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A NOTE ON IRRATIONALITY MEASURES

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Abstract

The paper deals with lower estimates for the irrationality measure of the sum of a special series. The result depends only on the form of convergence and does not make use of divisibility properties of integers or any algebraic identities.

1. Introduction

Let a and b be positive integers with $b \leq a$ such that a and b are coprime. Also let $\{f_n\}_{n=1}^{\infty}$ denote the Fibonacci sequence and let $\{l_n\}_{n=1}^{\infty}$ denote the Lucas sequence. Matala-Aho and Prévost [3] found interesting results concerning the irrationality measures of the sums of the series $\sum_{n=1}^{\infty} \frac{1}{f_{an+b}}$ and $\sum_{n=1}^{\infty} \frac{1}{l_{an+b}}$. Results concerning lower bounds for the irrationality measure of the sum of an infinite series whose terms are rational numbers appear also in Duverney [1] or Hančl and Filip [2] for instance. Recently Sondow [4] has given a new estimate for the irrationality measure of the number e. In the sequel, for a real number x we will use $[x]$ to denote the greatest integer less than or equal x.

We prove the following.

Proposition 1. *Let* $x^{\frac{2+4(e^{\pi}-1)}{(e^{\pi}-1)}} > 3$. *Then the sum of the series*

$$\sum_{n=1}^{\infty} \frac{1}{\left[2^{x^{(3+\sin\log n)n}}\right]}$$

has irrationality measure greater or equal to $x^{\frac{2+4(e^{\pi}-1)}{(e^{\pi}-1)}} - 1 > x^4 - 1$.

It is unclear to the authors if there exists a sequence $\{a_n\}_{n=1}^{\infty}$ of positive integers with $\limsup_{n\to\infty} a_n^{\frac{1}{3^n}} = 1$ such that for every sequence of positive integers $\{c_n\}_{n=1}^{\infty}$ the sum of the series $\sum_{n=1}^{\infty} \frac{1}{a_n c_n}$ has irrationality measure greater than 2.

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2. Main results

The following theorem is used in the estimation of irrationality measures of series for which the terms have large denominators.

THEOREM 1. *Let* α, β, γ, ν *and* m *be real numbers such that* $0 < \gamma$, $0 \le \nu < 1$, $0 < \beta < \alpha < \log_2\left(\frac{m}{1-\nu} + 1\right)$ *and* $2 \le m$. *Let* n_0 *be a positive integer. Assume that* $\{a_n\}_{n=1}^{\infty}$ *and* $\{b_n\}_{n=1}^{\infty}$ *are sequences of positive integers, with* $\{a_n\}_{n=n_0}^{\infty}$ *increasing, such that for every* $n \ge n_0$

$$b_n < a_n^{\nu} \log_2^{\gamma} a_n \tag{1}$$

and

$$a_n > 2^n. \tag{2}$$

Suppose that there exists positive real number k *with*

$$k < \frac{(\alpha - \beta)}{\log_2\left(\frac{m}{1-\nu} + 1\right) - \alpha} \tag{3}$$

such that for infinitely many n

$$a_n < 2^{2^{\beta n}} \tag{4}$$

and

$$a_{n+[kn]} > 2^{2^{\alpha(n+[kn])}}. \tag{5}$$

Then the number $\sum_{n=1}^{\infty} \frac{b_n}{a_n}$ *is irrational and its irrationality measure is greater than or equal to* m.

EXAMPLE 2.1. As an immediate consequence of Theorem 1 we obtain that the sum of the series

$$\sum_{n=1}^{\infty} \frac{2^{10^{10n-1}} + 3}{2^{[10^{(10+|\cos \log n|)n}]} + 5}$$

has irrationality measure greater or equal to $\frac{9}{10}\left(10^{\frac{1+11(e^{\pi/2}-1)}{(e^{\pi/2}-1)}} - 1\right) > 9 \cdot 10^{10}$.

If the numerators of the terms of the series are not large then we can use the following corollary to estimate the measure of irrationality.

COROLLARY 1. *Let* α, β, γ *and* m *be real numbers with* $0 < \beta < \alpha < 1$, $0 < \gamma$ *and* $2 \le m$. *Let* n_0 *be a positive integer. Assume that* $\{a_n\}_{n=1}^{\infty}$ *and*

$\{b_n\}_{n=1}^{\infty}$ are sequences of positive integers, with $\{a_n\}_{n=n_0}^{\infty}$ is increasing, such that for every $n \geq n_0$

$$b_n < \log_2^{\gamma} a_n$$

and

$$a_n > 2^n.$$

Suppose that there exists a positive real number k with

$$k < \frac{(\alpha - \beta)}{1 - \alpha}$$

such that for infinitely many n

$$a_n < 2^{(m+1)^{\beta n}} \qquad \text{and} \qquad a_{n+[kn]} > 2^{(m+1)^{\alpha(n+[kn])}}.$$

Then the number $\sum_{n=1}^{\infty} \frac{b_n}{a_n}$ is irrational and its irrationality measure is greater than or equal to m.

EXAMPLE 2.2. As an immediate consequence of Corollary 1 we obtain that the sum of the series

$$\sum_{n=1}^{\infty} \frac{1}{\left[2^{10^{(10+\cos\log n)n}}\right]}$$

has irrationality measure greater or equal to $10^{\frac{2+11(e^{\pi}-1)}{(e^{\pi}-1)}} - 1 > 10^{11}$.

3. Proofs

PROOF OF THEOREM 1. Assume that δ is a sufficiently small positive real number. Set $M = m - 2\delta(1-\nu)$. Let $N = N(\delta)$ be a positive integer greater than n_0, satisfying (4), (5). Also assume N is large enough to ensure that the function $\frac{\log_2^{\gamma} x}{x^{1-\nu}}$ is decreasing for $x > a_N$, that

$$a_N^N \geq \prod_{n=1}^{N} a_n \tag{6}$$

that for every $R \geq N$

$$\sum_{\log_2 a_R < n} \frac{n^{\gamma}}{2^{n(1-\nu)}} \leq \frac{\log_2^{1+\gamma} a_R}{a_R^{1-\nu}} \tag{7}$$

and

$$\frac{2\log_2^{1+\gamma} a_R}{a_R^{\frac{\frac{\delta^2}{M^2}(1-\nu)^3}{\left(1+\frac{\delta}{M}(1-\nu)\right)^2}}} < 1 \tag{8}$$

and that

$$\frac{(\alpha-\beta)}{\log_2\left(\frac{m}{1-\nu}+1\right)-\alpha}<\frac{(\alpha-\beta)-\frac{\log_2 N}{N}}{\log_2\left(\frac{m}{1-\nu}+1-\delta\right)-\alpha}. \tag{9}$$

Observe that we can suppose (6) is true because from (2) we know that we have $\lim_{n\to\infty} a_n=\infty$ and from the fact that $\{a_n\}_{n=n_0}^{\infty}$ is increasing and the fact that $n_0<N$ we obtain that $a_{n_0}\le a_N\le a_{N+1}\le\cdots$.

Note that for each δ there are infinitely many N with the properties (4)–(9), $N>n_0$ and with the fact that the function $\frac{\log_2^{\gamma} x}{x^{1-\nu}}$ is decreasing for $x>a_N$. Fix N and let us define the finite sequence $\{c_t\}_{t=N}^{N+[kN]}$ as follows

$$c_t=\begin{cases} a_t^t, & \text{if } t=N\\ a_t^{\frac{1}{\left(\frac{M}{1-\nu}+1+\delta\right)^{t-N}}}, & \text{if } t=N+1,N+2,\dots,N+[kN]. \end{cases}$$

Set

$$c_T=\max_{t=N,N+1,\dots,N+[kN]} c_t. \tag{10}$$

Note that $T=T(N)$. If $c_T=c_N$ then from (4) and (5) we obtain that

$$\begin{aligned} 2^{N2^{\beta N}} &> a_N^N=c_N\ge c_{N+[kN]}=a_{N+[kN]}^{\frac{1}{\left(\frac{M}{1-\nu}+1+\delta\right)^{[kN]}}}\\ &> 2^{\frac{2^{\alpha(N+[kN])}}{\left(\frac{M}{1-\nu}+1+\delta\right)^{[kN]}}}=2^{2^{\alpha(N+[kN])-[kN]\log_2\left(\frac{M}{1-\nu}+1+\delta\right)}}. \end{aligned}$$

Applying $\log_2$ twice to the above inequality we get

$$\log_2 N+\beta N>\alpha(N+[kN])-[kN]\log_2\left(\frac{M}{1-\nu}+1+\delta\right).$$

Thus

$$\begin{aligned} -\frac{\log_2 N}{N}+(\alpha-\beta) &< \frac{[kN]}{N}\left(\log_2\left(\frac{M}{1-\nu}+1+\delta\right)-\alpha\right)\\ &= \frac{[kN]}{N}\left(\log_2\left(\frac{m}{1-\nu}+1-\delta\right)-\alpha\right)\\ &< k\cdot\left(\log_2\left(\frac{m}{1-\nu}+1-\delta\right)-\alpha\right). \end{aligned}$$

Hence

$$\frac{(\alpha-\beta)-\frac{\log_2 N}{N}}{\log_2\left(\frac{m}{1-\nu}+1-\delta\right)-\alpha}<k.$$

This and (9) are in contradiction to (3). Therefore $c_T \neq c_N$ and

$$c_T \geq \max_{j=N,N+1,\dots,T-1} c_j.$$

From this and from the fact that the sequence $\{a_n\}_{n=n_0}^{\infty}$ is increasing we obtain that

$$\begin{aligned} a_T &\geq \big(\max_{j=N,N+1,\dots,T-1} c_j\big)^{\left(\frac{M}{1-\nu}+1+\delta\right)^{T-N}} \\ &> \prod_{i=N}^{T-1}\big(\max_{j=N,N+1,\dots,T-1} c_j\big)^{\left(\frac{M}{1-\nu}+\delta\right)\cdot\left(\frac{M}{1-\nu}+1+\delta\right)^{i-N}} \end{aligned} \tag{11}$$

where the second inequality comes from the fact that

$$\begin{aligned} \frac{\left(\frac{M}{1-\nu}+1+\delta\right)^{T-N}}{\left(\frac{M}{1-\nu}+1+\delta\right)-1} > \frac{\left(\frac{M}{1-\nu}+1+\delta\right)^{T-N}-1}{\left(\frac{M}{1-\nu}+1+\delta\right)-1} &= \left(\frac{M}{1-\nu}+1+\delta\right)^{T-N-1} \\ &+\left(\frac{M}{1-\nu}+1+\delta\right)^{T-N-2} \\ &+\cdots+1. \end{aligned}$$

Because $\{a_n\}_{n=n_0}^{\infty}$ is increasing, N is large and greater than n_0 and inequalities (6) and (11) yield

$$\begin{aligned} a_T &> \left(\prod_{i=N}^{T-1}\big(\max_{j=N,N+1,\dots,T-1} c_j\big)^{\left(\frac{M}{1-\nu}+1+\delta\right)^{i-N}}\right)^{\frac{M}{1-\nu}+\delta} \geq \left(\prod_{i=N}^{T-1} c_i^{\left(\frac{M}{1-\nu}+1+\delta\right)^{i-N}}\right)^{\frac{M}{1-\nu}+\delta} \\ &= \left(a_N^N \prod_{i=N+1}^{T-1} a_i\right)^{\frac{M}{1-\nu}+\delta} \geq \left(\prod_{i=1}^{T-1} a_i\right)^{\frac{M}{1-\nu}+\delta}. \end{aligned}$$

This implies that

$$a_T^{1-\nu}=\left(a_T^{\frac{1+\frac{\delta}{M}(1-\nu)}{1+\frac{\delta}{M}(1-\nu)}}\right)^{1-\nu}=a_T^{\frac{1-\nu}{1+\frac{\delta}{M}(1-\nu)}}\cdot a_T^{\frac{\frac{\delta}{M}(1-\nu)^2}{1+\frac{\delta}{M}(1-\nu)}}>a_T^{\frac{\frac{\delta}{M}(1-\nu)^2}{1+\frac{\delta}{M}(1-\nu)}}\cdot\left(\prod_{i=1}^{T-1} a_i\right)^{M}. \tag{12}$$

Now we will prove that

$$\sum_{n=T}^{\infty}\frac{b_n}{a_n}<\frac{2\log_2^{1+\gamma} a_T}{a_T^{1-\nu}}. \tag{13}$$

From (1), (2), (7), the fact that $\{a_n\}_{n=n_0}^{\infty}$ is an increasing sequence of positive integers (thus $a_{n_0} \le a_N \le a_{T-1} \le a_T \le \cdots$) and the fact that the function $\frac{\log_2^\gamma x}{x^{1-\nu}}$ is decreasing for $x > a_T$ we obtain that

$$\sum_{n=T}^{\infty} \frac{b_n}{a_n} < \sum_{n=T}^{\infty} \frac{\log_2^\gamma a_n}{a_n^{1-\nu}} = \sum_{T \le n \le \log_2 a_T} \frac{\log_2^\gamma a_n}{a_n^{1-\nu}} + \sum_{\log_2 a_T < n} \frac{\log_2^\gamma a_n}{a_n^{1-\nu}}$$
$$< \frac{\log_2^{1+\gamma} a_T}{a_T^{1-\nu}} + \sum_{\log_2 a_T < n} \frac{\log_2^\gamma a_n}{a_n^{1-\nu}} < \frac{\log_2^{1+\gamma} a_T}{a_T^{1-\nu}} + \sum_{\log_2 a_T < n} \frac{n^\gamma}{2^{n(1-\nu)}}$$
$$\le \frac{2\log_2^{1+\gamma} a_T}{a_T^{1-\nu}}.$$

Thus (13) holds. Now inequalities (12) and (13) imply that

(14)
$$\left|\sum_{n=1}^{\infty} \frac{b_n}{a_n} - \sum_{n=1}^{T-1} \frac{b_n}{a_n}\right| = \left|\sum_{n=1}^{\infty} \frac{b_n}{a_n} - \frac{\prod_{n=1}^{T-1} a_n \sum_{n=1}^{T-1} \frac{b_n}{a_n}}{\prod_{n=1}^{T-1} a_n}\right| = \left|\sum_{n=T}^{\infty} \frac{b_n}{a_n}\right|$$
$$\le \frac{2\log_2^{1+\gamma} a_T}{a_T^{1-\nu}} < \frac{2\log_2^{1+\gamma} a_T}{a_T^{\frac{\frac{\delta}{M}(1-\nu)^2}{1+\frac{\delta}{M}(1-\nu)}} \cdot \left(\prod_{i=1}^{T-1} a_i\right)^M} = \frac{2\log_2^{1+\gamma} a_T}{(a_T^{1-\nu})^{\frac{\frac{\delta}{M}(1-\nu)}{1+\frac{\delta}{M}(1-\nu)}} \cdot \left(\prod_{i=1}^{T-1} a_i\right)^M}$$
$$< \frac{2\log_2^{1+\gamma} a_T}{a_T^{\frac{\frac{\delta^2}{M^2}(1-\nu)^3}{\left(1+\frac{\delta}{M}(1-\nu)\right)^2}} \cdot \left(\prod_{i=1}^{T-1} a_i\right)^{M+\frac{\delta(1-\nu)}{1+\frac{\delta}{M}(1-\nu)}}}.$$

Let us put $q_T = \prod_{n=1}^{T-1} a_n$, $p_T = \prod_{n=1}^{T-1} a_n \sum_{n=1}^{T-1} \frac{b_n}{a_n}$ and $\epsilon = \frac{\delta(1-\nu)}{1+\frac{\delta}{M}(1-\nu)}$. From (8) and (14) we obtain that

$$\left|\sum_{n=1}^{\infty} \frac{b_n}{a_n} - \frac{p_T}{q_T}\right| < \frac{1}{q_T^{M+\epsilon}}. \tag{15}$$

The fact that $M + 2\delta(1-\nu) = m \ge 2$, where δ is sufficiently small, and that for each δ we can find infinitely many pairs $(p_T, q_T) = (p_{T(N)}, q_{T(N)})$ satisfying (15) imply that the number $\sum_{n=1}^{\infty} \frac{b_n}{a_n}$ is irrational and its irrationality measure is greater than or equal to m.

PROOF OF COROLLARY 1. It is enough to set $\nu = 0, \alpha = \alpha_P \cdot \log_2(m+1)$ and $\beta = \beta_P \cdot \log_2(m+1)$ in Theorem 1 where α_P, β_P are constants in Corollary 1.

PROOF OF PROPOSITION 1. For the function $\sin\log n$ we have that $\sin\log(ne^{\pi})$ is about $-\sin\log n$. Now set $k = e^{\pi} - 1$, $\alpha = \frac{(1+\epsilon)(4-2\epsilon)(e^{\pi}-1)}{2+4(e^{\pi}-1)}$ and $\beta = \frac{(1+\epsilon)(2+\epsilon)(e^{\pi}-1)}{2+4(e^{\pi}-1)}$ and $m = x^{\frac{2+4(e^{\pi}-1)}{e^{\pi}-1}\cdot\frac{1}{1+\epsilon}}$ in Corollary 1. Because we can take ϵ sufficiently small we obtain Proposition 1.

PROOF OF EXAMPLE 2.1. For the function $\cos\log n$ we have that $\cos\log(ne^{\frac{\pi}{2}})$ is about $-\sin\log n$. Now set $\nu = \frac{1}{10}$, $k = e^{\frac{\pi}{2}} - 1$, $\alpha = (11-\epsilon)\log_2 10$ and $\beta = (10+\epsilon)\log_2 10$ and $m = \frac{9}{10}\big(10^{\frac{1+11(e^{\pi/2}-1)}{(e^{\pi/2}-1)(1+\epsilon)}}\big)$ in Theorem 1. And let us take ϵ sufficiently small and the proof is complete.

The arguments in Example 2.2 are similar to the proof of Proposition 1.

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