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INCLUSIONS OF UNITAL C^*-ALGEBRAS OF INDEX-FINITE TYPE WITH DEPTH 2 INDUCED BY SATURATED ACTIONS OF FINITE DIMENSIONAL C^*-HOPF ALGEBRAS

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Abstract

Let B be a unital C^*-algebra and H a finite dimensional C^*-Hopf algebra with its dual C^*-Hopf algebra H^0. We suppose that there is a saturated action of H on B and we denote by A its fixed point C^*-subalgebra of B. Let E be the canonical conditional expectation from B onto A. In the present paper, we shall give a necessary and sufficient condition that there are a weak action of H^0 on A and a unitary cocycle σ of $H^0 \otimes H^0$ to A satisfying that there is an isomorphism π of $A \rtimes_\sigma H^0$ onto B, which is the twisted crossed product of A by the weak action of H^0 on A and the unitary cocycle σ, such that $F = E \circ \pi$, where F is the canonical conditional expectation from $A \rtimes_\sigma H^0$ onto A.

1. Introduction

Let $A \subset B$ be an irreducible and index-finte inclusion of unital C^*-algebras of depth 2. If A and B are factors then B is isomorphic to a crossed product of A by an (outer) action of a finite dimensional Kac (C^*-Hopf) algebra ([8], [13]). Izumi showed in [2] that if A and B are unital simple C^*-algebras then there is an action of a finite dimensional C^*-Hopf algebra H on B such that A is the fixed point C^*-subalgebra B^H. But B is not represented by a crossed product of A by an action of the dual C^*-Hopf algebra H^0 in general ([4], [5]). We give an example as follows: Let θ be an irrational number in (0, 1) and A_θ the corresponding irrational rotation C^*-algebra generated by unitary elements $\{u, v\}$ satisfying $uv = e^{2\pi\theta} vu$. For $n \in \mathsf{N}$ we define an outer automorphism σ by $\sigma(u) = e^{\frac{2\pi}{n} i} u$ and $\sigma(v) = v$ and an action α of $\mathsf{Z}/n\mathsf{Z}$ by $\alpha_k = \sigma^k$ for $k = 0, 1, 2, \ldots, n-1$. It is easy to see that the fixed point C^*-subalgebra is $A_{n\theta}$ generated by u^n and v. Then the inclusion $A_{n\theta} \subset A_\theta$ is irreducible, of depth 2 and of Watatani index n. The dual group of $\mathsf{Z}/n\mathsf{Z}$ is also $\mathsf{Z}/n\mathsf{Z}$. Suppose that A_θ is isomorphic to a crossed product $A_{n\theta} \rtimes_\beta \mathsf{Z}/n\mathsf{Z}$ of $A_{n\theta}$ by an outer action

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β of $\mathsf{Z}/n\mathsf{Z}$ on $A_{n\theta}$. Let w be a unitary element in $A_{n\theta} \rtimes_\beta \mathsf{Z}/n\mathsf{Z}$ implementing β, i.e., $w^n = 1$ and $w^k x w^{k*} = \beta_k(x)$ for $x \in A_{n\theta}$ ($k = 0, 1, 2, \ldots, n-1$). We can define a trace $\tilde{\tau}$ on $A_{n\theta} \rtimes_\beta \mathsf{Z}/n\mathsf{Z}$ by $\tilde{\tau}(w^k) = 0$ for $k = 1, 2, \ldots, n-1$ and $\tilde{\tau}(x) = \tau(x)$ for $x \in A_{n\theta}$, where τ is the unique tracial state on $A_{n\theta}$. Put $p = \frac{1}{n}\sum_{k=0}^{n-1} w^k$. Then p is a projection in $A_{n\theta} \rtimes_\beta \mathsf{Z}/n\mathsf{Z}$ with $\tilde{\tau}(p) = \frac{1}{n}$. Since A_θ has the unique tracial state and its values for projections in A_θ is $(\theta\mathsf{Z} + \mathsf{Z}) \cap [0, 1]$, A_θ is not isomorphic to $A_{n\theta} \rtimes_\beta \mathsf{Z}/n\mathsf{Z}$.

In factor cases, Kosaki ([7]) gave a necessary and sufficient condition for B to be characterized by a crossed product by a finite group as follows:

$$A' \cap B = \mathsf{C}1 \text{ and } A' \cap B_1 \text{ is commutative,} \tag{1.1}$$

where B_1 is the C^*-basic construction for $A \subset B$. We can see that the above example $A_{n\theta} \subset A_\theta$ satisfies Condition (1.1). So this characterization does not hold in C^*-algebras. However, A_θ can be represented by a twisted crossed product $A_{n\theta} \rtimes_{\beta,w} \mathsf{Z}/\mathsf{Z}n$ of $A_{n\theta}$ by a twisted action (β, w). In the previous paper [6], we showed that B is described by a twisted crossed product of A by its twisted action of a finite group if and only if the inclusion $A \subset B$ satisfies Condition (1.1) and all the minimal projections in $A' \cap B_1$ are Murray-von Neumann equivalent in B_1.

In [1], Blattner, Cohen and Montgomery defined a weak action of Hopf algebras, which is a generalization of twisted group actions. Let H be a finite dimensional C^*-Hopf algebra. We suppose that there is a saturated action of H on B defined in Szymański and Peligrad [14]. Let A be the fixed point C^*-subalgebra B^H and E the canonical conditional expectation from B onto A. Let $B \rtimes H$ be the crossed product of B by the action of H on B, which is defined in [14]. In [14], they showed that $B \rtimes H$ is isomorphic to B_1, the C^*-basic construction induced by E. Let ρ be the coaction of B_1 to $B_1 \otimes H$ defined by $\rho(b \rtimes h) = \sum_{(h)} (b \rtimes h_{(1)}) \otimes h_{(2)}$ for $b \in B$ and $h \in H$, where we identify B_1 with $B \rtimes H$ and $\Delta(h) = \sum_{(h)} h_{(1)} \otimes h_{(2)}$, Δ is the comultiplication of H. Our main result, Theorem 6.4, is that B can be represented by the twisted crossed product $A \rtimes_\sigma H^0$ if and only if $\rho(e_A)$ and $e_A \otimes 1$ are Murray-von Neumann equivalent, written $\rho(e_A) \sim e_A \otimes 1$ in $B_1 \otimes H$, where H^0 is the dual C^*-Hopf algebra of H and e_A is the Jones projection for $A \subset B$.

This paper is organized as follows: Section 2 consists of preliminaries containing definitions and fundamental matters of finite dimensional C^*-Hopf algebras and their weak actions. In Section 3 we define a unitary cocycle for a weak action of a finite dimensional C^*-Hopf algebra and discuss about a twisted crossed product. In Section 4 we suppose that $\rho(e_A) \sim e_A \otimes 1$ in $B_1 \otimes H$. We prove that there is a unitary element $u \in B \otimes H$ such that $\rho(e_A) = u^*(e_A \otimes 1)u$ and we give some properties of this unitary element. In Section 5 we construct a weak action of H^0 on A under the condition that $\rho(e_A) \sim e_A \otimes 1$ in $B_1 \otimes H$. In Section 6 we prove the main result, Theorem 6.4. Using this theorem we prove that B can be represented by a crossed product $A \rtimes H^0$ if and only if there is a tunnel construction $P \subset A$ for $A \subset B$ (Proposition 6.8). From this result, we can see that B always can be represented by $A \rtimes H^0$ and that any unitary cocycle is coboundary if $A \subset B$ are factors ([9], [11]).

2. Preliminaries on finite dimensional C^*-Hopf algebras

Following [14], we shall state the definition of a finite dimensional C^*-Hopf algebra and its basic properties. Throughout this paper, H denotes a finite dimensional C^*-Hopf algebra.

DEFINITION 2.1. We say that a finite dimensional C^*-algebra H is *a C^*-Hopf algebra* if H has the following properties.

(1) There exist linear maps;
  (a) comultiplication $\Delta : H \longrightarrow H \otimes H$,
  (b) counit $\epsilon : H \longrightarrow \mathsf{C}$,
  (c) antipode $S : H \longrightarrow H$, Δ and ϵ are C^*-algebra homomorphisms and S is a $*$- preserving antimultiplicative involution. We have $\Delta(1) = 1 \otimes 1$, $\epsilon(1) = 1$ and $S(1) = 1$, where 1 is the unit element in H.

(2) The following identities hold;
  (a) $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$,
  (b) $(\epsilon \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \epsilon) \circ \Delta = \mathrm{id}$, where $\mathsf{C} \otimes H$ and $H \otimes \mathsf{C}$ are identified with H,
  (c) $(m \circ (S \otimes \mathrm{id}))(\Delta(h)) = \epsilon(h) = (m \circ (\mathrm{id} \otimes S))(\Delta(h))$ for any $h \in H$, where $m : H \otimes H \longrightarrow H$ denotes the multiplication.

We shall use Sweedler's notation $\Delta(h) = h_{(1)} \otimes h_{(2)}$ for $h \in H$ which suppresses a possible summation when we write the comultiplications. By [14, Theorem 2.2] or Woronowicz [16], there is the Haar trace τ on H such that $(\tau \otimes \mathrm{id})(\Delta(h)) = \tau(h)1 = (\mathrm{id} \otimes \tau)(\Delta(h))$ for any $h \in H$.

Let H^0 be the linear space of linear functionals on H. By [14, Proposition 2.3], H^0 has a C^*-Hopf algebra structure. We regard H^0 as a finite dimensional C^*-Hopf algebra by this structure. We denote its comultiplication, counit, antipode and so on by Δ^0, ϵ^0, S^0 and so on.

Since H is a finite dimensional C^*-algebra, $H \cong \oplus_{k=1}^N M_{d_k}(\mathsf{C})$ as C^*-algebras, where for each $n \in \mathsf{N}$, we denote by $M_n(\mathsf{C})$ the $n \times n$-matrix algebra over C. Let $\{v_{ij}^k\}_{ij=1}^{d_k}$ be a system of matrix units of a C^*-subalgebra of H isomorphic to $M_{d_k}(\mathsf{C})$ for $k = 1, 2, \ldots, N$. Also, let $\{w_{ij}^k \mid k = 1, 2, \ldots, N, i, j =$

$1, 2, \ldots, d_k\}$ be a basis of H satisfying [14, Theorem 2.2,2]. We call it *a system of comatrix units* of H. By [14, Theorem 2.2] or [16], there is a minimal and central projection e in H, called *the distinguished projection* such that $he = \epsilon(h)e$ for any $h \in H$. By [14, Proposition 2.10] and the discussions below it,

$$\Delta(e) = \sum_{i,j,k} \frac{1}{d_k} v_{ji}^k \otimes S(v_{ij}^k) = \sum_{i,j,k} \frac{1}{d_k} S(v_{ji}^k) \otimes v_{ij}^k.$$

Also, by the above equations and Definition 2.1,

$$e = \sum_{i,j,k} \frac{1}{d_k} \epsilon(v_{ji}^k) v_{ij}^k = \sum_{i,j,k} \frac{1}{d_k} \epsilon(v_{ji}^k) S(v_{ij}^k).$$

Furthermore, we note that the Haar trace τ on H is the distinguished projection in H^0.

Next, following Blattner, Cohen and Montgomery [1, Definitions 1.1 and 2.1] and [14, Definition 2.4], we shall define an action and a coaction of a finite dimensional C^*-Hopf algebra H on a unital C^*-algebra A.

DEFINITION 2.2. By *a weak action* of H on A, we mean a bilinear map $(h, x) \mapsto h \cdot x$ of $H \times A$ to A such that for $h \in H$, $x, y \in A$

(1) $h \cdot xy = (h_{(1)} \cdot x)(h_{(2)} \cdot y)$,

(2) $h \cdot 1 = \epsilon(h)1$,

(3) $1 \cdot x = x$,

(4) $(h \cdot x)^* = S(h^*) \cdot x^*$.

By *an action* of H on A, we mean a weak action such that $h \cdot (l \cdot x) = (hl) \cdot x$ for $h, l \in H$, $x \in A$.

Let $\mathrm{Hom}(H, A)$ be the linear space of all linear maps from H to A. Then by Sweedler [12, pp. 69-70] it becomes a unital *-algebra as follows: For any $f, g \in \mathrm{Hom}(H, A)$ $(fg)(h) = f(h_{(1)})g(h_{(2)})$, $f^*(h) = f(S(h^*))^*$, where ϵ, the counit in H is the unit element in $\mathrm{Hom}(H, A)$. We call it *a unital convolution *- algebra.* Then as mentioned in [1, pp. 163], there is an isomorphism $\imath$ of $A \otimes H^0$ onto $\mathrm{Hom}(H, A)$ defined by $\imath(x \otimes \phi)(h) = \phi(h)x$ for any $x \in A$, $h \in H$ and $\phi \in H^0$ since H is finite dimensional.

DEFINITION 2.3. A weak action of H on A is *inner* if there is a unitary element $U \in \mathrm{Hom}(H, A)$ such that for any $h \in H$ and $x \in A$, $h \cdot x = U(h_{(1)})xU^*(h_{(2)})$. We say that U *implements* the weak action.

DEFINITION 2.4. By *a weak coaction* of H on A, we mean a linear map $\rho : A \to A \otimes H$ such that for $x, y \in A$,

(1) $\rho(xy) = \rho(x)\rho(y)$,
(2) $\rho(1) = 1 \otimes 1$,
(3) $(\mathrm{id} \otimes \epsilon)(\rho(x)) = x$,
(4) $\rho(x^*) = \rho(x)^*$.

By *a coaction* of H on A, we mean a weak coaction such that

$$(\rho \otimes \mathrm{id}) \circ \rho = (\mathrm{id} \otimes \Delta) \circ \rho.$$

If H acts on A in the saturated fashion defined in [14], then its fixed point C^*-subalgebra A^H of A is defined by

$$A^H = \{x \in A \mid h \cdot x = \epsilon(h)x \text{ for any } h \in H\}.$$

Also, we can define a conditional expectation E from A onto A^H with $\mathrm{Index}(E) = \dim(H)$ by $E(x) = e \cdot x$ for $x \in A$ by [14], where $\mathrm{Index}(E)$ is the Watatani index of E. We call E *the canonical* conditional expectation of A onto A^H.

3. A twisted crossed product of a unital C^*-algebra by a finite dimensional C^*-Hopf algebra

Modifying [14] and [1], we shall define a twisted crossed product of a unital C^*-algebra by a finite dimensional C^*-Hopf algebra. Let H be a finite dimensional C^*-Hopf algebra and A a unital C^*-algebra. In the same way as in Section 2, $\mathrm{Hom}(H \otimes H, A)$ becomes a unital convolution *-algebra as follows: For any $f, g \in \mathrm{Hom}(H \otimes H, A)$ $(fg)(h, l) = f(h_{(1)}, l_{(1)})g(h_{(2)}, l_{(2)})$, $f^*(h, l) = f(S(h^*), S(l^*))^*$, where $\epsilon \otimes \epsilon$ is the unit element in $\mathrm{Hom}(H \otimes H, A)$. We suppose that there is a weak action of H on A.

DEFINITION 3.1. Let $\sigma : H \otimes H \longrightarrow A$ be a bilinear map. The bilinear map σ is *a unitary cocycle* for the weak action of H on A if σ satisfies the following:

(1) In the unital convolution *- algebra $\mathrm{Hom}(H \otimes H, A)$, σ is a unitary element,
(2) σ is normal, that is, for any $h \in H$, $\sigma(h, 1) = \sigma(1, h) = \epsilon(h)1$,
(3) (cocycle condition) For any $h, l, m \in H$

$$[h_{(1)} \cdot \sigma(l_{(1)}, m_{(1)})]\sigma(h_{(2)}, l_{(2)}m_{(2)}) = \sigma(h_{(1)}, l_{(1)})\sigma(h_{(2)}l_{(2)}, m),$$

(4) (twisted modular condition) For any $h, l \in H$, $x \in A$,

$$[h_{(1)} \cdot (l_{(1)} \cdot x)]\sigma(h_{(2)}, l_{(2)}) = \sigma(h_{(1)}, l_{(1)})(h_{(2)}l_{(2)} \cdot x).$$

We suppose that σ is a unitary cocycle for the weak action of H on A. Let $A \rtimes_\sigma H$ be a unital *-algebra, called *a twisted crossed product* of A by H, where underlying space is $A \otimes H$. We denote by $x \rtimes h$ the element induced by $x \in A$ and $h \in H$. Its multiplication and *-operation are given by

$$(x \rtimes h)(y \rtimes l) = x(h_{(1)} \cdot y)\sigma(h_{(2)}, l_{(1)}) \rtimes h_{(3)}l_{(2)}$$

$$(x \rtimes h)^* = \sigma(S(h_{(2)}), h_{(1)})^*(h^*_{(3)} \cdot x^*) \rtimes h^*_{(4)}$$

for any $x, y \in A$ and $h, l \in H$. By [1, Corollary 4.6], $A \rtimes_\sigma H$ is a unital algebra. It is necessary to show that $A \rtimes_\sigma H$ is a *-algebra. We shall show it. Before doing it, we note the following:

REMARK 3.2. We identify A with the C^*-subalgebra $A \rtimes 1$ of $A \rtimes_\sigma H$. Also, if σ is trivial, that is, for any $h, l \in H$ $\sigma(h, l) = \epsilon(h)\epsilon(l)1$, then $A \rtimes_\sigma H$ is the ordinary crossed product, $A \rtimes H$ which is defined in [14]. Furthermore, in the ordinary crossed product, we can also identify H with the C^*-subalgebra $1 \rtimes H$ of $A \rtimes H$.

LEMMA 3.3. *For any* $h, l \in H$*,*

(1) $h \cdot \sigma(S(l_{(2)}), l_{(1)}) = \sigma(h_{(1)}, S(l_{(3)}))\sigma(h_{(2)}S(l_{(2)}), l_{(1)})$

(2) $h \cdot \sigma(l_{(1)}, S(l_{(2)})) = \sigma(h_{(1)}, l_{(1)})\sigma(h_{(2)}l_{(2)}, S(l_{(3)}))$

PROOF. For any $h, l \in H$,

$$\begin{aligned} h \cdot \sigma(S(l_{(2)}), l_{(1)}) &= (\epsilon(h_{(2)})h_{(1)}) \cdot \sigma(S(l_{(2)}), l_{(1)}) \\ &= [h_{(1)} \cdot \sigma(S(l_{(2)}), l_{(1)})]\sigma(h_{(2)}, 1) \\ &= [h_{(1)} \cdot \sigma(S(l_{(3)}), l_{(1)})]\sigma(h_{(2)}, \epsilon(l_{(2)})) \\ &= [h_{(1)} \cdot \sigma(S(l_{(4)}), l_{(1)})]\sigma(h_{(2)}, S(l_{(3)})l_{(2)}) \\ &= \sigma(h_{(1)}, S(l_{(3)}))\sigma(h_{(2)}S(l_{(2)}), l_{(1)}) \end{aligned}$$

by Definition 3.1(3). In the same way, we obtain Equation (2).

LEMMA 3.4. *For any* $x, y \in A$ *and* $h \in H$*,* $((x \rtimes 1)(y \rtimes h))^* = (y \rtimes h)^*(x \rtimes 1)^*$.

PROOF. By routine computations,

$$\begin{aligned} (y \rtimes h)^*(x \rtimes 1)^* &= \sigma(S(h_{(2)}), h_{(1)})^*(h^*_{(3)} \cdot y^*)(h^*_{(4)} \cdot x^*)\sigma(h^*_{(5)}, 1) \rtimes h^*_{(6)} \\ &= \sigma(S(h_{(2)}), h_{(1)})^*(h^*_{(3)} \cdot y^*x^*) \rtimes h^*_{(4)} \\ &= ((x \rtimes 1)(y \rtimes h))^*. \end{aligned}$$

LEMMA 3.5. *For any* $x \in A$ *and* $h \in H$, $((1 \rtimes h)(x \rtimes 1))^* = (x \rtimes 1)^*(1 \rtimes h)^*$.

PROOF. By Definition 3.1(4),

$$\begin{aligned}((1 \rtimes h)(x \rtimes 1))^* &= [(S(h_{(4)}) \cdot (h_{(1)} \cdot x))\sigma(S(h_{(3)}), h_{(2)})]^* \rtimes h^*_{(5)} \\ &= [\sigma(S(h_{(4)}), h_{(1)})(S(h_{(3)})h_{(2)} \cdot x)]^* \rtimes h^*_{(5)} \\ &= (\sigma(S(h_{(2)}), h_{(1)})x)^* \rtimes h^*_{(3)} = (x \rtimes 1)^*(1 \rtimes h)^*.\end{aligned}$$

LEMMA 3.6. *For any* $h, l \in H$, $((1 \rtimes h)(1 \rtimes l))^* = (1 \rtimes l)^*(1 \rtimes h)^*$.

PROOF. By routine calculations,

$$\begin{aligned}&((1 \rtimes h)(1 \rtimes l))^* \\ &\qquad = [(S(h_{(4)}l_{(4)}) \cdot \sigma(h_{(1)}, l_{(1)}))\sigma(S(h_{(3)}l_{(3)}), h_{(2)}l_{(2)})]^* \rtimes (h_{(5)}l_{(5)})^*.\end{aligned}$$

Using Definition 3.1(3),

$$\begin{aligned}((1 \rtimes h)(1 \rtimes l))^* &= [\sigma(S(h_{(4)}l_{(3)}), h_{(1)})\sigma(S(h_{(3)}l_{(2)})h_{(2)}, l_{(1)})]^* \rtimes (h_{(5)}l_{(4)})^* \\ &= \sigma(S(l_{(2)}), l_{(1)})^*\sigma(S(l_{(3)})S(h_{(2)}), h_{(1)})^* \rtimes l^*_{(4)}h^*_{(3)}.\end{aligned}$$

On the other hand,

$$\begin{aligned}&(1 \rtimes l)^*(1 \rtimes h)^* \\ &\qquad = \sigma(S(l_{(2)}), l_{(1)})^*[S(l_{(3)}) \cdot \sigma(S(h_{(2)}), h_{(1)})]^*\sigma(l^*_{(4)}, h^*_{(3)}) \rtimes l^*_{(5)}h^*_{(4)}.\end{aligned}$$

Using Lemma 3.3(1) and that $\sigma^*\sigma = \epsilon \otimes \epsilon$,

$$\begin{aligned}&(1 \rtimes l)^*(1 \rtimes h)^* \\ &\quad = \sigma(S(l_{(2)}), l_{(1)})^* \\ &\qquad \times [\sigma(S(l_{(4)}), S(h_{(3)}))\sigma(S(l_{(3)})S(h_{(2)}), h_{(1)})]^*\sigma(l^*_{(5)}, h^*_{(4)}) \rtimes l^*_{(6)}h^*_{(5)} \\ &\quad = \sigma(S(l_{(2)}), l_{(1)})^*\sigma(S(l_{(3)})S(h_{(2)}), h_{(1)})^* \rtimes l^*_{(4)}h^*_{(3)}.\end{aligned}$$

Therefore we obtain the conclusion.

LEMMA 3.7. *For any* $h \in H$, $(1 \rtimes h)^{**} = 1 \rtimes h$.

PROOF. Using Lemma 3.3(1) and that $\sigma^*\sigma = \epsilon \otimes \epsilon$, by routine computations,

$$\begin{aligned}(1 \rtimes h)^{**} &= \sigma(S(h_{(4)})^*, h^*_{(3)})^*[h_{(5)} \cdot \sigma(S(h_{(2)}), h_{(1)})] \rtimes h_{(6)} \\ &= \sigma(S(h^*_{(5)}), h^*_{(4)})^*\sigma(h_{(6)}, S(h_{(3)}))\sigma(h_{(7)}S(h_{(2)}), h_{(1)}) \rtimes h_{(8)} \\ &= \epsilon(h_{(4)})\epsilon(S(h_{(3)}))\sigma(h_{(5)}S(h_{(2)}), h_{(1)}) \rtimes h_{(6)} = 1 \rtimes h.\end{aligned}$$

PROPOSITION 3.8. *The unital algebra $A \rtimes_\sigma H$ is a *-algebra.*

PROOF. We have only to show that for any $x, y \in A$ and $h, l \in H$,

$$((x \rtimes h)(y \rtimes l))^* = (y \rtimes l)^*(x \rtimes h)^*, \quad (x \rtimes h)^{**} = x \rtimes h.$$

Since $(1 \rtimes h)(y \rtimes l)$ is a finite sum of elements in the form $z \rtimes k$, where $z \in A$, $k \in H$, by Lemma 3.4

$$\begin{aligned}((x \rtimes h)(y \rtimes l))^* &= ((x \rtimes 1)(1 \rtimes h)(y \rtimes l))^* \\ &= ((1 \rtimes h)(y \rtimes 1)(1 \rtimes l))^*(x \rtimes 1)^*.\end{aligned}$$

Also, we can write that $(1 \rtimes h)(y \rtimes 1) = \sum_i z_i \rtimes k_i$, where $z_i \in A$, $k_i \in H$ for any i. Hence

$$((1 \rtimes h)(y \rtimes 1)(1 \rtimes l))^* = \sum_i ((z_i \rtimes 1)(1 \rtimes k_i)(1 \rtimes l))^*.$$

Since $(1 \rtimes k_i)(1 \rtimes l)$ is also a finite sum of elements in the form $z \rtimes k$, where $z \in A$, $k \in H$, by Lemmas 3.4, 3.5 and 3.6,

$$\begin{aligned}((1 \rtimes h)(y \rtimes 1)(1 \rtimes l))^* &= \sum_i (1 \rtimes l)^*(1 \rtimes k_i)^*(z_i \rtimes 1)^* \\ &= \sum_i (1 \rtimes l)^*((z_i \rtimes 1)(1 \rtimes k_i))^* \\ &= (1 \rtimes l)^*(y \rtimes 1)^*(1 \rtimes h)^*.\end{aligned}$$

Thus by Lemma 3.4,

$$((x \rtimes h)(y \rtimes l))^* = (1 \rtimes l)^*(y \rtimes 1)^*(1 \rtimes h)^*(x \rtimes 1)^* = (y \rtimes l)^*(x \rtimes h)^*.$$

Furthermore, by the above discussion and Lemma 3.7,

$$(x \rtimes h)^{**} = (x \rtimes 1)^{**}(1 \rtimes h)^{**} = x \rtimes h.$$

Modifying [14], we shall define a C^*-norm in $A \rtimes_\sigma H$. We suppose that A acts on a Hilbert space $\mathscr{H}$ faithfully and non-degenerately. Let $l^2(\tau, H)$ be a Hilbert space induced by the Haar trace τ on H and B the C^*-algebra of all bounded linear operatros on a Hibert space $\mathscr{H} \otimes l^2(\tau, H)$. Let θ be a map from $A \rtimes_\sigma H$ to B defined by for any $x \in A$, $h, l \in H$ and $\xi \in \mathscr{H}$,

$$\theta(x \rtimes h)(\xi \otimes l) = (S(h_{(3)}l_{(2)}) \cdot x)\sigma(S(h_{(2)}l_{(1)}), h_{(1)})\xi \otimes h_{(4)}l_{(3)}.$$

We shall show that θ is a faithful representation of the *-algebra $A \rtimes_\sigma H$ to B.

LEMMA 3.9. *For any $x, y \in A$ and $h \in H$, $\theta((x \rtimes 1)(y \rtimes h)) = \theta(x \rtimes 1)\theta(y \rtimes h)$.*

PROOF. By routine calculations, for any $\xi \in \mathscr{H}$ and $l \in l^2(\tau, H)$,

$$\begin{aligned}
&\theta(x \rtimes 1)\theta(y \rtimes h)(\xi \otimes l) \\
&= [S(h_{(4)}l_{(3)}) \cdot x][S(h_{(3)}l_{(2)}) \cdot y]\sigma(S(h_{(2)}l_{(1)}), h_{(1)})\xi \otimes h_{(5)}l_{(4)} \\
&= [S(h_{(3)}l_{(2)}) \cdot xy]\sigma(S(h_{(2)}l_{(1)}), h_{(1)})\xi \otimes h_{(4)}l_{(3)} \\
&= \theta((x \rtimes 1)(y \rtimes h))(\xi \otimes l).
\end{aligned}$$

LEMMA 3.10. *For any $x \in A$ and $h, k \in H$, $\theta((1 \rtimes h)(x \rtimes k)) = \theta(1 \rtimes h)\theta(x \rtimes k)$.*

PROOF. By Definition 3.1(3) and (4) for any $\xi \in \mathscr{H}$ and $l \in l^2(\tau, H)$,

$$\begin{aligned}
&\theta((1 \rtimes h)(x \rtimes k))(\xi \otimes l) \\
&\quad = [S(h_{(6)}k_{(5)}l_{(3)}) \cdot (h_{(1)} \cdot x)][S(h_{(5)}k_{(4)}l_{(2)}) \cdot \sigma(h_{(2)}, k_{(1)})] \\
&\qquad\quad \times \sigma(S(h_{(4)}k_{(3)}l_{(1)}), h_{(3)}k_{(2)})\xi \otimes h_{(7)}k_{(6)}l_{(4)} \\
&\quad = [S(h_{(4)}k_{(4)}l_{(3)}) \cdot (h_{(1)} \cdot x)]\sigma(S(h_{(3)}k_{(3)}l_{(2)}), h_{(2)})\sigma(S(k_{(2)}l_{(1)}), k_{(1)})\xi \\
&\qquad\quad \otimes h_{(5)}k_{(5)}l_{(4)} \\
&\quad = \sigma(S(h_{(4)}k_{(4)}l_{(3)}), h_{(1)})[S(h_{(3)}k_{(3)}l_{(2)})h_{(2)} \cdot x]\sigma(S(k_{(2)}l_{(1)}), k_{(1)})\xi \\
&\qquad\quad \otimes h_{(5)}k_{(5)}l_{(4)} \\
&\quad = \sigma(S(h_{(2)}k_{(4)}l_{(3)}), h_{(1)})[S(k_{(3)}l_{(2)}) \cdot x]\sigma(S(k_{(2)}l_{(1)}), k_{(1)})\xi \otimes h_{(3)}k_{(5)}l_{(4)} \\
&\quad = \theta(1 \rtimes h)\theta(x \rtimes k)(\xi \otimes l).
\end{aligned}$$

LEMMA 3.11. *For any $x \in A$ and $h \in H$, $\theta((x \rtimes 1)^*) = \theta(x \rtimes 1)^*$, $\theta((1 \rtimes h)^*) = \theta(1 \rtimes h)^*$.*

PROOF. For any $\xi, \eta \in \mathscr{H}$ and $l, k \in H$,

$$(\theta((x \rtimes 1)^*)(\xi \otimes l)|\eta \otimes k) = ([S(l_{(1)}) \cdot x^*]\xi|\eta)\tau(l_{(2)}k^*).$$

Also, by [14, Theorem 2.2],

$$\begin{aligned}
&(\theta(x \rtimes 1)^*(\xi \otimes l)|\eta \otimes k) \\
&= (\xi \otimes l|\theta(x \rtimes 1)(\eta \otimes k)) = (\xi|[S(k_{(1)}) \cdot x]\eta)\tau(lk^*_{(2)}) \\
&= (\xi|[S(k_{(1)}) \cdot x]\eta)\tau(\epsilon(l_{(1)})l_{(2)}k^*_{(2)}) = (\xi|[S(k_{(1)})S(l^*_{(2)})l^*_{(1)}\tau(k_{(2)}l^*_{(3)}) \cdot x]\eta) \\
&= (\xi|[\tau(S(l^*_{(3)}k_{(2)}))S(l^*_{(2)}k_{(1)})l^*_{(1)} \cdot x]\eta) = (\xi|[\tau(l^*_{(2)}k)l^*_{(1)} \cdot x]\eta) \\
&= ([S(l_{(1)}) \cdot x^*]\xi|\eta)\tau(l_{(2)}k^*).
\end{aligned}$$

Thus we obtain the first equation in the above. Furthermore, using Lemma 3.3(1) and that $\sigma^*\sigma = \epsilon \otimes \epsilon$, by routine computations,

$$\begin{aligned}
&(\theta((1 \rtimes h)^*)(\xi \otimes l)|\eta \otimes k) \\
&\quad = ([S(h^*_{(5)}l_{(2)}) \cdot \sigma(S(h_{(2)}), h_{(1)})^*]\sigma(S(h^*_{(4)}l_{(1)}), h_{(3)})\xi|\eta)\tau(h^*_{(6)}l_{(3)}k^*) \\
&\quad = (\xi|\sigma^*(l^*_{(1)}h_{(5)}, S(h_{(4)}))\sigma(l^*_{(2)}h_{(6)}, S(h_{(3)})) \\
&\qquad\quad \times \sigma(l^*_{(3)}h_{(7)}S(h_{(2)}), h_{(1)})\eta)\tau(h^*_{(8)}l_{(4)}k^*) \\
&\quad = (\xi|\sigma(\tau(kl^*_{(2)}h_{(2)})l^*_{(1)}, h_{(1)})\eta) \\
&\quad = (\xi|\sigma(\tau(k_{(2)}\epsilon(k_{(1)})l^*_{(2)}h_{(3)}\epsilon(h_{(2)}))l^*_{(1)}, h_{(1)})\eta) \\
&\quad = (\xi|\sigma(S(k_{(1)})S(h_{(2)})\tau(h_{(4)}k_{(3)}l^*_{(2)})h_{(3)}k_{(2)}l^*_{(1)}, h_{(1)})\eta).
\end{aligned}$$

By [14, Theorem 2.2],

$$\begin{aligned}
(\theta((1 \rtimes h)^*)(\xi \otimes l)|\eta \otimes k) &= (\xi|\sigma(S(h_{(2)}k_{(1)}), h_{(1)})\eta)\tau(lk^*_{(2)}h^*_{(3)}) \\
&= (\theta(1 \rtimes h)^*(\xi \otimes l)|\eta \otimes k).
\end{aligned}$$

Let V be a linear map from H to $A \rtimes_\sigma H$ defined by $V(h) = 1 \rtimes h$ for any $h \in H$. By easy computations $V \in \mathrm{Hom}(H, A \rtimes_\sigma H)$. Moreover, we have the following properties:

LEMMA 3.12. (i) *The element V is a unitary one in* $\mathrm{Hom}(H, A \rtimes_\sigma H)$.
(ii) *For any $x \in A$ and $h, l \in H$,*

$$\begin{aligned}
(h \cdot x) \rtimes 1 &= V(h_{(1)})(x \rtimes 1)V^*(h_{(2)}), \\
\sigma(h, l) \rtimes 1 &= V(h_{(1)})V(l_{(1)})V^*(h_{(2)}l_{(2)}).
\end{aligned}$$

PROOF. Since σ is a unitary element in $\mathrm{Hom}(H \otimes H, A \rtimes_\sigma H)$,

$$(V^*V)(h) = \sigma(h^*_{(3)}, S(h_{(4)}))^*\sigma(S(h_{(2)}), h_{(5)}) \rtimes S(h_{(1)})h_{(6)} = \epsilon(h) \rtimes 1.$$

Furthermore, by Lemma 3.3(2),

$$\begin{aligned}
(VV^*)(h) &= (S(h^*_{(1)}) \cdot \sigma(h^*_{(4)}, S(h_{(5)})))^*\sigma(h_{(2)}, S(h_{(3)})) \rtimes 1 \\
&= \sigma(S(h^*_{(1)})h^*_{(6)}, S(h^*_{(7)}))^*\sigma(S(h^*_{(2)}), h^*_{(5)})^*\sigma(h_{(3)}, S(h_{(4)})) \rtimes 1 \\
&= \epsilon(h) \rtimes 1.
\end{aligned}$$

Hence V is a unitary element in $\mathrm{Hom}(H, A \rtimes_\sigma H)$. Also, since $\sigma^*\sigma = \epsilon \otimes \epsilon$, by Lemma 3.3(1)

$$\begin{aligned}
&V(h_{(1)})(x \rtimes 1)V^*(h_{(2)}) \\
&\quad = (h_{(1)} \cdot x)[S(h^*_{(2)}) \cdot \sigma(h^*_{(5)}, S(h^*_{(6)}))]^* \sigma(h_{(3)}, S(h_{(4)})) \rtimes 1 \\
&\quad = (h_{(1)} \cdot x)[\sigma(S(h^*_{(3)}), h_{(6)})\sigma(S(h^*_{(2)})h^*_{(7)}, S(h^*_{(8)}))]^* \sigma(h_{(4)}, S(h_{(5)})) \rtimes 1 \\
&\quad = (h \cdot x) \rtimes 1.
\end{aligned}$$

Furthermore, since $\sigma^*\sigma = \epsilon \otimes \epsilon$, by Lemma 3.3(2)

$$\begin{aligned}
&V(h_{(1)})V(l_{(1)})V^*(h_{(2)}l_{(2)}) \\
&\quad = \sigma(h_{(1)}, l_{(1)})[\sigma(S(h_{(3)}l_{(3)})^*, (h_{(6)}l_{(6)})^*) \\
&\qquad \times \sigma(S(h_{(2)}l_{(2)})^*(h_{(7)}l_{(7)})^*, S(h_{(8)}l_{(8)})^*)]^* \sigma(h_{(4)}l_{(4)}, S(h_{(5)}l_{(5)})) \rtimes 1 \\
&\quad = \sigma(h_{(1)}, l_{(1)})\sigma(S(h_{(2)}l_{(2)})^*(h_{(5)}l_{(5)})^*, S(h_{(6)}l_{(6)})^*)^* \\
&\qquad \times (\sigma^*\sigma)(h_{(3)}l_{(3)}, S(h_{(4)}l_{(4)})) \rtimes 1 \\
&\quad = \sigma(h, l) \rtimes 1.
\end{aligned}$$

This lemma means that the action of H on A is inner in $A \rtimes_\sigma H$. Using the above lemmas we shall show the following proposition:

PROPOSITION 3.13. *The map θ is an injective representation of $A \rtimes_\sigma H$ to* B.

PROOF. It is immediate by Lemmas 3.9, 3.10 and 3.11 that θ is a representation of $A \rtimes_\sigma H$ to B. We have only to show that θ is injective. Let $\{w^k_{ij}\}$ be a system of comatrix units of H and let $x = \sum_{i,j,k} x_{ijk} \rtimes w^k_{ij}$, where $x_{ijk} \in A$. We suppose that $\theta(x) = 0$. Then for any $\xi \in \mathscr{H}$,

$$0 = \theta(x)(\xi \otimes 1) = \sum_{i,j,k,t_1,t_2,t_3} [S(w^k_{t_2t_3}) \cdot x_{ijk}]\sigma(S(w^k_{t_1t_2}), w^k_{it_1})\xi \otimes w^k_{t_3j}$$

by [14, Theorem 2.2,2]. Since $\{w^k_{ij}\}$ is a basis of H and A acts on $\mathscr{H}$ faithfully and non-degenerately, $\sum_{i,t_1,t_2}[S(w^k_{t_2t_3}) \cdot x_{ijk}]\sigma(S(w^k_{t_1t_2}), w^k_{it_1}) = 0$ for k, t_3, j. Thus $\sum_{i,t_1,t_2}[S(w^k_{t_2s}) \cdot x_{ijk}]\sigma(S(w^k_{t_1t_2}), w^k_{it_1}) \rtimes 1 = 0$ for any j, k, s. By Lemma 3.12

$$\begin{aligned}
&0 \\
&= \sum_{i,t_1,t_2,t_3,t_4} V(S(w^k_{t_4s}))(x_{ijk} \rtimes 1)V^*(S(w^k_{t_3t_4}))V(S(w^k_{t_2t_3}))V(w^k_{it_1}) \times V^*(\epsilon(w^k_{t_1t_2})) \\
&= \sum_{i,t_1} V(S(w^k_{t_1s}))(x_{ijk} \rtimes 1)V(w^k_{it_1}).
\end{aligned}$$

Since s is arbitrary and $V^*V = \epsilon$, for any r

$$0 = \sum_{i,t_1,s} V^*(S(w^k_{sr}))V(S(w^k_{t_1 s}))(x_{ijk} \rtimes 1)V(w^k_{it_1}) = \sum_i (x_{ijk} \rtimes 1)V(w^k_{ir}).$$

Since r is arbitrary and $VV^* = \epsilon$, for any p

$$0 = \sum_{i,r}(x_{ijk} \rtimes 1)V(w^k_{ir})V^*(w^k_{rp}) = \sum_i (x_{ijk} \rtimes 1)\epsilon(w^k_{ip}).$$

Since $\epsilon(w^k_{ip}) = \delta_{ip}$ by [14, Theorem 2.2,2], $x_{pjk} \rtimes 1 = 0$ for any p, j, k, where δ_{ip} is the Kronecker delta. Hence θ is injective.

Let F be a linear map from $A \rtimes_\sigma H$ to A defined by $F(x \rtimes h) = \tau(h)x$ for any $x \in A$ and $h \in H$. In the same way as in [14, Proposition 2.8], we can see that F is a conditional expectation from $A \rtimes_\sigma H$ onto A.

LEMMA 3.14. *The conditional expectation F is faithful.*

PROOF. We show this lemma in the same way as in [14, Proposition 2.8]. Let $\{h_j\}$ be a basis of H such that $\tau(h_i h_j^*) = \delta_{ij}$, where δ_{ij} is the Kronecker delta. Let $x = \sum_i x_i \rtimes h_i$ in $A \rtimes_\sigma H$, where $x_i \in A$ for any i. We suppose that $F(xx^*) = 0$. Then by Definition 3.1(4) and [14, Theorem 2.2,1]

$$\begin{aligned}
0 &= \sum_{i,j} F((x_i \rtimes h_i)(x_j \rtimes h_j)^*) \\
&= \sum_{i,j,(h_i),(h_j)} x_i \tau(h_{i(4)} h^*_{j(5)})[h_{i(1)} \cdot \sigma(S(h_{j(2)}), h_{j(1)})^*] \\
&\qquad \times [h_{i(2)} \cdot (h^*_{j(3)} \cdot x_j)]\sigma(h_{i(3)}, h_{j(4)}) \\
&= \sum_{i,j,(h_i),(h_j)} x_i \tau(h_{i(4)} h^*_{j(5)})[S(h^*_{i(1)}) \cdot \sigma(S(h_{j(2)}), h_{j(1)})]^* \sigma(h_{i(2)}, h^*_{j(3)}) \\
&\qquad \times (h_{i(3)} h^*_{j(4)} \cdot x_j) \\
&= \sum_{i,j,(h_i),(h_j)} x_i \tau(h_{i(3)} h^*_{j(4)})[S(h^*_{i(1)}) \cdot \sigma(S(h_{j(2)}), h_{j(1)})]^* \sigma(h_{i(2)}, h^*_{j(3)}) x_j^*.
\end{aligned}$$

Since $\sigma^*\sigma = \epsilon \otimes \epsilon$, by Lemma 3.3(1) $0 = \sum_{i,j} \tau(h_i h_j^*) x_i x_j^* = \sum_i x_i x_i^*$. Hence $x_i = 0$ for any i. Thus F is faithful.

PROPOSITION 3.15. *The unital $*$-algebra $\theta(A \rtimes_\sigma H)$ is closed in* B.

PROOF. We note that for any $x \in A \rtimes_\sigma H$, $\|F(x)\| \leq \|\theta(x)\|$. Indeed, we can write $x = \sum_i x_i \rtimes h_i$, where $x_i \in A$, $h_i \in H$ for any i. Since $\mathrm{id} \otimes \tau$ can be regarded as a contractive linear map from $\mathscr{H} \otimes l^2(\tau, H)$ to $\mathscr{H}$, for any $\xi \in \mathscr{H}$,

$$\begin{aligned}\|\theta(x)(\xi \otimes 1)\| &= \left\| \sum_{i,(h_i)} [S(h_{i(3)}) \cdot x_i] \sigma(S(h_{i(2)}), h_{i(1)}) \xi \otimes h_{i(4)} \right\| \\ &\geq \left\| \sum_{i,(h_i)} [S(h_{i(3)}) \cdot x_i] \sigma(S(h_{i(2)}), h_{i(1)}) \xi \otimes \tau(h_{i(4)}) \right\| \\ &= \left\| \sum_i (\tau(h_i) x_i \xi) \otimes 1 \right\| = \|F(x)\xi\|\end{aligned}$$

by [14, Theorem 2.2]. Hence $\|F(x)\| \leq \|\theta(x)\|$ for any $x \in A \rtimes_\sigma H$. Thus we can obtain this proposition in the same way as in [14, Proposition 2.15].

By Proposition 3.15, we can regard $A \rtimes_\sigma H$ as a C^*-subalgebra of B and $A \rtimes_\sigma H$ is independent of the choice of a Hilbert space $\mathscr{H}$. We call it *the twisted crossed product* of a unital C^*-algebra A by a weak action H on A and a unitary cocycle σ. Following [14, Definition 2.7], we define the dual action of H^0 on $A \rtimes_\sigma H$.

Definition 3.16. There is the dual action of H^0 on $A \rtimes_\sigma H$ defined by

$$\phi \cdot (x \rtimes h) = x \rtimes (\phi \rightharpoonup h)$$

for $x \in A$, $h \in H$, $\phi \in H^0$, where $\rightharpoonup$ is the Sweedler's arrow which is the action of H^0 on H defined in [14, Example 2.5].

It is necessary to check that the above is an action. But we can easily do it. Also, we have the following lemma:

Lemma 3.17. *The following statements hold.*

(1) $F(x \rtimes h) = \tau \cdot (x \rtimes h)$ *for any* $x \in A$ *and* $h \in H$,

(2) $A = (A \rtimes_\sigma H)^{H^0}$, *where* $(A \rtimes_\sigma H)^{H^0}$ *is the fixed point* C^*-*subalgebra of* $A \rtimes_\sigma H$ *for the action of* H^0 *on* $A \rtimes_\sigma H$.

Proof. This is immediate by routine computations.

By Lemma 3.17(1), we can see that $1 \rtimes \tau$ is the Jones projection induced by F.

Proposition 3.18. *Let* $\{w_{ij}^k\}$ *be a system of comatrix units of* H. *Then*

$$\{((\sqrt{d_k} \rtimes w_{ij}^k)^*, \sqrt{d_k} \rtimes w_{ij}^k)\}_{i,j,k}$$

is a quasi-basis for F *and* $\mathrm{Index}(F) = \dim(H)$.

PROOF. For any $x \in A \rtimes_\sigma H$, we can write that $x = \sum_{i,j,k} x_{ijk} \rtimes w_{ij}^k$, where $x_{ijk} \in A$ for any i, j, k. Since F is an A-A-bimodule map, in order to prove the first statement, we have only to show that for any i_0, j_0, k_0,

$$\sum_{i,j,k} F((1 \rtimes w_{i_0 j_0}^{k_0})(1 \rtimes w_{ij}^k)^*)(1 \rtimes w_{ij}^k) = \frac{1}{d_{k_0}} \rtimes w_{i_0 j_0}^{k_0}.$$

By routine computations and [14, Theorem 2.2,2],

$$\begin{aligned}\sum_{i,j,k} & F((1 \rtimes w_{i_0 j_0}^{k_0})(1 \rtimes w_{ij}^k)^*)(1 \rtimes w_{ij}^k) \\ &= \sum_{i,t_1,t_2,s_1,s_2} \frac{1}{d_{k_0}} [S(w_{i_0 s_1}^{k_0})^* \cdot \sigma(w_{t_2 t_1}^{k_0 *}, S(w_{t_1 i}^{k_0})^*)]^* \sigma(w_{s_1 s_2}^{k_0}, S(w_{s_2 t_2}^{k_0})) \rtimes w_{i j_0}^{k_0}.\end{aligned}$$

We change the sufixes as follows: We change t_2, t_1 and i to s_3, s_4 and s_5, respectively. Then since $\sigma^* \sigma = \epsilon \otimes \epsilon$, by Lemma 3.3(2) and routine computations,

$$\begin{aligned}\sum_{i,j,k} & F((1 \rtimes w_{i_0 j_0}^{k_0})(1 \rtimes w_{ij}^k)^*)(1 \rtimes w_{ij}^k) \\ &= \sum_{s_1,\ldots,s_5} \frac{1}{d_{k_0}} [S(w_{i_0 s_1}^{k_0})^* \cdot \sigma(w_{s_3 s_4}^{k_0 *}, S(w_{s_4 s_5}^{k_0})^*)]^* \sigma(w_{s_1 s_2}^{k_0}, S(w_{s_2 s_3}^{k_0})) \rtimes w_{s_5 j_0}^{k_0} \\ &= \sum_{s_1,\ldots,s_7} \frac{1}{d_{k_0}} \sigma(S(w_{i_0 s_1}^{k_0})^* w_{s_5 s_6}^{k_0 *}, S(w_{s_6 s_7}^{k_0 *}))^* \sigma(S(w_{s_1 s_2}^{k_0})^*, w_{s_4 s_5}^{k_0 *})^* \\ &\qquad \times \sigma(w_{s_2 s_3}^{k_0}, S(w_{s_3 s_4}^{k_0})) \rtimes w_{s_7 j_0}^{k_0} \\ &= \frac{1}{d_{k_0}} \rtimes w_{i_0 j_0}^{k_0}.\end{aligned}$$

Furthermore, since $1 \rtimes w_{ij}^k = V(w_{ij}^k)$ for any i, j, k, by [14, Theorem 2.2] and Lemma 3.12,

$$\begin{aligned}\mathrm{Index}(F) &= \sum_{i,j,k} d_k V(w_{ij}^k)^* V(w_{ij}^k) = \sum_{ij,k} d_k V^*(w_{ji}^k) V(w_{ij}^k) = \sum_{j,k} d_k \epsilon(w_{jj}^k) \\ &= \dim(H).\end{aligned}$$

We denote by B the twisted crossed product $A \rtimes_\sigma H$. Then we can define the dual action of H^0 on B in the same way as in Definition 3.16. Also, we can define a coaction ρ of H^0 on $B \rtimes H^0$ by $\rho(x \rtimes \phi) = (x \rtimes \phi_{(1)}) \otimes \phi_{(2)}$ for any $x \in B$ and $\phi \in H^0$. We can easily check that ρ is a coaction of H^0 on $B \rtimes H^0$.

PROPOSITION 3.19. *We have that* $\rho(1_B \rtimes \tau) \sim (1_B \rtimes \tau) \otimes 1^0$ *in* $\big(B \rtimes H^0\big) \otimes H^0$*, where* 1_B *and* 1^0 *are the unit elements in* B *and* H^0*, respectively.*

PROOF. Let V be a unitary element in $\mathrm{Hom}(H, B)$ defined by $V(h) = 1 \rtimes h$ for any $h \in H$. We regard it as an element in $\mathrm{Hom}(H, B \rtimes H^0)$. Also, there is an isomorphism $\imath$ of $(B \rtimes H^0) \otimes H^0$ onto $\mathrm{Hom}(H, B \rtimes H^0)$ defined after Definition 2.2. Hence we regard $\rho(1 \rtimes \tau)$ and $(1 \rtimes \tau) \otimes 1^0$ as elements in $\mathrm{Hom}(H, B \rtimes H^0)$. We denote them by $\rho(1 \rtimes \tau)^\wedge$ and $((1 \rtimes \tau) \otimes 1^0)^\wedge$, respectively. Then by direct computations, for any $h \in H$,

$$\begin{aligned}(((1 \rtimes \tau) \otimes 1^0)^\wedge V)(h) &= (1 \rtimes \tau)\epsilon(h_{(1)})(V(h_{(2)}) \rtimes 1^0) \\ &= (1 \rtimes h_{(1)}) \rtimes \tau_{(1)}(h_{(2)})\tau_{(2)}.\end{aligned}$$

On the other hand,

$$\begin{aligned}(V\rho(1 \rtimes \tau)^\wedge)(h) &= (V(h_{(1)}) \rtimes 1^0)(1 \rtimes \tau_{(2)}(h_{(2)})\tau_{(1)}) \\ &= (1 \rtimes h_{(1)}) \rtimes \tau_{(2)}(h_{(2)})\tau_{(1)}.\end{aligned}$$

Furthermore, for any $h, l \in H$,

$$\begin{aligned}(\tau_{(2)}\tau_{(1)}(h))(l) &= \tau_{(1)}(h)\tau_{(2)}(l) = \tau(hl) = \tau(lh) \\ &= \tau_{(1)}(l)\tau_{(2)}(h) = (\tau_{(2)}(h)\tau_{(1)})(l).\end{aligned}$$

Since H^0 can be identified with $1_B \rtimes H^0$, a C^*-subalgebra of $B \rtimes H^0$,

$$V\rho(1 \rtimes \tau)^\wedge = ((1 \rtimes \tau) \otimes 1^0)^\wedge V.$$

Since V is a unitary element in $\mathrm{Hom}(H, B)$ by Lemma 3.12(i), so is V in $\mathrm{Hom}(H, B \rtimes H^0)$. Therefore, $\rho(1 \rtimes \tau) \sim (1 \rtimes \tau) \otimes 1^0$ in $(B \rtimes H^0) \otimes H^0$.

4. Construction of a unitary element and its properties

As mentioned in Introduction, let B be a unital C^*-algebra and H a finite dimensional C^*-Hopf algebra. We suppose that there is a saturated action of H on B defined in [14]. Let A be its fixed point C^*-subalgebra of B and E the canonical conditional expectation from B onto A. Let $B \rtimes H$ be the crossed product of B by the action of H on B and we denote it by B_1. In the same way as in Section 3, we can define a coaction ρ of H on B_1 by for any $x \in B$ and $h \in H$

$$\rho(x \rtimes h) = (x \rtimes h_{(1)}) \otimes h_{(2)}.$$

We suppose that $\rho(1 \rtimes e) \sim (1 \rtimes e) \otimes 1$ in $B_1 \otimes H$, where e is the distinguished projection in H. We note that $1 \rtimes e$ is the Jones projection induced by E. We shall show that the above condition is a necessary and sufficient one we stated in Introduction. Since $\rho(1 \rtimes e) \sim (1 \rtimes e) \otimes 1$ in $B_1 \otimes H$, there is a partial isometry $w \in B_1 \otimes H$ such that $w^* w = \rho(1 \rtimes e)$, $ww^* = (1 \rtimes e) \otimes 1$. Let $\{v_{ij}^k\}_{i,j,k}$ be a system of matrix units of H. We write $w = \sum_{i,j,k} x_{ij}^k \otimes v_{ij}^k$, where $x_{ij}^k \in B_1$ for any i, j, k. Moreover, we write $x_{ij}^k = \sum_{i_1,j_1,k_1} (b_{ij}^k)_{i_1 j_1}^{k_1} \rtimes v_{i_1 j_1}^{k_1}$ for any i, j, k, where $(b_{ij}^k)_{i_1 j_1}^{k_1} \in B$ for any i, j, k, i_1, j_1, k_1. Then $((1 \rtimes e) \otimes 1)w = \sum_{i,j,k} (1 \rtimes e) x_{ij}^k \otimes v_{ij}^k$. Also, since $\{v_{ij}^k\}$ is a system of matrix units of H, by Equation (5) in [14]

$$(1 \rtimes e) x_{ij}^k = \sum_{i_1,j_1,k_1,i_2} \frac{1}{d_{k_1}} S(v_{i_1 i_2}^{k_1}) \cdot (b_{ij}^k)_{i_1 j_1}^{k_1} \rtimes v_{i_2 j_1}^{k_1}.$$

Let $u_{ij}^k = \sum_{i_1,j_1,k_1} S(v_{i_1 j_1}^{k_1}) \cdot (b_{ij}^k)_{i_1 j_1}^{k_1} \in B$ for any i, j, k. Then by routine calculations, for any i, j, k,

$$(1 \rtimes e)(u_{ij}^k \rtimes 1) = \sum_{i_1,j_1,k_1,i_2} \frac{1}{d_{k_1}} S(v_{j_1 i_2}^{k_1}) S(v_{i_1 j_1}^{k_1}) \cdot (b_{ij}^k)_{i_1 j_1}^{k_1} \rtimes v_{i_2 j_1}^{k_1} = (1 \rtimes e) x_{ij}^k.$$

Thus

$$\begin{aligned} ((1 \rtimes e) \otimes 1)w &= \sum_{i,j,k} (1 \rtimes e)(u_{ij}^k \rtimes 1) \otimes v_{ij}^k \\ &= ((1 \rtimes e) \otimes 1)\Bigl(\sum_{i,j,k} (u_{ij}^k \rtimes 1) \otimes v_{ij}^k\Bigr). \end{aligned}$$

Let $u = \sum_{i,j,k} u_{ij}^k \otimes v_{ij}^k \in B \otimes H$. Since we identify B with $B \rtimes 1$, a C^*-subalgebra of B_1, we identify $B \otimes H$ with $(B \rtimes 1) \otimes H$. If we do so, $u = \sum_{i,j,k} (u_{ij}^k \rtimes 1) \otimes v_{ij}^k$. Hence by the above equation, we obtain that

$$\begin{aligned} ((1 \rtimes e) \otimes 1)w &= ((1 \rtimes e) \otimes 1)u, \\ u^*((1 \rtimes e) \otimes 1)u &= w^*((1 \rtimes e) \otimes 1)w = \rho(1 \rtimes e). \end{aligned}$$

On the other hand, $u^*((1 \rtimes e) \otimes 1)u = \sum_{i,j,k,j_1} (u_{ij}^{k*} \rtimes e)(u_{ij_1}^k \rtimes 1) \otimes v_{jj_1}^k$. Since $\rho(1 \rtimes e) = \sum_{i,j,k} \frac{1}{d_k}((1 \rtimes S(v_{ji}^k)) \otimes v_{ij}^k)$, we can see that for any j, j_1, k,

$$\frac{1}{d_k}(1 \rtimes S(v_{j_1 j}^k)) = \sum_i (u_{ij}^{k*} \rtimes e)(u_{ij_1}^k \rtimes 1). \tag{4.1}$$

Now, we shall show that u is a unitary element in $B \otimes H$. By [14, Proposition 2.8], we can define the canonical faithful conditional expectation E_1 from B_1 onto B by $E_1(x \rtimes h) = \tau(h)x$ for any $x \in B$ and $h \in H$, where we identify B with $B \rtimes 1$ and τ is the Haar trace on H.

LEMMA 4.1. *We have that* $u^*u = 1 \otimes 1$ *in* $B \otimes H$.

PROOF. Since we regard $B \otimes H$ as a C^*-subalgebra of $B_1 \otimes H$, we shall show that $u^*u = (1 \rtimes 1) \otimes 1$. By Equation (4.1) and the definition of E_1, for any j, j_1, k,

$$\sum_i u_{ij}^{k*} u_{ij_1}^k = \frac{1}{\tau(e)d_k} \tau(S(v_{j_1 j}^k)).$$

Using the the above equation,

$$\begin{aligned} u^*u &= \sum_{j,k,j_1} \frac{1}{\tau(e)d_k} (\tau(S(v_{j_1 j}^k)) \rtimes 1) \otimes v_{jj_1}^k \\ &= (1 \rtimes 1) \otimes \sum_{j,k,j_1} \frac{1}{\tau(e)d_k} \tau(S(v_{j_1 j}^k)) v_{jj_1}^k \\ &= (1 \rtimes 1) \otimes \frac{1}{\tau(e)} (\tau \otimes \mathrm{id}) \left(\sum_{j,k,j_1} \frac{1}{d_k} S(v_{j_1 j}^k) \otimes v_{jj_1}^k \right) \\ &= (1 \rtimes 1) \otimes \frac{1}{\tau(e)} (\tau \otimes \mathrm{id})(\Delta(e)) = (1 \rtimes 1) \otimes 1. \end{aligned}$$

Therefore we obtain the conclusion.

PROPOSITION 4.2. *The element* u *is a unitary one in* $B \otimes H$.

PROOF. By Lemma 4.1, it suffices to show that $uu^* = 1 \otimes 1$. First,

$$((1 \rtimes e) \otimes 1)uu^*((1 \rtimes e) \otimes 1) = (1 \rtimes e) \otimes 1 = \sum_{i,k} (1 \rtimes e) \otimes v_{ii}^k.$$

On the other hand,

$$\begin{aligned} ((1 \rtimes e) \otimes 1)uu^*((1 \rtimes e) \otimes 1) &= \sum_{i,j,k,i_1} (1 \rtimes e)(u_{ij}^k u_{i_1 j}^{k*} \rtimes 1)(1 \rtimes e) \otimes v_{ii_1}^k \\ &= \sum_{i,j,k,i_1} (e \cdot (u_{ij}^k u_{i_1 j}^{k*}) \rtimes e) \otimes v_{ii_1}^k \\ &= \sum_{i,j,k,i_1} (E(u_{ij}^k u_{i_1 j}^{k*}) \rtimes e) \otimes v_{ii_1}^k \end{aligned}$$

by [14, Proposition 2.12]. Thus

$$\sum_j E(u_{ij}^k u_{i_1 j}^{k*}) \rtimes e = \begin{cases} 0 & \text{if } i \neq i_1 \\ 1 \rtimes e & \text{if } i = i_1 \end{cases}$$

for any k. Hence using E_1,

$$\sum_j E(u_{ij}^k u_{i_1 j}^{k*}) = \begin{cases} 0 & \text{if } i \neq i_1 \\ 1 & \text{if } i = i_1 \end{cases}$$

for any k. It follows by the above equation that

$$(E \otimes \mathrm{id})(uu^*) = \sum_{i,j,k,i_1} E(u_{ij}^k u_{i_1 j}^{k*}) \otimes v_{ii_1}^k = \sum_{i,k} 1 \otimes v_{ii}^k = 1 \otimes 1.$$

Since $E \otimes \mathrm{id}$ is faithful, $uu^* = 1 \otimes 1$.

Also, we have the following proposition:

PROPOSITION 4.3. *The set* $\left\{\left(\sqrt{d_k} u_{ij}^{k*}, \sqrt{d_k} u_{ij}^k\right)\right\}_{i,j,k}$ *is a quasi-basis for* E.

PROOF. We note that if $j = j_1$ in Equation (4.1), we obtain that for any j, k,

$$\frac{1}{d_k}(1 \rtimes S(v_{jj}^k)) = \sum_i (u_{ij}^{k*} \rtimes e)(u_{ij}^k \rtimes 1).$$

Since $1 \rtimes e$ is the Jones projection in B_1 induced by E, for any $x \in B$,

$$\begin{aligned} (1 \rtimes e)\left\{\sum_{i,j,k} d_k E(xu_{ij}^{k*})u_{ij}^k \rtimes 1\right\} &= \sum_{i,j,k} d_k (1 \rtimes e)(E(xu_{ij}^{k*})u_{ij}^k \rtimes 1) \\ &= \sum_{i,j,k} d_k (1 \rtimes e)(xu_{ij}^{k*} \rtimes 1)(1 \rtimes e)(u_{ij}^k \rtimes 1) \\ &= \sum_{i,j,k} d_k (1 \rtimes e)(x \rtimes 1)(u_{ij}^{k*} \rtimes e)(u_{ij}^k \rtimes 1) \\ &= \sum_{j,k} (1 \rtimes e)(x \rtimes 1)(1 \rtimes S(v_{jj}^k)) \\ &= (1 \rtimes e)(x \rtimes 1). \end{aligned}$$

Therefore we obtain the conclusion.

Moreover, we have the following:

LEMMA 4.4. *For any* $h \in H$, $\sum_{i,j,k}(h \cdot u_{ij}^k) \otimes v_{ij}^k = \sum_{i,j,k} u_{ij}^k \otimes v_{ij}^k h.$

PROOF. Let $\{w_{ij}^k\}$ be a system of comatrix units of H. We note that

$$((1 \rtimes e) \otimes 1)u = u\rho(1 \rtimes e).$$

Then by [14, Theorem 2.2],

$$((1 \rtimes e) \otimes 1)u = \sum_{i,j,k,i_1,j_1,k_1} \frac{d_{k_1}}{\dim(H)}((w^{k_1}_{i_1 j_1} \cdot u^k_{ij}) \rtimes w^{k_1}_{j_1 i_1}) \otimes v^k_{ij}.$$

Also,

$$u\rho(1 \rtimes e) = \sum_{i,j,k,i_1,j_1,k_1} \frac{d_{k_1}}{\dim(H)}((u^k_{ij} \rtimes w^{k_1}_{i_1 j_1}) \otimes v^k_{ij} w^{k_1}_{j_1 i_1}).$$

Thus since $\{w^k_{ij}\}$ is a basis of H, we obtain that for any i_1, j_1, k_1,

$$\sum_{i,j,k}(w^{k_1}_{j_1 i_1} \cdot u^k_{ij}) \otimes v^k_{ij} = \sum_{i,j,k} u^k_{ij} \otimes v^k_{ij} w^{k_1}_{j_1 i_1}.$$

Therefore we obtain the conclusion.

In the rest of this section, we are devoted to the properties of u.

LEMMA 4.5. *Let $x \in (B \rtimes 1) \otimes H$. If $x((1 \rtimes e) \otimes 1) = ((1 \rtimes e) \otimes 1)x$, then $x \in (A \rtimes 1) \otimes H$.*

PROOF. This is immediate by routine computations.

LEMMA 4.6. *For any $a \in A$,*

$$u((a \rtimes 1) \otimes 1)u^*((1 \rtimes e) \otimes 1) = ((1 \rtimes e) \otimes 1)u((a \rtimes 1) \otimes 1)u^*.$$

PROOF. Since u is a unitary element in $(B \rtimes 1) \otimes H$ by Proposition 4.2, we have only to show that for any $a \in A$,

$$((a \rtimes 1) \otimes 1)u^*((1 \rtimes e) \otimes 1)u = u^*((1 \rtimes e) \otimes 1)u((a \rtimes 1) \otimes 1).$$

Since $u^*((1 \rtimes e) \otimes 1)u = \rho(1 \rtimes e)$, for any $a \in A$,

$$\begin{aligned} ((a \rtimes 1) \otimes 1)u^*((1 \rtimes e) \otimes 1)u &= ((a \rtimes 1) \otimes 1)\rho(1 \rtimes e) \\ &= \rho((a \rtimes 1)(1 \rtimes e)) \\ &= \rho((1 \rtimes e)(a \rtimes 1)) \\ &= \rho(1 \rtimes e)((a \rtimes 1) \otimes 1) \\ &= u^*((1 \rtimes e) \otimes 1)u((a \rtimes 1) \otimes 1). \end{aligned}$$

PROPOSITION 4.7. *For any $a \in A$, $u(a \otimes 1)u^* \in A \otimes H$.*

PROOF. This is immediate by Lemmas 4.5 and 4.6.

Let $z = \sum_{i,j,k} \epsilon(v_{ij}^k) u_{ij}^k \in B$.

LEMMA 4.8. *The element z is a unitary one in A.*

PROOF. We note that $\mathrm{id} \otimes \epsilon$ is a homomorphihsm of $B \otimes H$ onto B. Since u is a unitary element in $B \otimes H$, so is $z = (\mathrm{id} \otimes \epsilon)(u)$ in B. Also, we have that $((1 \rtimes e) \otimes 1)u = u\rho(1 \rtimes e)$. Since $(\mathrm{id} \otimes \epsilon) \circ \rho = \mathrm{id}$,

$$(1 \rtimes e)z = (\mathrm{id} \otimes \epsilon)(((1 \rtimes e) \otimes 1)u) = (\mathrm{id} \otimes \epsilon)(u\rho(1 \rtimes e)) = z(1 \rtimes e).$$

Thus z is in A.

REMARK 4.9. For any unitary element $a \in A$, we can see that

$$\{(\sqrt{d_k} a u_{ij}^{k*}, \sqrt{d_k} u_{ij}^k a^*)\}_{i,j,k}$$

is a quasi-basis for E by the above proposition and easy computations. Especially $\{(\sqrt{d_k} z u_{ij}^{k*}, \sqrt{d_k} u_{ij}^k z^*)\}_{i,j,k}$ is a quasi-basis for E.

Let $U = u(z^* \otimes 1)$ which is used in the next section. Clearly U is a unitary element in $B \otimes H$.

5. A weak action of the dual C^*-Hopf algebra and a unitary cocycle

As mentioned in Section 2, there is an isomorphism $\imath$ of $B \otimes H$ onto the unital convolution *-algebra $\mathrm{Hom}(H^0, B)$ defined by for any $x \in B$, $h \in H$ and $\phi \in H^0$,

$$\imath(x \otimes h)(\phi) = \phi(h)x.$$

For any $x \in B \otimes H$, we denote by $x^\wedge$ an element $\imath(x) \in \mathrm{Hom}(H^0, B)$. We constructed a unitary element $U \in B \otimes H$ in the previous section. Then $U^\wedge$ is a unitary element in $\mathrm{Hom}(H^0, B)$. Let $\{\phi_{mn}^r\}$ be the dual basis of H^0 corresponding to a system of matrix units $\{v_{ij}^k\}$ of H. Then it is a system of comatrix units of H^0. By [14, Theorem 2.2], $\Delta^0(\phi_{mn}^r) = \sum_t \phi_{mt}^r \otimes \phi_{tn}^r$ for any m, n, r. Hence we can see that $U^\wedge(\phi_{mn}^r) = u_{mn}^r z^*$ and $U^{\wedge *}(\phi_{mn}^r) = z u_{nm}^{r*}$ for any m, n, r.

LEMMA 5.1. *We define $\phi \cdot x = U^\wedge(\phi_{(1)}) x U^{\wedge *}(\phi_{(2)})$ for any $x \in B$ and $\phi \in H^0$. Then $(\phi, x) \mapsto \phi \cdot x$ is a weak inner action of H^0 on B.*

PROOF. Since $U^\wedge(1^0) = 1$, by [1, Lemma 1.4], it suffices to show that $(\phi \cdot x)^* = S^0(\phi^*) \cdot x^*$ for any $x \in B$, $\phi \in H^0$. Thus we have only to show that $(\phi_{mn}^r \cdot x)^* = S^0(\phi_{mn}^{r*}) \cdot x^*$ for any $x \in B$ and m, n, r. Indeed

$$(\phi_{mn}^r \cdot x)^* = \sum_t (u_{mt}^r z^* x z u_{nt}^{r*})^* = \phi_{nm}^r \cdot x^* = (S^0 \circ S^0)(\phi_{nm}^r) \cdot x^* = S^0(\phi_{mn}^{r*}) \cdot x^*.$$

LEMMA 5.2. *For any $a \in A$ and $\phi \in H^0$, $\phi \cdot a \in A$.*

PROOF. For any $a \in A$, $u(a \otimes 1)u^* = \sum_{i,j,k,i_1} u_{ij}^k a u_{i_1 j}^{k*} \otimes v_{ii_1}^k$. Thus by Proposition 4.7, $\sum_j u_{ij}^k a u_{i_1 j}^{k*} \in A$ for any $a \in A$ and i, i_1, k. Hence for any r, m, n and $a \in A$

$$\phi_{mn}^r \cdot a = \sum_t u_{mt}^r z^* a z u_{nt}^{r*} \in A.$$

Therefore we obtain the conclusion.

COROLLARY 5.3. *The map $H^0 \times A \longrightarrow A : (\phi, a) \mapsto \phi \cdot a$ is a weak action of H^0 on A, where $a \in A$ and $\phi \in H^0$.*

PROOF. This is immediate by Lemmas 5.1 and 5.2.

Following [1, Example 4.11], we shall construct a unitary cocycle of $H^0 \otimes H^0$ to B. Let σ be a bilinear map from $H^0 \otimes H^0$ to B defind by for any $\phi, \psi \in H^0$,

$$\sigma(\phi, \psi) = U^\wedge(\phi_{(1)}) U^\wedge(\psi_{(1)}) U^{\wedge *}(\phi_{(2)} \psi_{(2)}).$$

By [1, Example 4.11], σ satisfies Conditions (2), (3) and (4) of Definition 3.1 of a unitary cocyle for the weak inner action of H^0 on B.

LEMMA 5.4. *For any $\phi, \psi \in H^0$, $\sigma(\phi, \psi) \in A$.*

PROOF. For any $\phi \in H^0$,

$$(U^*((1 \rtimes e) \otimes 1))^\wedge(\phi) = U^{\wedge *}(\phi_{(1)}) \epsilon^0(\phi_{(2)})(1 \rtimes e) = U^{\wedge *}(\phi)(1 \rtimes e),$$

$$\begin{aligned}(\rho(1 \rtimes e) U^*)^\wedge(\phi) &= ((1 \rtimes e_{(1)}) \otimes e_{(2)})^\wedge(\phi_{(1)}) U^{\wedge *}(\phi_{(2)}) \\ &= (\mathrm{id} \otimes \phi_{(1)}) \rho(1 \rtimes e) U^{\wedge *}(\phi_{(2)}).\end{aligned}$$

Since $U^*((1 \rtimes e) \otimes 1) = \rho(1 \rtimes e) U^*$, we obtain that

$$U^{\wedge *}(\phi)(1 \rtimes e) = (\mathrm{id} \otimes \phi_{(1)}) \rho(1 \rtimes e) U^{\wedge *}(\phi_{(2)}). \tag{5.1}$$

Also, for any $\phi \in H^0$,

$$(U \rho(1 \rtimes e))^\wedge(\phi) = U^\wedge(\phi_{(1)})(1 \rtimes e_{(1)}) \phi_{(2)}(e_{(2)}),$$

$$(((1 \rtimes e) \otimes 1) U)^\wedge(\phi) = (1 \rtimes e) \epsilon^0(\phi_{(1)}) U^\wedge(\phi_{(2)}) = (1 \rtimes e) U^\wedge(\phi).$$

Since $((1 \rtimes e) \otimes 1) U = U \rho(1 \rtimes e)$, we obtain that

$$(1 \rtimes e) U^\wedge(\phi) = U^\wedge(\phi_{(1)})(1 \rtimes e_{(1)}) \phi_{(2)}(e_{(2)}). \tag{5.2}$$

Furthermore, by Equation (5.2)

$$\rho((1 \rtimes e)U^\wedge(\phi)) = \rho(U^\wedge(\phi_{(1)})(1 \rtimes e_{(1)})\phi_{(2)}(e_{(2)})).$$

Since $U^\wedge(\phi) \in B$ and $\rho(U^\wedge(\phi)) = U^\wedge(\phi) \otimes 1$, we see that

$$(5.3) \qquad (1 \rtimes e_{(1)})U^\wedge(\phi) \otimes e_{(2)} = U^\wedge(\phi_{(1)})(1 \rtimes e_{(1)})\phi_{(2)}(e_{(3)}) \otimes e_{(2)},$$

where we identify B with $B \rtimes 1$. Now, we shall show that any $\phi, \psi \in H^0$,

$$(1 \rtimes e)(\sigma(\phi, \psi) \rtimes 1) = (\sigma(\phi, \psi) \rtimes 1)(1 \rtimes e).$$

First, by Equation (5.2)

$$(1 \rtimes e)(\sigma(\phi, \psi) \rtimes 1) = U^\wedge(\phi_{(1)})(1 \rtimes e_{(1)})\phi_{(2)}(e_{(2)})U^\wedge(\psi_{(1)})U^{\wedge *}(\phi_{(3)}\psi_{(2)}).$$

Moreover, by Equation (5.3),

$$\begin{aligned}(1 \rtimes e)&(\sigma(\phi, \psi) \rtimes 1) \\ &= U^\wedge(\phi_{(1)})\phi_{(2)}(e_{(2)})U^\wedge(\psi_{(1)})(1 \rtimes e_{(1)})\psi_{(2)}(e_{(3)})U^{\wedge *}(\phi_{(3)}\psi_{(3)}).\end{aligned}$$

On the other hand, by Equation (5.1)

$$\begin{aligned}(\sigma(\phi, \psi) &\rtimes 1)(1 \rtimes e) \\ &= U^\wedge(\phi_{(1)})U^\wedge(\psi_{(1)})(\mathrm{id} \otimes \phi_{(2)}\psi_{(2)})\rho(1 \rtimes e)U^{\wedge *}(\phi_{(3)}\psi_{(3)}) \\ &= U^\wedge(\phi_{(1)})U^\wedge(\psi_{(1)})(1 \rtimes e_{(1)})\phi_{(2)}(e_{(2)})\psi_{(2)}(e_{(3)})U^{\wedge *}(\phi_{(3)}\psi_{(3)}).\end{aligned}$$

It follows that for any $\phi, \psi \in H^0$, $(1 \rtimes e)(\sigma(\phi, \psi) \rtimes 1) = (\sigma(\phi, \psi) \rtimes 1)(1 \rtimes e)$. Therefore we obtain the conclusion.

LEMMA 5.5. *The element σ is a unitary one in* $\mathrm{Hom}(H^0 \otimes H^0, A)$.

PROOF. By Lemma 5.4, it suffices to show that $\sigma^*\sigma = \sigma\sigma^* = \epsilon^0 \otimes \epsilon^0$. For any $\phi, \psi \in H^0$, we see that $(\sigma^*\sigma)(\phi, \psi) = (\sigma\sigma^*)(\phi, \psi) = \epsilon^0(\phi)\epsilon^0(\psi)$ by routine computations. Therefore we obtain the conclusion.

PROPOSITION 5.6. *The element σ is a unitary cocycle for the weak action of H^0 on A.*

PROOF. This is immediate by Lemma 5.5 and [1, Example 4.11].

6. A twisted crossed product induced by an inclusion of unital C^*-algebras of depth 2

In this section we suppose that there is a saturated action of a finite dimensional C^*-Hopf algebra H on a unital C^*-algebra B. Also, we suppose that A is the fixed point C^*-subalgebra of B for the action of H and that $\rho(1 \rtimes e) \sim (1 \rtimes e) \otimes 1$ in $B_1 \otimes H$, where $B_1 = B \rtimes H$. Furthermore, we suppose that ρ is the coaction of H on B_1 induced by the action of H on B. By the previous section, we can construct the weak action of the dual C^*-Hopf algebra H^0 on A and the unitary cocycle $\sigma \in \mathrm{Hom}(H^0 \otimes H^0, A)$ using the unitary element $U \in B \otimes H$ defined at the end of Section 4. By Section 3 we can construct the twisted crossed product $A \rtimes_\sigma H^0$ of A by the weak action of H^0. Also, we can define the dual action of H on $A \rtimes_\sigma H^0$. Let π be a map from $A \rtimes_\sigma H^0$ to B defined by $\pi(a \rtimes \phi) = aU^\wedge(\phi)$ for any $a \in A$ and $\phi \in H^0$. Then the following proposition holds:

PROPOSITION 6.1. *With the above notations, π is an epimorphism of $A \rtimes_\sigma H^0$ onto B satisfying that $h \cdot \pi(x) = \pi(h \cdot x)$ for any $x \in A \rtimes_\sigma H^0$ and $h \in H$.*

PROOF. Clearly π is a linear map from $A \rtimes_\sigma H^0$ to B. For any $a, b \in A$ and $\phi, \psi \in H^0$,

$$\begin{aligned}\pi((a \rtimes \phi)(b \rtimes \psi)) &= a(\phi_{(1)} \cdot b)\sigma(\phi_{(2)}, \psi_{(1)})U^\wedge(\phi_{(3)}\psi_{(2)}) \\ &= aU^\wedge(\phi_{(1)})b\epsilon^0(\phi_{(2)})U^\wedge(\psi_{(1)})\epsilon^0(\phi_{(3)}\psi_{(2)}) \\ &= aU^\wedge(\phi)bU^\wedge(\psi) = \pi(a \rtimes \phi)\pi(b \rtimes \psi).\end{aligned}$$

Also,

$$\begin{aligned}\pi((a \rtimes \phi)^*) &= \pi[\sigma(S(\phi_{(2)}), \phi_{(1)})^*(\phi^*_{(3)} \cdot a^*) \rtimes \phi^*_{(4)}] \\ &= U^{\wedge *}(\epsilon^0(\phi_{(2)}))^*U^\wedge(\phi_{(1)})^*U^{\wedge *}(\phi^*_{(4)})U^\wedge(\phi^*_{(5)})a^*\epsilon^0(\phi^*_{(6)}) \\ &= U^\wedge(\phi)^*a^* = \pi(a \rtimes \phi)^*.\end{aligned}$$

Thus π is a homomorphism of $A \rtimes_\sigma H^0$ to B. For any $x \in B$, we can write

$$x = \sum_{i,j,k} d_k E(xzu^{k*}_{ij})u^k_{ij}z^*$$

by Proposition 4.3 and Remark 4.9. Put $y = \sum_{i,j,k} d_k E(xzu^{k*}_{ij}) \rtimes \phi^k_{ij}$. Then $y \in A \rtimes_\sigma H^0$ and $\pi(y) = \sum_{i,j,k} d_k E(xzu^{k*}_{ij})U^\wedge(\phi^k_{ij}) = x$ since $U^\wedge(\phi^k_{ij}) = u^k_{ij}z^*$ for any i, j, k. Hence π is surjective. Furthermore, for any $a \in A$, $h \in H$, $\phi \in H^0$,

$$\begin{aligned}\pi(h \cdot (a \rtimes \phi)) = \phi_{(2)}(h)aU^\wedge(\phi_{(1)}) &= \sum_{i,j,k,(\phi)} \phi_{(1)}(v^k_{ij})\phi_{(2)}(h)au^k_{ij}z^* \\ &= \sum_{i,j,k} \phi(v^k_{ij}h)au^k_{ij}z^*\end{aligned}$$

since $U=\sum_{i,j,k}u_{ij}^k z^*\otimes v_{ij}^k$. On the other hand,

$$\begin{aligned}h\cdot\pi(a\rtimes\phi)&=\sum_{i,j,k}a(h\cdot u_{ij}^k\phi(v_{ij}^k)z^*)=\sum_{i,j,k}a(\mathrm{id}\otimes\phi)((h\cdot u_{ij}^k)\otimes v_{ij}^k)z^*\\&=\sum_{i,j,k}a(\mathrm{id}\otimes\phi)(u_{ij}^k\otimes v_{ij}^k h)z^*=\sum_{i,j,k}\phi(v_{ij}^k h)au_{ij}^k z^*\end{aligned}$$

by Lemma 4.4. Therefore $\pi(h\cdot(a\rtimes\phi))=h\cdot\pi(a\rtimes\phi)$ for any $a\in A$, $h\in H$, $\phi\in H^0$.

COROLLARY 6.2. *With the same notations as above, $F=E\circ\pi$, where F is the canonical conditional expectation from $A\rtimes_\sigma H^0$ onto A.*

PROOF. For any $a\in A$, $\pi(a\rtimes 1^0)=aU^\wedge(1^0)=a$. Hence for any $a\in A$, $\phi\in H^0$,

$$(E\circ\pi)(a\rtimes\phi)=e\cdot\pi(a\rtimes\phi)=\pi(e\cdot(a\rtimes\phi))=F(a\rtimes\phi)$$

by Proposition 6.1, where we identify A with $A\rtimes 1^0$.

PROPOSITION 6.3. *With the same notations as above, π is an isomorphism of $A\rtimes_\sigma H^0$ onto B.*

PROOF. By Proposition 6.1, we have only to show that π is injective. We suppose that $\pi(x)=0$ for an element $x\in A\rtimes_\sigma H^0$. Then $F(x^*x)=E(\pi(x^*)\pi(x))=0$ by Corollary 6.2. Since F is faithful by Lemma 3.14, $x=0$.

The following theorem is the main result:

THEOREM 6.4. *Let B be a unital C^*-algebra and H a finite dimensional C^*-Hopf algebra acting on B in the saturated fashion. Let A be the fixed point C^*-subalgebra of B for the action of H on B and E the canonical conditional expectation from B onto A. Let e be a minimal and central projection in H, which is called the distinguished projection and ρ the coaction of H on $B\rtimes H$, the crossed product of B by the action of H on B, which is induced by the action of H on B. Then the following are equivalent:*

(1) *We have that $\rho(1\rtimes e)\sim(1\rtimes e)\otimes 1$ in $(B\rtimes H)\otimes H$,*
(2) *There are a weak action of H^0 on A and a unitary cocycle σ of $H^0\otimes H^0$ to A satisfying that there is an isomorphism π of $A\rtimes_\sigma H^0$ onto B such that $F=E\circ\pi$,*

where H^0 is the dual C^*-Hopf algebra of H and F is the canonical conditional expectation from $A\rtimes_\sigma H^0$ onto A.

PROOF. This is immediate by Propositions 3.19, 6.3 and Corollary 6.2.

Let $A \subset B$ be an irreducible inclusion of unital C^*-algebras and E a condtional expectation from B onto A which is index-finite and of depth 2. Then in [2] Izumi pointed the following: There is a finite dimensional C^*-Hopf algebra H acting on B such that $A = B^H$ and $E(x) = e \cdot x$ for any $x \in B$. We note that the action of H on B is saturated by [14]. Let ρ be the coaction of H on $B \rtimes H$ defined in the same way as in Section 3. We call ρ the *coaction of H on $B \rtimes H$ induecd* by the inclusion $A \subset B$.

COROLLARY 6.5. *Let $A \subset B$ be an irreducible inclusion of unital C^*-algebras and E a condtional expectation from B onto A which is index-finite and of depth 2. Let H be a finite dimensional C^*-Hopf algebra acting on B in the saturated fashion such that the inclusion $A \subset B$ can be identified with the inclusion $B^H \subset B$. Let ρ be the coaction of H on $B \rtimes H$ induced by $A \subset B$. Furthermore, let e be the distinguished projection in H. Then the following are equivalent:*

(1) *We have that $\rho(1 \rtimes e) \sim (1 \rtimes e) \otimes 1$ in $(B \rtimes H) \otimes H$,*

(2) *There are a weak action of H^0 on A and a unitary cocycle σ of $H^0 \otimes H^0$ to A satisfying that there is an isomorphism π of $A \rtimes_\sigma H^0$ onto B such that $F = E \circ \pi$,*

where H^0 is the dual C^*-Hopf algebra of H and F is the canonical conditional expectation from $A \rtimes_\sigma H^0$ onto A.

PROOF. This is immediate by Theorem 6.4.

We shall give another application of Theorem 6.4. Let A be a unital C^*-algebra. We suppose that A has cancellation and the unique tracial state τ_A. Let τ_{A*} be the homomorphism of $K_0(A)$ to R induced by τ_A. Also, we suppose that τ_{A*} is injective. Irrational rotation C^*-algebras, UHF-algebras and AFD II_1-factors have the above properties.

LEMMA 6.6. *Let A be as above and let B be a unital C^*-algebra which is strongly Morita equivalent to A. Then B has the following properties:*

(1) *B has cancellation,*

(2) *B has the unique tracial state τ_B,*

(3) *Let τ_{B*} be the homomorphism of $K_0(B)$ to R. Then τ_{B*} is injective.*

PROOF. Since unital C^*-algebras A and B are strongly Morita equivalent, B is isomorphic to a full corner of some full matrix algebra over A by Rieffel [10, Proposition 2.1]. By this fact and [10, Proposition 2.2], we can obtain the conclusion.

COROLLARY 6.7. *With the same notations and assumptions as Theorem 6.4, we suppose that A has cancellation and the unique tracial state τ_A and that the homomorphism τ_{A*} of $K_0(A)$ to $\mathbf{R}$ induced by τ_A is injective. Then there are a weak action of H^0 on A and a unitary cocycle σ of $H^0 \otimes H^0$ to A satisfying that there is an isomorphism π of $A \rtimes_\sigma H^0$ onto B such that $F = E \circ \pi$, where H^0 is the dual C^*-Hopf algebra and F is the canonical conditional expectation from $A \rtimes_\sigma H^0$ onto A.*

PROOF. Let ρ be the coaction of H on $B_1 = B \rtimes H$ induced by the action of H on B and e the distinguished projection in H. Then by [14, Definition 4.2], B_1 is strongly Morita equivalent to A. Thus by Lemma 6.6(2), B_1 has the unique tracial state τ_{B_1}. Recall that $H \cong \oplus_{k=1}^N M_{d_k}(\mathbf{C})$ as C^*-algebras. We identify H with $\oplus_{k=1}^N M_{d_k}(\mathbf{C})$. For $k = 1, 2, \ldots, N$, let p_k be a minimal central projection in H and π_k a homomorphism of $B_1 \otimes H$ onto $B_1 \otimes M_{d_k}(\mathbf{C})$ defined by $\pi_k(x) = x((1 \rtimes 1) \otimes p_k)$ for any $x \in B_1 \otimes H$. Let Tr_k be the unique tracial state on $M_{d_k}(\mathbf{C})$ and let $\tau_k = \tau_{B_1} \otimes Tr_k$ for $k = 1, 2, \ldots, N$. Let τ_{k*} be the homomorphism of $K_0(B_1 \otimes M_{d_k}(\mathbf{C}))$ to $\mathbf{R}$ induced by τ_k for $k = 1, 2, \ldots, N$. Since $\tau_k \circ \pi_k \circ \rho$ is a tracial state on B_1, $\tau_{B_1} = \tau_k \circ \pi_k \circ \rho$. Thus for $k = 1, 2, \ldots, N$,

$$\begin{aligned}\tau_{k*}([\rho(1 \rtimes e)((1 \rtimes 1) \otimes p_k)]) &= (\tau_k \circ \pi_k \circ \rho)(1 \rtimes e) = \tau_{B_1}(1 \rtimes e) \\ &= \tau_k((1 \rtimes e) \otimes p_k) \\ &= \tau_{k*}([((1 \rtimes e) \otimes 1)((1 \rtimes 1) \otimes p_k)]).\end{aligned}$$

Since τ_{k*} is injective for $k = 1, 2, \ldots, N$, by Lemma 6.6(3) in $K_0(B_1 \otimes M_{d_k}(\mathbf{C}))$,

$$[\rho(1 \rtimes e)((1 \rtimes 1) \otimes p_k)] = [((1 \rtimes e) \otimes 1)((1 \rtimes 1) \otimes p_k)].$$

Since $B_1 \otimes M_{d_k}(\mathbf{C})$ has cancellation by Lemma 6.6(1), we have

$$\rho(1 \rtimes e)((1 \rtimes 1) \otimes p_k) \sim ((1 \rtimes e) \otimes 1)((1 \rtimes 1) \otimes p_k)$$

in $B_1 \otimes M_{d_k}(\mathbf{C})$ for $k = 1, 2, \ldots, N$. Hence $\rho(1 \rtimes e) \sim (1 \rtimes e) \otimes 1$ in $B_1 \otimes H$. Therefore we obtain the conclusion by Theorem 6.4.

Also, we have the following similar result to Theorem 6.4.

PROPOSITION 6.8. *With the same notations and assumptions as Corollary 6.5, the following are equivalent:*

(1) *There are a C^*-subalgebra P of A with the common unit and a conditional expectation G from A onto P, which is index-finite, satisfying that there is an isomorphism π of P_1 onto B such that $G_1 = E \circ \pi$,*

where P_1 is the C^*-basic construction induced by G and G_1 is the dual conditional expectation of G from P_1 onto A.

(2) *There is a saturated action of H^0 on A satisfying that there is an isomorphism π of $A \rtimes H^0$ onto B such that $F = E \circ \pi$.*

PROOF. (1) $\Rightarrow$ (2): Let ρ, B_1 and e be as in the proof of Corollary 6.7. Since the inclusion $A \subset B$ is of depth 2, so is the inclusion $P \subset A$. Hence since $P' \cap B_1$ is isomorphic to some full matrix algebra over $\mathbf{C}$, $P' \cap B_1$ has the properties (1), (2) and (3) in Lemma 6.6. In the same way as in the proof of Corollary 6.7, $\rho(1 \rtimes e) \sim (1 \rtimes e) \otimes 1$ in $(P' \cap B_1) \otimes H$. Thus there is a partial isometry $w \in (P' \cap B_1) \otimes H$ such that $w^* w = \rho(1 \rtimes e)$, $ww^* = (1 \rtimes e) \otimes 1$. Also, in the same way as in Section 4, there is a unitary element $U \in (P' \cap B) \otimes H$ such that $((1 \rtimes e) \otimes 1)w = ((1 \rtimes e) \otimes 1)U$. In the same discussions as in Sections 4 and 5, we can define a weak action of H^0 on A and a unitary cocycle σ of $H^0 \otimes H^0$ to A by for any $x \in A$ and $\phi, \psi \in H^0$,

$$\phi \cdot x = U^\wedge(\phi_{(1)}) x U^{\wedge *}(\phi_{(2)})$$
$$\sigma(\phi, \psi) = U^\wedge(\phi_{(1)}) U^\wedge(\psi_{(1)}) U^{\wedge *}(\phi_{(2)}\psi_{(2)}) \in A,$$

where $U^\wedge$ is a unitary element in $\mathrm{Hom}(H^0, P' \cap B)$ induced by U. We note that since $U \in (P' \cap B) \otimes H$, $U^\wedge(\phi) \in P' \cap B$ for any $\phi \in H^0$. Let e_P be the Jones projection induced by $P \subset A$. Since e_P is a minimal and central projection in $P' \cap B$, for any $x \in P' \cap B$, there is the unique element $c(x) \in \mathbf{C}$ such that $xe_P = e_P x = c(x)e_P$. We regard c as a map $x \in P' \cap B \mapsto c(x) \in \mathbf{C}1$. Then c is a homomorphism of $P' \cap B$ to $\mathbf{C}$. Let $c^\wedge$ be a homomorphism of H^0 to $\mathbf{C}1$ defined by $c^\wedge = c \circ U^\wedge$. By easy computations, we can see that $c^\wedge$ is a unitary element in $\mathrm{Hom}(H^0, \mathbf{C}1)$ with $c^\wedge(1^0) = 1$. Furthermore, for any $\phi, \psi \in H^0$,

$$\begin{aligned}
\sigma(\phi, \psi)e_P &= U^\wedge(\phi_{(1)}) U^\wedge(\psi_{(1)}) U^{\wedge *}(\phi_{(2)}\psi_{(2)}) e_P \\
&= U^\wedge(\phi_{(1)}) U^\wedge(\psi_{(1)}) e_P c^{\wedge *}(\phi_{(2)}\psi_{(2)}) \\
&= c^\wedge(\phi_{(1)}) c^\wedge(\psi_{(1)}) c^{\wedge *}(\phi_{(2)}\psi_{(2)}) e_P.
\end{aligned}$$

Since $\sigma(\phi, \psi) \in A$, $\sigma(\phi, \psi) = c^\wedge(\phi_{(1)}) c^\wedge(\psi_{(1)}) c^{\wedge *}(\phi_{(2)}\psi_{(2)})$ for any $\phi, \psi \in H^0$. Let $W = c^{\wedge *} U^\wedge \in \mathrm{Hom}(H^0, B)$. Then for any $x \in A$ and $\phi \in H^0$

$$W(\phi_{(1)}) x W^*(\phi_{(2)}) = c^{\wedge *}(\phi_{(1)})(\phi_{(2)} \cdot x) c^\wedge(\phi_{(3)}) \in A.$$

Thus by easy computations, we can see that the map

$$A \times H^0 \ni (x, \phi) \mapsto W(\phi_{(1)}) x W^*(\phi_{(2)}) \in A$$

is an action of H^0 on A. We denote by $A \rtimes H^0$ the crossed product of A by the above action of H^0 on A. Let Φ be a map from $A \rtimes H^0$ to $A \rtimes_\sigma H^0$

defined by for any $x \in A$ and $\phi \in H^0$, $\Phi(x \rtimes \phi) = xc^{\wedge *}(\phi_{(1)}) \rtimes \phi_{(2)}$, where $x \rtimes \phi \in A \rtimes H^0$. Then by routine computations, Φ is an isomorphism of $A \rtimes H^0$ onto $A \rtimes_\sigma H^0$ satisying that $F' = E \circ \Phi$, where F' is the canonical conditional expectations from $A \rtimes H^0$ onto A. (2) $\Rightarrow$ (1): Let $P = A^{H^0}$, the fixed point C^*-subalgebra of A for the action of H^0 on A. Then P is the desired C^*-subalgebra of A.

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