, which implies that A is a C_α-group.

COROLLARY 3.6. *Suppose $n < \omega$, G and A are groups and $f : G \to A$ is an ω_1-bijective homomorphism. If G is an n-Σ-group, then A is an n-Σ-group.*

Notice that in Example 2.3, A is a 1-Σ-group (in fact, it is a dsc group of length $\omega + 1$), but G is not a 1-Σ-group (since any separable 1-Σ-group is, in fact, a direct sum of cyclics). This shows that the implications in the last two results cannot be reversed, even in the case where $n = 1$.

4. $p^{\omega+n}$-projective groups

The following elementary consequence of Wallace's Theorem has frequently been found useful (see, e.g., [1] and [15]). In fact, we include a separate proof since the result can be approached directly:

COROLLARY 4.1. *Suppose A is a separable group with a subgroup G such that A/G is countable. Then G is a direct sum of cyclic groups iff A is a direct sum of cyclic groups.*

PROOF. Note that if A is a direct sum of cyclics, it immediately follows that G is, as well, so assume G is a direct sum of cyclics. We may write

$A = G + C$ for some countable subgroup C of A. There is, therefore, a decomposition $G = G_1 \oplus G_2$, where G_2 is countable and $C \cap G \subseteq G_2$. Thus, $C + G_2$ is a countable group with no elements of infinite height, being a subgroup of A, and hence $C + G_2$ is a direct sum of cyclics. We claim that $A = G_1 \oplus (C + G_2)$: Clearly $G_1 + C + G_2 = G + C = A$, and if $g_1 = c + g_2$ (where each symbol represents an element of the corresponding subgroup), then $c = g_1 - g_2 \in C \cap G \subseteq G_2$ implies that $g_1 = 0$, proving the claim. Therefore, since G_1 and $C + G_2$ are direct sums of cyclics, the same will be true of A.

If $n < \omega$, then a group A is $p^{\omega+n}$*-projective* iff for all groups X we have $p^{\omega+n}\operatorname{Ext}(A, X) = \{0\}$ or, equivalently, $p^n \operatorname{Pext}(A, X) = \{0\}$. A more concrete characterization of this notion is given by Corollary 6.5 of [22], which states that A is $p^{\omega+n}$-projective iff there is a subgroup P of $A[p^n]$ such that A/P is a direct sum of cyclics. One easy consequence of this is that an arbitrary subgroup of a $p^{\omega+n}$-projective is also $p^{\omega+n}$-projective. By Theorem 5 of [11], if A and A' are $p^{\omega+n}$-projectives, then $A \cong A'$ iff there is an isometry $A[p^n] \cong A'[p^n]$. These groups have been studied extensively (e.g., [15]).

In [9], Dieudonné gave an example showing that in the last corollary, the hypothesis of countability is necessary (see, for example, [10], v. II, p. 16, Exercise 11). In fact, if A is any $p^{\omega+n}$-projective group, then A has a subgroup $P \subseteq A[p^n]$ (which must be a direct sum of cyclics) such that A/P is also a direct sum of cyclics. On the other hand, there are many separable $p^{\omega+n}$-projective groups which are not direct sums of cyclics. This connection is developed in the following generalization of Corollary 4.1 (see also [3], [4], [8]):

THEOREM 4.2. *Suppose $n < \omega$, G and A are separable groups and $f : G \to A$ is an ω_1-bijective homomorphism. Then G is $p^{\omega+n}$-projective iff A is $p^{\omega+n}$-projective.*

PROOF. Before beginning, note that if A is a separable group, then $A[p^n]$ will be a closed subgroup of A, so if P is a subgroup of $A[p^n]$, then the p-adic closure, $\overline{P}$, will be contained in $A[p^n]$, i.e., $p^\omega(A/P) \subseteq A[p^n]/P$, which implies that A/P has length at most $\omega + n$.

As usual, if I is the image of f, then by considering the natural factorization $G \to I \to A$, we may break the argument into two cases, where f is actually injective (and ω_1-surjective), and where f is actually surjective (and ω_1-injective).

Suppose first that f is injective; in fact, assume G is a subgroup of A and A/G is countable. If A is $p^{\omega+n}$-projective, it immediately follows that G is, as well. Conversely, suppose G is $p^{\omega+n}$-projective. Let P be a subgroup of $G[p^n]$

such that G/P is a direct sum of cyclics. Then $L = A/P$ must be reduced, and in fact, of length at most $\omega + n$. Since $(A/P)/(G/P) \cong A/G$ is countable, it follows from Wallace's Theorem that $L = A/P$ is a dsc group, and hence that $L/p^{\omega}L$ is a direct sum of cyclics. However, if $\overline{P}$ is the p-adic closure of P in A, then $\overline{P} \subseteq A[p^n]$ and $A/\overline{P} \cong (A/P)/(\overline{P}/P) \cong L/p^{\omega}L$ is a direct sum of cyclics, showing that A is $p^{\omega+n}$-projective.

Suppose next that f is surjective; in fact, assume K is a countable subgroup of G with $A = G/K$. Suppose first that G is $p^{\omega+n}$-projective. Let P be a subgroup of $G[p^n]$ such that G/P is a direct sum of cyclics. Let $G/P = C_1 \oplus C_2$, where C_2 is a countable subgroup containing $K' = (K+P)/P$. If $P' = (K+P)/K$, then P' is a subgroup of $A[p^n]$, and

$$\begin{aligned} A/P' &= (G/K)/([K+P]/K) \\ &\cong G/[K+P] \\ &\cong (G/P)/([K+P]/P) \\ &\cong C_1 \oplus (C_2/K') \end{aligned}$$

is a dsc group. Therefore, if $\overline{P}'$ is the p-adic closure of P' in A, then $\overline{P}' \subseteq A[p^n]$, and

$$A/\overline{P}' \cong (A/P')/(\overline{P}'/P') \cong (A/P')/p^{\omega}(A/P')$$

is a direct sum of cyclics, so that A is $p^{\omega+n}$-projective.

Finally, suppose $A = G/K$ and A is $p^{\omega+n}$-projective. Let P be a subgroup of $A[p^n]$ such that $A/P = \oplus_{i\in I} C_i$ where each C_i is cyclic. Let $P_1 \leq G$ be the subgroup containing K such that $P = P_1/K$; note $p^n P_1 \subseteq K \leq P_1$ and $G/P_1 \cong A/P = \oplus_{i\in I} C_i$. By a standard "back-and-forth" argument, there is a countable pure subgroup L of G containing K and a countable subset $J \subseteq I$ such that $[L + P_1]/P_1 \cong \oplus_{i\in J} C_i$. Note that since L is a countable separable group, it is, in fact, a direct sum of cyclics. If $A' = G/L$ and $P' = [L+P_1]/L$, then

$$\begin{aligned} A'/P' &= (G/L)/([L+P_1]/L) \\ &\cong G/[L+P_1] \\ &\cong (G/P_1)/([L+P_1]/P_1) \\ &\cong \oplus_{i\in I} C_i / \oplus_{i\in J} C_i \\ &\cong \oplus_{i\in I-J} C_i \end{aligned}$$

is a direct sum of cyclics. Since $p^n P_1 \subseteq L$, and hence $p^n P' = 0$, we can conclude that A' is $p^{\omega+n}$-projective. Now, for any group X, by Theorem 53.7 of [10], the pure-exact sequence

$$0 \to L \to G \to A' \to 0$$

determines a corresponding right-exact sequence:

$$\operatorname{Pext}(A', X) \to \operatorname{Pext}(G, X) \to \operatorname{Pext}(L, X) \to 0.$$

Since L is a direct sum of cyclics, we have $\operatorname{Pext}(L, X) = 0$, and since A' is $p^{\omega+n}$-projective we have $p^n \operatorname{Pext}(A', X) = 0$. Therefore, $p^n \operatorname{Pext}(G, X) = 0$ for all X, which means that G is $p^{\omega+n}$-projective, as required.

5. κ-Q-groups and weakly ω_1-separable groups

If κ is an uncountable cardinal, then by a slight extension of the terminology of [21], we say G is a *κ-Q-group* if for every infinite subgroup $C \subseteq G$, $|C| < \kappa$ implies $|\overline{C}| = |C|$, where $\overline{C}$ denotes the closure of C in the p-adic topology on G (so $\overline{C}/C = p^\omega(G/C)$). Following [21], a separable ω_1-Q-group is said to be *weakly ω_1-separable*. Finally, a separable group is a *Q-group* iff it is a κ-Q-group for all uncountable κ.

THEOREM 5.1. *Suppose κ is an uncountable cardinal, G and A are separable groups and $f : G \to A$ is an ω_1-bijective homomorphism. Then G is a κ-Q-group iff A is a κ-Q-group.*

PROOF. As usual, we break this into two arguments corresponding to when f is assumed to be injective and surjective. Suppose first that f is injective, and in fact, assume G is a subgroup of A with A/G being countable. If A is a κ-Q-group, and C is an infinite subgroup of G with $|C| < \kappa$, then the closure of C in G is contained in the closure of C in A. Since the latter has the same cardinality as C, so must the former, and hence G is also a κ-Q-group.

On the other hand, assume that it is G that is a κ-Q-group, and that C is an infinite subgroup of A with $|C| < \kappa$. We can certainly expand C without altering its cardinality so that $A = G + C$ and $C \cap G$ are infinite, so we assume that these two conditions hold. Note that these assumptions guarantee that $G/(C \cap G) \cong (G + C)/C = A/C$. Therefore, if $\overline{C \cap G}$ is the closure of $C \cap G$ in G and $\overline{C}$ is the closure of C in A, we have,

$$\overline{C \cap G}/(C \cap G) = p^\omega(G/(C \cap G)) \cong p^\omega(A/C) = \overline{C}/C.$$

Now, since G is a κ-Q-group, we can conclude,

$$|\overline{C}/C| = |\overline{C \cap G}/(C \cap G)| \leq |\overline{C \cap G}| = |C \cap G| \leq |C|.$$

This, in turn, implies that $|\overline{C}| = |C|$, showing that A is, in fact, a κ-Q-group.

Assume now that f is surjective, and in fact, assume $A = G/K$, where K is countable. Suppose first that G is a κ-Q-group. If C is an infinite subgroup of A with $|C| < \kappa$, then let C_0 be the subgroup of G containing K defined by the equation $C = C_0/K$. If $\overline{C}$ is the closure of C in A and $\overline{C}_0$ is the closure of C_0 in G, then $\overline{C}_0/C_0 = p^{\omega}(G/C_0) \cong p^{\omega}(A/C) = \overline{C}/C$ so $|\overline{C}/C| = |\overline{C}_0/C_0| \leq |C_0| = |C|$, so that $|\overline{C}| = |C|$, as required.

Conversely, suppose that A is a κ-Q-group and C_0 is an infinite subgroup of G with $|C_0| < \kappa$. Replacing C_0 by $C_0 + K$ does not alter its cardinality, so we may assume $K \subseteq C_0$. Again, by possibly expanding C_0 without altering its cardinality, we may assume $C = C_0/K$ is infinite. Therefore, the argument of the last paragraph shows that $|\overline{C}_0| = |\overline{C}| = |C| = |C_0|$, as required.

COROLLARY 5.2. *Suppose G and A are separable groups and* $f : G \to A$ *is an* ω_1*-bijective homomorphism. Then*

(a) *G is weakly* ω_1*-separable iff A is weakly* ω_1*-separable;*

(b) *G is a Q-group iff A is a Q-group.*

Again following [21], a separable group is ω_1*-separable* if every countable subset is contained in a countable summand. One might ask if the analogue to Corollary 5.2(a) holds for ω_1-separable groups. The difficulty of this question is illustrated by two facts from [21]: Assuming Martin's Axiom (MA) and the denial of the Continuum Hypothesis ($\neg$ CH), if A is an ω_1-separable group of cardinality $\aleph_1$ and G is a pure and dense subgroup of A with A/G countable, then G is ω_1-separable, and in fact, $A \cong G$ (Theorem 2.6). (In fact, under (MA + $\neg$ CH) the classes of weakly ω_1-separable groups and ω_1-separable groups both of cardinality $\aleph_1$, coincide, so a result analogous to Corollary 5.2(a) holds for this class.) Moreover, in [21] was also showed that if G is a pure and closed subgroup of the separable group A, then A is weakly ω_1-separable iff G and A/G are weakly ω_1-separable (Theorem 1.5). On the other hand, in the constructible universe (V=L), if A is an ω_1-separable group of cardinality $\aleph_1$ and A is not a direct sum of cyclic groups, then there is a pure subgroup G of A with $A/G \cong \mathbf{Z}_{p^\infty}$ and G is not ω_1-separable (Theorem 3.2).

6. σ-summable and n-Honda groups

If H is a group containing a subgroup K, then the *height spectrum* (of K in H) is defined to be the collection of ordinals $\{\mathrm{ht}_H(x) : x \in K\}$. We say K is *height-finite* if it has finite height spectrum. We begin with the following technical, but useful, lemma.

LEMMA 6.1. *Suppose S is the height spectrum of a subgroup K of a group H and F is a finite subgroup of H, and S′ is the height spectrum of* $F + K$*. Then the set* $S' \setminus S$ *is finite.*

PROOF. If we assume, by way of contradiction, that $S' \setminus S$ is infinite, then there is an infinite set $\{f_i + k_i : i < \omega\}$ of elements of $F + K$ such that

$\{\mathrm{ht}(f_i + k_i) : i < \omega\}$ is an infinite set of elements of $S' \setminus S$. Since F is finite, there are distinct $i, j < \omega$ such that $f_i = f_j$ and $\mathrm{ht}(f_i + k_i) < \mathrm{ht}(f_j + k_j)$. This implies that $\mathrm{ht}(k_i - k_j) = \mathrm{ht}((f_i + k_i) - (f_j + k_j)) = \min\{\mathrm{ht}(f_i + k_i), \mathrm{ht}(f_j + k_j)\} = \mathrm{ht}(f_i + k_i) \in S' \setminus S$. However, this contradicts that $k_i - k_j \in K$, so that $\mathrm{ht}(k_i - k_j) \in S$.

The last result has the following immediate consequences:

COROLLARY 6.2. *Suppose H is a reduced group of length λ, F and K are subgroups of H, and F is finite.*

(a) *If λ is a limit ordinal and there is an ordinal $\alpha < \lambda$ such that $K \cap p^\alpha H = \{0\}$, then there is an ordinal $\beta < \lambda$ such that $(K + F) \cap p^\beta H = \{0\}$.*

(b) *If K is height-finite, then the same holds for $F + K$.*

A group A of length λ is called *σ-summable* if $A[p] = \cup_{i<\omega} A_i$, where for all $i < \omega$, $A_i \subseteq A_{i+1}$ and $A_i \cap p^{\alpha_i} A = 0$ for some $\alpha_i < \lambda$ (see [19]). Note the similarity of this property to the classical Kulikov's criterion describing when a group is a direct sums of cyclics; it follows that a separable group is σ-summable iff it is a direct sum of cyclics. It is well-known (see, for example, [19]) that all totally projective groups whose length is a limit ordinal of countable cofinality are σ-summable. More generally, if λ is a limit ordinal of countable cofinality and A is a direct sum of groups of length less than λ, then A is σ-summable.

Although we can prove our next result using the original definition (cf., [3]), the following criterion, due to Hill ([14]), is slightly more convenient.

HILL'S CRITERION 6.3 ([14]). *A group A of length λ is σ-summable iff $A = \cup_{i<\omega} \Gamma_i$, where for all $i < \omega$, $\Gamma_i \subseteq \Gamma_{i+1}$ and there is an ordinal $\alpha_i < \lambda$ such that $\Gamma_i \cap p^{\alpha_i} A = \{0\}$.*

Our next result was first established in [3]; nevertheless, we include a different, more conceptual, proof.

PROPOSITION 6.4. *Suppose A is a reduced group of limit length λ and G is a σ-summable isotype subgroup of A such that A/G is countable. Then A is σ-summable.*

PROOF. Let C be a countable subgroup of A such that $A = C + G$. Write $C = \cup_{i<\omega} C_i$, where each C_i is a finite subgroup and $C_i \subseteq C_{i+1}$. Referring to Hill's criterion, if μ is the length of G, we can write $G = \cup_{i<\omega} \Gamma_i$, where for every $i < \omega$, $\Gamma_i \subseteq \Gamma_{i+1}$ and there is an ordinal $\alpha_i < \mu$ so that $\Gamma_i \cap p^{\alpha_i} G = \{0\}$. Hence, $\Gamma_i \cap p^{\alpha_i} A = \{0\}$ and $\alpha_i < \lambda$ since $\mu \leq \lambda$. Therefore, if $\Gamma'_i = \Gamma_i + C_i$, then $A = \cup_{i<\omega} \Gamma'_i$ where for each $i < \omega$, $\Gamma'_i \subseteq \Gamma'_{i+1}$, and Corollary 6.2(a) implies that there is an ordinal $\beta_i < \lambda$ with $\Gamma'_i \cap p^{\beta_i} A = \{0\}$. Finally, a second application of Hill's criterion allows us to conclude that A is σ-summable, as required.

We now construct an example which shows that the hypothesis that G be *isotype* cannot be removed. To this end, we pause for the following simple observation:

PROPOSITION 6.5. *If A is group of length λ and $\alpha < \lambda$, then A is σ-summable iff $p^\alpha A$ is σ-summable.*

PROOF. Suppose $\lambda = \alpha + \gamma$. Assuming first that A is σ-summmable, then using Hill's criterion, suppose A is the union of Γ_i's, where for every $i < \omega$, there is an ordinal $\beta_i < \lambda$ such that $p^{\beta_i} A \cap \Gamma_i = 0$. Then $p^\alpha A$ has length γ, and by setting $\Gamma'_i = p^\alpha A \cap \Gamma_i$, we have $p^\alpha A = \cup_{i<\omega}\Gamma'_i$. If $\beta'_i = \beta_i - \alpha$ when $\beta_i \geq \alpha$ and $\beta'_i = 0$ when $\beta_i < \alpha$, then $\beta'_i < \gamma$ and $p^{\beta'_i}(p^\alpha A) \cap \Gamma'_i \subseteq p^{\beta_i} A \cap \Gamma_i = 0$, as required.

Conversely, if $p^\alpha A$ is σ-summable, then $(p^\alpha A)[p] = \cup_{i<\omega} A_i$, where for all $i < \omega$, $A_i \subseteq A_{i+1}$ and $A_i \cap p^{\alpha_i}(p^\alpha A) = 0$ for some $\alpha_i < \gamma$. If $A[p]$ is the valuated direct sum $V \oplus (p^\alpha A)[p]$, and $A'_i = V \oplus A_i$, then $A[p] = \cup_{i<\omega} A'_i$, where for all $i < \omega$, $A'_i \subseteq A'_{i+1}$ and if $\alpha'_i = \alpha + \alpha_i < \alpha + \gamma = \lambda$, then $A'_i \cap p^{\alpha'_i} A = 0$.

EXAMPLE 6.6. Let $\overline{B}$ be an unbounded torsion-complete group with B a countable direct sum of cyclic groups. One easily constructs a group L such that $p^\omega L = \overline{B}$ and $L/p^\omega L$ is countable. (For example, if C is a countable group with $p^\omega C = B$, then we can let L be the torsion subgroup of the completion of C in the $\omega \cdot 2$ topology, i.e., the topology using $\{p^{\omega+i} C : i < \omega\}$ as a neighborhood base of 0. Alternatively, this follows from Theorem 76.1 of [10].) Suppose now that M is any group such that $p^\omega M$ is an unbounded direct sum of cyclics. If we let $A = M \oplus L$ and $G = M \oplus p^\omega L = M \oplus \overline{B}$, then it is easy to see that the following properties hold:

(1) Both A and G have length $\omega \cdot 2 = \omega + \omega$.

(2) G is a (non-isotype) subgroup of A and $A/G \cong L/\overline{B} = L/p^\omega L$ is countable.

(3) G is σ-summable (this follows from Proposition 6.5, since $p^\omega G = p^\omega M$ is a direct sum of cyclics and hence σ-summable).

(4) A is not σ-summable (this also follows from Proposition 6.5, since $p^\omega A = p^\omega M \oplus \overline{B}$ is not a direct sum of cyclics and hence not σ-summable).

If $0 < n < \omega$, then a reduced group A will be called *n-Honda* if $A[p^n] = \cup_{i<\omega} A_i$, where for every $i < \omega$, $A_i \subseteq A_{i+1}$ and A_i is height-finite in A.

Clearly, an n-Honda group is m-Honda for all $m \leq n$. Since an n-Honda group clearly has countable length, by Honda's criterion (see, for instance, [10], Theorem 84.1), being 1-Honda is equivalent to the usual notion of summability, and therefore, for any $n \geq 1$, an n-Honda group must be summable. There exists a summable group of length less than $\omega \cdot 2$ which is not 2-Honda (see [5] and [6]). Notice also that summable groups of countable limit length are themselves σ-summable.

PROPOSITION 6.7. *Suppose A is a reduced group such that G is an isotype subgroup of A and A/G is countable. Then G is n-Honda iff A is n-Honda.*

PROOF. Note that if A is n-Honda and G is an arbitrary isotype subgroup of A, it easily follows that G is n-Honda (if A_i satisfies the definition for A, then one easily checks $G_i = A_i \cap G$ satisfies the definition for G). So assume it is G that is n-Honda (and of course, A/G is countable).

Observe that $A[p^n]/G[p^n] \cong (A/G)[p^n]$ is at most countable. Hence $A[p^n] = G[p^n] + C$, where $C \leq A[p^n]$ is countable. Let $G[p^n] = \cup_{i<\omega} G_i$, where for each $i < \omega$, $G_i \subseteq G_{i+1}$ and G_i are height-finite in G, whence in A. In addition, let $C = \cup_{i<\omega} C_i$, where for each $i < \omega$, $C_i \subseteq C_{i+1}$ and C_i is finite. Then $A[p^n] = \cup_{i<\omega} A_i$, where $A_i = G_i + C_i$. Certainly, $A_i \subseteq A_{i+1}$ and by Corollary 6.2(b) we have that all A_i are height-finite in A. So, A is n-Honda, finishing the proof.

Our final example shows that in Proposition 6.7, the requirement that G be isotype in A cannot be omitted.

EXAMPLE 6.8. As in Example 6.6, let L be a group such that $p^\omega L = \overline{B}$ where $\overline{B}$ is an unbounded torsion-complete group with B a countable direct sum of cyclic groups, and such that $L/p^\omega L$ is countable. Next, let G be a group so that $\overline{B} \subseteq G$, $p^\omega G = \overline{B}[p]$ and $G/\overline{B}$ is a direct sum of cyclic groups. (To construct such a G, let H be a dsc group of length $\omega + 1$ such that there is a group isomorphism $\phi : p^\omega H \to \overline{B}[p]$, and let $G = [H \oplus \overline{B}]/\{(x, \phi(x)) : x \in p^\omega H\}$, so G is the sum of H and $\overline{B}$ along ϕ.) Finally, let A be the result of identifying $\overline{B}$ in L and G, that is, $A = L + G$ with $L \cap G = \overline{B}$. We therefore have the following:

(1) $A/G \cong L/\overline{B} = L/p^\omega L$ is countable.

(2) G is summable (= 1-Honda): Indeed, since $G/\overline{B}$ is a direct sum of cyclics, we may write $G = \cup_{i<\omega} G_i$, where for each $i < \omega$ we have $\overline{B} \subseteq G_i \subseteq G_{i+1}$ and $G_i \cap p^i G \subseteq \overline{B}$. It follows that $G[p] = \cup_{i<\omega} G_i[p]$ with $G_i[p] \cap p^i G \subseteq \overline{B}[p] = (p^\omega G)[p]$. But $p^{\omega+1} G = 0$ and therefore all $G_i[p]$ are height-finite in G. So, by Honda's criterion, G is summable.

(3) A is not summable (= 1-Honda): Observe that $\overline{B} = p^\omega L \subseteq p^\omega A$ and $A/\overline{B} = (G/\overline{B}) \oplus (L/\overline{B})$ is a direct sum of cyclics, so that $p^\omega A = \overline{B}$.

Since $p^{\omega}A = \overline{B}$ is not summable, it follows that A is not summable, as claimed. Since $G \cap p^{\omega}A = \overline{B} \neq \overline{B}[p] = p^{\omega}G$, G is not isotype in A.

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