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C^*-ALGEBRAS OF LABELLED GRAPHS II – SIMPLICITY RESULTS

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Abstract

We prove simplicity and pure infiniteness results for a certain class of labelled graph C^*-algebras. We show, by example, that this class of unital labelled graph C^*-algebras is strictly larger than the class of unital graph C^*-algebras.

1. Introduction

This paper has two main aims. The first is to continue the development of the C^*-algebras of labelled graphs begun in [3] and the second is to provide a tractable example which illustrates why they are worthy of further study.

A labelled graph is a directed graph E in which the edges have been labelled by symbols coming from a countable alphabet. By considering the sequences of labels carried by the bi-infinite paths in E one obtains a shift space X; the labelled graph is then called a presentation of X. A directed graph is a (trivial) example of a labelled graph, and the shift space it presents is a shift of finite type (see [13]). In [3] we showed how to associate a C^*-algebra to a labelled space, which consists of a labelled graph together with a certain collection of subsets of vertices. By making suitable choices of the labelled spaces it was shown in [3, Proposition 5.1, Theorem 6.3] that the class of labelled graph C^*-algebras includes graph C^*-algebras, the ultragraph C^*-algebras of [21], [22] and the C^*-algebras of shift spaces in the sense of [14], [4].

In this paper we shall work almost exclusively with the labelled spaces which arise in connection with shift spaces. In particular we shall be interested in identifying key properties of our labelled spaces which allow us to prove results about the simplicity and pure infiniteness of the associated C^*-algebra (see Theorem 6.4 and Theorem 6.9).

Up to now, the examples of labelled spaces that we have considered have turned out to have C^*-algebras isomorphic to the C^*-algebra of the underlying directed graph (see [3, Theorem 6.6]). In this paper we turn our attention to

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the question of whether the class of C^*-algebras of labelled spaces that we are considering is strictly larger than the class of graph C^*-algebras. In section 7.1 we give presentations of the Dyck shifts D_N and show that their associated C^*-algebras cannot be unital graph C^*-algebras. In section 7.2 we present a labelled graph which presents an irreducible non-sofic shift, whose C^*-algebra is simple and purely infinite.

There have now been many papers on the C^*-algebras associated to shift spaces (see [7], [6], [14], [16], [5], [4], [2], [3] for example). A drawback to some of the approaches is that the canonical C^*-algebra associated to an irreducible shift space is often not simple (see [3, Remark 6.10]). We believe that an equally valid way to study the C^*-algebra associated to a shift space is to study the C^*-algebras of the various labelled graphs which present it. This belief is founded on the observation that the labelled graph $(E_1, \mathscr{L}_1)$ of Examples 5.4(i) is a presentation of an irreducible sofic shift (called the even shift) whose C^*-algebra is simple (see [3, Remark 6.10]) whereas the C^*-algebra associated to the even shift in [4] is not simple.

The work of Matsumoto on symbolic matrix systems and their associated λ-graph systems gives us an important method for studying shift spaces using labelled graphs (see [18], [15], [16], [17] amongst others). However, we feel that there is an extra facility afforded by our approach. Whilst λ-graph systems are indeed labelled graphs, they are quite complicated. This makes them difficult to visualise; for instance the labelled graphs in Examples 5.4(i) give rise to the same C^*-algebras as the ones for the symbolic matrix systems described on [16, p. 297]. Furthermore we believe that our presentations of the Dyck shifts in section 7.1 give us a more tractable way of studying them. Of equal importance is the fact that our labelled spaces are ideally suited to handle shift spaces over countably infinite alphabets.

The paper begins with a long section in which we describe many of the important concepts associated to labelled graphs and labelled spaces. The two main results of this section are Proposition 2.4 and Proposition 2.6. In Proposition 2.4 we give an important embellishment to the treatment of labelled spaces in [3] by identifying the basic objects in a labelled space, which we call the generalised vertices. In Proposition 2.6 we establish concrete connections between our work and that of Matsumoto by showing how to associate a symbolic matrix system to a labelled graph.

In section 3 we recall the definition of the C^*-algebra of a labelled space from [3]. In Proposition 3.4 we give a new description of the canonical spanning set for a labelled graph C^*-algebra in terms of generalised vertices. Then in Proposition 3.6 we use this new description to show the relationship between the C^*-algebra of a labelled graph and the λ-graph C^*-algebra of the associated symbolic matrix system.

In section 4 we give a description of the AF core of a labelled graph C^*-algebra before moving on to prove the Cuntz-Krieger uniqueness theorem (Theorem 5.5) in section 5. The central hypothesis to the Cuntz-Krieger uniqueness theorem for labelled graphs is the notion of disagreeability, which replaces the aperiodicity hypothesis in the corresponding theorem for directed graphs (see, for example [1, Theorem 3.1]).

In section 6 we give the simplicity and pure infiniteness results for labelled graph C^*-algebras. To prove the simplicity result (Theorem 6.4) we need a notion of cofinality appropriate for labelled graphs. The notion of cofinality for labelled graphs is much more subtle than that for directed graphs as many different infinite paths in the underlying directed graph can have the same labels. To prove the pure infiniteness result (Theorem 6.9) we need to examine how periodic paths arise in labelled graphs. The situtation is much more complicated than for directed graphs since periodic points in the shift space associated to a labelled graph need not arise from a loop in the underlying directed graph.

Finally in section 7 we provide two new examples of labelled graphs to which our main results apply. In section 7.1 we provide a labelled graph presentation of the Dyck shifts D_N. In Proposition 7.2 show that these presentations give rise to simple purely infinite labelled graph C^*-algebras. In Remark 7.3 we give a formula for the K-theory of our labelled graph C^*-algebras which demonstrates that the C^*-algebras we associate to Dyck shifts cannot be isomorphic to graph C^*-algebras. In section 7.2 we provide a presentation of an interesting new irreducible non-sofic shift whose labelled graph C^*-algebra is simple and purely infinite.

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2. Collected definitions and notation

Directed graphs

A directed graph E consists of a quadruple (E^0, E^1, r, s) where E^0 and E^1 are (not necessarily countable) sets of vertices and edges respectively and $r, s : E^1 \to E^0$ are maps giving the direction of each edge. A path $\lambda = e_1 \dots e_n$ is a sequence of edges $e_i \in E^1$ such that $r(e_i) = s(e_{i+1})$ for $i = 1, \dots, n-1$, we define $s(\lambda) = s(e_1)$ and $r(\lambda) = r(e_n)$. The collection of paths of length n in E is denoted E^n and the collection of all finite paths in E by E^*, so that $E^* = \bigcup_{n\geq 0} E^n$.

A *loop* in E is a path which begins and ends at the same vertex, that is $\lambda \in E^*$ with $s(\lambda) = r(\lambda)$. We say that E is *row-finite* if every vertex emits

finitely many edges. The graph E is called *transitive* if given any pair of vertices $u, v \in E^0$ there is a path $\lambda \in E^*$ with $s(\lambda) = u$ and $r(\lambda) = v$. We denote the collection of all infinite paths in E by E^∞.

Standing assumption 1

We will assume that our directed graphs E are *essential*: all vertices emit and receive edges (i.e., E has no sinks or sources).

Labelled graphs

A *labelled graph* $(E, \mathcal{L})$ over a countable alphabet $\mathcal{A}$ consists of a directed graph E together with a labelling map $\mathcal{L} : E^1 \to \mathcal{A}$. Without loss of generality we may assume that the map $\mathcal{L}$ is onto.

Let $\mathcal{A}^*$ be the collection of all *words* in the symbols of $\mathcal{A}$. The map $\mathcal{L}$ extends naturally to a map $\mathcal{L} : E^n \to \mathcal{A}^*$, where $n \geq 1$: for $\lambda = e_1 \dots e_n \in E^n$ put $\mathcal{L}(\lambda) = \mathcal{L}(e_1) \dots \mathcal{L}(e_n)$; in this case the path $\lambda \in E^n$ is said to be a *representative* of the *labelled path* $\mathcal{L}(e_1) \dots \mathcal{L}(e_n)$. Let $\mathcal{L}(E^n)$ denote the collection of all labelled paths in $(E, \mathcal{L})$ of length n where we write $|\alpha| = n$ if $\alpha \in \mathcal{L}(E^n)$. The set $\mathcal{L}^*(E) = \bigcup_{n \geq 1} \mathcal{L}(E^n)$ is the collection of all labelled paths in the labelled graph $(E, \mathcal{L})$. We may similarly extend $\mathcal{L}$ to E^∞.

The labelled graph $(E, \mathcal{L})$ is *left-resolving* if for all $v \in E^0$ the map $\mathcal{L} : r^{-1}(v) \to \mathcal{A}$ is injective. The left-resolving condition ensures that for all $v \in E^0$ the labels $\{\mathcal{L}(e) : r(e) = v\}$ of all incoming edges to v are all different. For α in $\mathcal{L}^*(E)$ we put

$$s_{\mathcal{L}}(\alpha) = \{s(\lambda) \in E^0 : \mathcal{L}(\lambda) = \alpha\} \text{ and } r_{\mathcal{L}}(\alpha) = \{r(\lambda) \in E^0 : \mathcal{L}(\lambda) = \alpha\},$$

so that $r_{\mathcal{L}}, s_{\mathcal{L}} : \mathcal{L}^*(E) \to 2^{E^0}$. We shall drop the subscript on $r_{\mathcal{L}}$ and $s_{\mathcal{L}}$ if the context in which it is being used is clear.

Let $(E, \mathcal{L})$ be a labelled graph. For $A \subseteq E^0$ and $\alpha \in \mathcal{L}^*(E)$ the *relative range of* α *with respect to* A is defined to be

$$r_{\mathcal{L}}(A, \alpha) = \{r(\lambda) : \lambda \in E^*,\ \mathcal{L}(\lambda) = \alpha,\ s(\lambda) \in A\}.$$

A collection $\mathcal{B} \subseteq 2^{E^0}$ of subsets of E^0 is said to be *closed under relative ranges for* $(E, \mathcal{L})$ if for all $A \in \mathcal{B}$ and $\alpha \in \mathcal{L}^*(E)$ we have $r(A, \alpha) \in \mathcal{B}$. If $\mathcal{B}$ is closed under relative ranges for $(E, \mathcal{L})$, contains $r(\alpha)$ for all $\alpha \in \mathcal{L}^*(E)$ and is also closed under finite intersections and unions, then we say that $\mathcal{B}$ is *accommodating* for $(E, \mathcal{L})$.

Let $\mathcal{E}^{0,-}$ denote the smallest subset of 2^{E^0} which is accommodating for $(E, \mathcal{L})$. Since $\mathcal{E}^{0,-}$ is generated by a countable family of subsets of E^0, under countable operations, it follows that $\mathcal{E}^{0,-}$ is countable, even though E^0 itself may be uncountable. Of course, 2^{E^0} is the largest accommodating collection of subsets for $(E, \mathcal{L})$.

Labelled spaces

A *labelled space* consists of a triple $(E, \mathscr{L}, \mathscr{B})$, where $(E, \mathscr{L})$ is a labelled graph and $\mathscr{B}$ is accommodating for $(E, \mathscr{L})$.

A labelled space $(E, \mathscr{L}, \mathscr{B})$ is *weakly left-resolving* if for every $A, B \in \mathscr{B}$ and every $\alpha \in \mathscr{L}^*(E)$ we have $r(A, \alpha) \cap r(B, \alpha) = r(A \cap B, \alpha)$.

REMARKS 2.1. (i) If $(E, \mathscr{L}, \mathscr{E}^{0,-})$ is weakly left-resolving then $\mathscr{E}^{0,-}$ is the closure of $\{r(\alpha) : \alpha \in \mathscr{L}^*(E)\}$ under finite unions and intersections (cf. [3, Remark 3.9]). Moreover, if $\alpha \in \mathscr{L}^*(E)$ and $A = \bigcup_{k=1}^m \bigcap_{i=1}^n r(\beta_{i,k}) \in \mathscr{E}^{0,-}$ where $\beta_{i,k} \in \mathscr{L}^*(E)$ for $i = 1, \ldots, n$ and $k = 1, \ldots, m$ then we have

$$r(A, \alpha) = \bigcup_{k=1}^{m} \bigcap_{i=1}^{n} r(\beta_{i,k}\alpha).$$

(ii) If $(E, \mathscr{L})$ is left-resolving then the labelled space $(E, \mathscr{L}, \mathscr{B})$ is weakly left-resolving for any $\mathscr{B}$.

For $\ell \geq 1$ and $A \subseteq E^0$ let $E^\ell A = \{\lambda \in E^\ell : r(\lambda) \in A\}$ and $AE^\ell = \{\lambda \in E^\ell : s(\lambda) \in A\}$. The labelled space $(E, \mathscr{L}, \mathscr{B})$ is *receiver set-finite* if for all $A \in \mathscr{B}$ and all $\ell \geq 1$ the set $\mathscr{L}(E^\ell A) := \{\mathscr{L}(\lambda) : \lambda \in E^\ell A\}$ is finite. In particular, the labelled space $(E, \mathscr{L}, \mathscr{B})$ is receiver set-finite if each $A \in \mathscr{B}$ receives only finitely labelled paths of length ℓ (even though A may receive infinitely many paths of each length ℓ). More generally, for $\ell \geq 1$ and $A \subseteq E^0$ let

$$\mathscr{L}(E^{\leq \ell}) = \bigcup_{j=1}^{\ell} \mathscr{L}(E^j) \text{ and } \mathscr{L}(E^{\leq \ell} A) = \bigcup_{j=1}^{\ell} \mathscr{L}(E^j A).$$

We say that the labelled space $(E, \mathscr{L}, \mathscr{B})$ is *set-finite* if for all $A \in \mathscr{B}$ the set $\mathscr{L}(AE^1) := \{\mathscr{L}(\lambda) : \lambda \in AE^1\}$ is finite. One may similarly define $\mathscr{L}(AE^n)$ (note that $\mathscr{L}(AE^n)$ was denoted L_A^n in [3]).

In this paper we shall focus exclusively on the (minimal) accommodating labelled space $(E, \mathscr{L}, \mathscr{E}^{0,-})$ associated to a labelled graph $(E, \mathscr{L})$. We do this in order to relate our work to that of Matsumoto (see [14], [15], [16], [17], [18]). One may extend the results in this paper to include more general labelled spaces, but since we have no examples in mind, we choose not to.

Standing assumption 2

We will assume that $(E, \mathscr{L}, \mathscr{E}^{0,-})$ is receiver set-finite, set-finite and weakly left-resolving.

REMARK 2.2. The conditions of set-finiteness and receiver set-finiteness are trivially satisfied by labelled spaces over finite alphabets. The condition

of set-finiteness for labelled spaces is the analogue of row-finiteness for directed graphs. Taken together the conditions of set-finiteness and receiver set-finiteness give us the analogue of local finiteness for directed graphs.

Generalised vertices

For $v \in E^0$ and $\ell \geq 1$ let

$$\Lambda_\ell(v) = \{\lambda \in \mathscr{L}(E^{\leq \ell}) : v \in r(\lambda)\} = \mathscr{L}(E^{\leq \ell}v).$$

The relation $\sim_\ell$ on E^0 is defined by $v \sim_\ell w$ if and only if $\Lambda_\ell(v) = \Lambda_\ell(w)$; hence $v \sim_\ell w$ if v and w receive exactly the same labelled paths of length at most ℓ. Evidently $\sim_\ell$ is an equivalence relation and we use $[v]_\ell$ to denote the equivalence class of $v \in E^0$. We call the $[v]_\ell$ *generalised vertices* as they play the same role in labelled spaces as vertices in a directed graph.

Set $\Omega_\ell = E^0/\sim_\ell$ and Ω be the disjoint union of the Ω_ℓ for $\ell \geq 1$. If the alphabet $\mathscr{A}$ is finite, then Ω_ℓ is finite. If there is $L \geq 1$ such that $\Omega_\ell = \Omega_L$ for all $\ell \geq L$, then the underlying shift $\mathsf{X}_{E,\mathscr{L}}$ is a sofic shift (see [4], [13]). Conversely, if X is a sofic shift then every presentation $(E, \mathscr{L})$ of the shift X has this property (see [13, Exercise 3.2.6]).

For $\ell \geq 1$ let $\mathscr{E}^{0,-}_\ell \subseteq \mathscr{E}^{0,-}$ be the smallest subset of 2^{E^0} which contains $r(\lambda)$ for all $\lambda \in \mathscr{L}(E^{\leq \ell})$ and is closed under finite intersections and unions. Evidently $\mathscr{E}^{0,-}_\ell \subseteq \mathscr{E}^{0,-}_{\ell+1}$. Following Remark 2.1(i) we have $\mathscr{E}^{0,-} = \bigcup_{\ell=1}^\infty \mathscr{E}^{0,-}_\ell$.

For $v \in E^0$ and $\ell \geq 1$, the equivalence class $[v]_\ell$ does not necessarily belong to $\mathscr{E}^{0,-}_\ell$; however, as we shall see in Proposition 2.4 (i), $[v]_\ell$ may be expressed as a difference of elements of $\mathscr{E}^{0,-}_\ell$. First we need the following technical lemma.

LEMMA 2.3. *Let $(E, \mathscr{L}, \mathscr{E}^{0,-})$ be a labelled space, $v \in E^0$ and $\ell \geq 1$.*

(i) *The set $\Lambda_\ell(v)$ is finite and $X_\ell(v) := \bigcap_{\lambda \in \Lambda_\ell(v)} r(\lambda) \in \mathscr{E}^{0,-}_\ell$. Moreover $[v]_\ell \subseteq X_\ell(v)$.*

(ii) *The set of labels $Y_\ell(v) := \bigcup_{w \in X_\ell(v)} \Lambda_\ell(w) \backslash \Lambda_\ell(v)$ is finite, and $r(Y_\ell(v)) \in \mathscr{E}^{0,-}_\ell$.*

PROOF. For the first statement let $A \in \mathscr{E}^{0,-}$ be such that $v \in A$. Since $(E, \mathscr{L}, \mathscr{E}^{0,-})$ is receiver set-finite $\mathscr{L}(E^j v) \subseteq \mathscr{L}(E^j A)$ is finite for all $j \geq 1$ and hence $\Lambda_\ell(v) = \bigcup_{j=1}^\ell \mathscr{L}(E^j v)$ is finite for all $\ell \geq 1$. It now follows that $X_\ell(v)$ is a finite intersection of elements of $\mathscr{E}^{0,-}_\ell$ and hence $X_\ell(v) \in \mathscr{E}^{0,-}_\ell$. Since $X_\ell(v)$ is the set of vertices which receive at least the same labelled paths as v up to length ℓ we certainly have $[v]_\ell \subseteq X_\ell(v)$.

For the second statement observe that $Y_\ell(v) = \mathscr{L}(E^{\leq \ell} X_\ell(v)) \backslash \Lambda_\ell(v)$. Since $(E, \mathscr{L}, \mathscr{E}^{0,-})$ is receiver set-finite and $X_\ell(v) \in \mathscr{E}^{0,-}$ the sets $\mathscr{L}(E^{\leq \ell} X_\ell(v))$

and $Y_\ell(v)$ must be finite. Note that $r(Y_\ell(v)) = \bigcup_{\mu\in Y_\ell(v)} r(\mu)$ belongs to $\mathscr{E}_\ell^{0,-}$ as it is a finite union of elements of $\mathscr{E}_\ell^{0,-}$.

The set $Y_\ell(v)$ denotes the additional labelled paths of length at most ℓ received by those vertices which receive at least the same labelled paths as v up to length ℓ.

PROPOSITION 2.4. *Let $(E, \mathscr{L}, \mathscr{E}^{0,-})$ be a labelled space, $v \in E^0$ and $\ell \geq 1$.*

(i) *We have $[v]_\ell = X_\ell(v)\backslash r(Y_\ell(v))$.*

(ii) *For every set $A \in \mathscr{E}_\ell^{0,-}$ we can find vertices $v_1, \ldots, v_m \in A$ such that $A = \bigcup_{i=1}^m [v_i]_\ell$.*

(iii) *There are $w_1, \ldots, w_n \in [v]_\ell$ such that $[v]_\ell = \bigcup_{i=1}^n [w_i]_{\ell+1}$.*

PROOF. For the first statement observe that $[v]_\ell$ consists of those vertices which receive exactly the labelled paths from $\Lambda_\ell(v)$ whereas other vertices in $X_\ell(v)$ may receive more labelled paths. Hence, to form $[v]_\ell$ we remove those vertices from $X_\ell(v)$ which receive different labelled paths of length ℓ from v – these are precisely the vertices in $r(Y_\ell(v))$.

For the second statement note that by Remark 2.1(i) any $A \in \mathscr{E}_\ell^{0,-}$ can be written as a finite union of elements of the form $B_k = \bigcap_{i=1}^n r(\beta_i)$ where $\beta_i \in \mathscr{L}(E^{\leq \ell})$. If $v_1 \in B_k$ then $[v_1]_\ell \subseteq B_k$ as v_1, and hence every vertex in $[v_1]_\ell$, must receive $\beta_1, \ldots, \beta_n$ and so lie in B_k. If $B_k \neq [v_1]_\ell$, there is $v_2 \in B_k$ with $\Lambda_\ell(v_1) \neq \Lambda_\ell(v_2)$. Again we have $[v_2]_\ell \subseteq B_k$. Since $(E, \mathscr{L}, \mathscr{E}^{0,-})$ is receiver set-finite $B_k \in \mathscr{E}_\ell^{0,-}$ receives only finitely many different labelled paths of length at most ℓ. Hence there are vertices $\{v_i : 1 \leq i \leq m\}$ in B_k such that $B_k = \bigcup_{i=1}^m [v_i]_\ell$ and our result is established.

For the final statement we observe that since $\mathscr{E}_\ell^{0,-} \subseteq \mathscr{E}_{\ell+1}^{0,-}$ the first statement shows that $[v]_\ell$ may be written as a difference $A\backslash B$ of elements of $\mathscr{E}_{\ell+1}^{0,-}$. The result then follows by applying the second statement to $A, B \in \mathscr{E}_{\ell+1}^{0,-}$ and noting that the $[w_i]_{\ell+1}$'s are disjoint.

Shift spaces

Let $(E, \mathscr{L})$ be a labelled graph. The subshift X_E is defined by $\mathsf{X}_E = \{x \in (E^1)^{\mathsf{Z}} : s(x_{i+1}) = r(x_i) \text{ for all } i \in \mathsf{Z}\}$. The subshift $(\mathsf{X}_{E,\mathscr{L}}, \sigma)$ is defined by

$$\mathsf{X}_{E,\mathscr{L}} = \{y \in \mathscr{A}^{\mathsf{Z}} : \text{ there exists } x \in \mathsf{X}_E \text{ such that } y_i = \mathscr{L}(x_i) \text{ for all } i \in \mathsf{Z}\},$$

where σ is the shift map $\sigma(y)_i = y_{i+1}$ for $i \in \mathsf{Z}$. The labelled graph $(E, \mathscr{L})$ is said to be a *presentation* of the shift space $\mathsf{X}_{E,\mathscr{L}}$ with language $\mathscr{L}^*(E)$.

We are primarily interested in one-sided shift spaces, namely

$$\mathsf{X}^+_{E,\mathscr{L}} = \{y \in \mathscr{A}^{\mathsf{N}} : \text{ there exists } x \in E^\infty \text{ such that } y_i = \mathscr{L}(x_i) \text{ for all } i \in \mathsf{N}\}$$

and we restrict the shift map to $\mathsf{X}^+_{E,\mathscr{L}}$. For an infinite labelled path $x \in \mathsf{X}^+_{E,\mathscr{L}}$ we define $s_{\mathscr{L}}(x)$ to be the set of all $v \in E^0$ for which there is an infinite path $\widehat{x} \in E^\infty$ with $s(\widehat{x}) = v$ and $\mathscr{L}(\widehat{x}) = x$. The infinite path $\hat{x}$ is said to be a *representative* of x.

An infinite labelled path $x \in \mathsf{X}^+_{E,\mathscr{L}}$ is *periodic* if $\sigma^n x = x$ for some $n \geq 1$. A path which is not periodic is called *aperiodic*.

EXAMPLE 2.5. If E is a directed graph then we may consider it as a labelled graph when endowed with the trivial labelling $\mathscr{L}_t$. In this case $\mathscr{E}^{0,-}$ consists of all finite subsets of E^0 (see [3, Examples 4.3(i)]) and $[v]_\ell = \{v\}$ for all $\ell \geq 1$. We shall identify $\mathscr{L}_t^*(E)$ with E^* and $\mathsf{X}^+_{E,\mathscr{L}_t}$ with E^∞.

Symbolic Matrix Systems

Essential symbolic matrix systems are defined in [16, §2]. To a left-resolving labelled graph $(E, \mathscr{L})$ over a finite alphabet we associate matrices $(M(E)_{\ell,\ell+1}, I(E)_{\ell,\ell+1})_{\ell\geq 1}$ as follows: For $\ell \geq 1$, write $\Omega_\ell = \{[v_i]_\ell : i = 1, \ldots, m(\ell)\}$, then $I(E)_{\ell,\ell+1}$ is a $m(\ell) \times m(\ell+1)$ matrix with entries 0, 1 determined by

$$I(E)_{\ell,\ell+1}([v_i]_\ell, [w_j]_{\ell+1}) = \begin{cases} 1 & \text{if } [w_j]_{\ell+1} \subset [v_i]_\ell \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

The symbolic matrix $M(E)_{\ell,\ell+1}$ is the same size as $I(E)_{\ell,\ell+1}$ with entries determined as follows: For $v \in E^0$ let $\langle v\rangle_\ell$ denote the collection of labelled paths of length exactly ℓ which arrive at v. Since $(E, \mathscr{L})$ is left-resolving we may partition the set of labelled paths of length $\ell + 1$ arriving at w to write $\langle w\rangle_{\ell+1}$ as the disjoint union

$$\langle w\rangle_{\ell+1} = \bigcup_{e\in r^{-1}(w)} \langle s(e)\rangle_\ell \mathscr{L}(e),$$

where $\langle s(e)\rangle_\ell \mathscr{L}(e)$ denotes the set of labelled paths of length $\ell + 1$ formed by the juxtaposition of the symbol $\mathscr{L}(e)$ at the end of each labelled path in $\langle s(e)\rangle_\ell$. Since all vertices in $[v_i]_\ell$ and $[w_j]_{\ell+1}$ receive the same labelled paths of length ℓ and $\ell + 1$ respectively we may unambiguously define

$$M(E)_{\ell,\ell+1}([v_i]_\ell, [w_j]_{\ell+1}) = \sum_{e\in s^{-1}(v_i)\cap r^{-1}(w_j)} \mathscr{L}(e) \tag{2}$$

where the right hand-side is treated as a formal sum.

PROPOSITION 2.6. *Let $(E, \mathscr{L})$ be a left-resolving labelled graph over a finite alphabet. Then the matrices $(M(E)_{\ell,\ell+1}, I(E)_{\ell,\ell+1})_{\ell\geq 1}$ defined above form an essential symbolic matrix system.*

PROOF. If suffices to check that the matrices $(M(E)_{\ell,\ell+1}, I(E)_{\ell,\ell+1})_{\ell\geq 1}$ satisfy the conditions on [16, p .290]: Since E is essential it is straightforward to check from the definition of $I(E)_{\ell,\ell+1}$ and $M(E)_{\ell,\ell+1}$ that conditions (1), (2), (2-a), (2-b), (3), (5-i) and (5-ii) are satisfied. It remains to check that for $\ell \geq 1$ we have $M(E)_{\ell,\ell+1}I(E)_{\ell+1,\ell+2} = I(E)_{\ell,\ell+1}M(E)_{\ell+1,\ell+2}$.

For $\ell \geq 1$ we form the entry $M(E)_{\ell,\ell+1}I(E)_{\ell+1,\ell+2}([u_i]_\ell, [w_k]_{\ell+1})$ as follows: For each $[v_j]_{\ell+1}$ which receives an edge from $[u_i]_\ell$, the entry is the formal sum of the labels received by the unique $[v_j]_{\ell+1}$ of which $[w_k]_{\ell+2}$ is a subset. In which case

$$M(E)_{\ell,\ell+1}I(E)_{\ell+1,\ell+2}([u_i]_\ell, [w_k]_{\ell+2}) = \sum_{e\in s^{-1}(u_i)\cap r^{-1}(w_k)} \mathscr{L}(e).$$

On the other hand, to form the entry $I(E)_{\ell,\ell+1}M(E)_{\ell+1,\ell+2}([u_i]_\ell, [w_k]_{\ell+2})$ we take each $[v_j]_{\ell+1}$ which is a subset of $[u_i]_\ell$ and then formally sum the labels of the edges to $[w_k]_{\ell+1}$. In which case

$$\begin{aligned} I(E)_{\ell,\ell+1}M(E)_{\ell+1,\ell+2}([u_i]_\ell, [w_k]_{\ell+2}) &= \sum_{[v_j]_{\ell+1}\subseteq[u_i]_\ell} \sum_{e\in s^{-1}(v_j)\cap r^{-1}(w_k)} \mathscr{L}(e) \\ &= \sum_{e\in s^{-1}(u_i)\cap r^{-1}(w_k)} \mathscr{L}(e). \end{aligned}$$

Hence for $\ell \geq 1$ we have $M(E)_{\ell,\ell+1}I(E)_{\ell+1,\ell+2} = I(E)_{\ell,\ell+1}M(E)_{\ell+1,\ell+2}$ as required.

3. C^*-algebras of labelled spaces

We recall from [3] the definition of the C^*-algebra associated to the labelled space $(E, \mathscr{L}, \mathscr{E}^{0,-})$.

DEFINITION 3.1. Let $(E, \mathscr{L}, \mathscr{E}^{0,-})$ be a labelled space. A representation of $(E, \mathscr{L}, \mathscr{E}^{0,-})$ consists of projections $\{p_A : A \in \mathscr{E}^{0,-}\}$ and partial isometries $\{s_a : a \in \mathscr{A}\}$ with the properties that

(i) If $A, B \in \mathscr{E}^{0,-}$ then $p_A p_B = p_{A\cap B}$ and $p_{A\cup B} = p_A + p_B - p_{A\cap B}$, where $p_\emptyset = 0$.

(ii) If $a \in \mathscr{A}$ and $A \in \mathscr{E}^{0,-}$ then $p_A s_a = s_a p_{r(A,a)}$.

(iii) If $a, b \in \mathscr{A}$ then $s_a^* s_a = p_{r(a)}$ and $s_a^* s_b = 0$ unless $a = b$.

(iv) For $A \in \mathscr{E}^{0,-}$ we have

$$p_A = \sum_{a\in\mathscr{L}(AE^1)} s_a p_{r(A,a)} s_a^*. \tag{3}$$

REMARK 3.2. If the directed graph E contains sinks then we need to modify condition (iv) above (note that $\mathscr{E}^{0,-}$ is different in this case (see [3, Definition 3.8])). The original definition [3, Definition 4.1] was in error since it would lead to degeneracy of the vertex projections for sinks. We thank Toke Carlsen for pointing this out to us. If A contains a finite number of sinks and $\mathscr{B} = \mathscr{E}^{0,-}$ or $\mathscr{E}^0$, then we obtain the relation

$$p_A = \sum_{a \in \mathscr{L}(AE^1)} s_a p_{r(A,a)} s_a^* + \sum_{v \in A : v \text{ is a sink}} p_v.$$

DEFINITION 3.3. Let $(E, \mathscr{L}, \mathscr{E}^{0,-})$ be a labelled space, then $C^*(E, \mathscr{L}, \mathscr{E}^{0,-})$ is the universal C^*-algebra generated by a representation of $(E, \mathscr{L}, \mathscr{E}^{0,-})$.

The universal property of $C^*(E, \mathscr{L}, \mathscr{E}^{0,-})$ allows us to define a strongly continuous action γ of T on $C^*(E, \mathscr{L}, \mathscr{E}^{0,-})$ called the *gauge action* (see [3, Section 5]). As in [20, Proposition 3.2] we denote by Φ the conditional expectation of $C^*(E, \mathscr{L}, \mathscr{E}^{0,-})$ onto the fixed point algebra $C^*(E, \mathscr{L}, \mathscr{E}^{0,-})^\gamma$. If $(E, \mathscr{L}, \mathscr{E}^{0,-})$ is a labelled space then by [3, Lemma 4.4] we have

$$C^*(E, \mathscr{L}, \mathscr{E}^{0,-}) = \overline{\operatorname{span}}\{s_\alpha p_A s_\beta^* : \alpha, \beta \in \mathscr{L}^*(E), A \in \mathscr{E}^{0,-}\}.$$

Indeed, we can write down a more informative spanning set for $C^*(E, \mathscr{L}, \mathscr{E}^{0,-})$.

PROPOSITION 3.4. *Let $(E, \mathscr{L}, \mathscr{E}^{0,-})$ be a labelled space. Then*

$$C^*(E, \mathscr{L}, \mathscr{E}^{0,-}) = \overline{\operatorname{span}}\{s_\alpha p_{[v]_\ell} s_\beta^* : \alpha, \beta \in \mathscr{L}^*(E), [v]_\ell \in \Omega_\ell\}$$

where

$$p_{[v]_\ell} := p_{X_\ell(v)} - p_{r(Y_\ell(v))} p_{X_\ell(v)} = \sum_{a \in \mathscr{L}([v]_\ell E^1)} s_a p_{r([v]_\ell, a)} s_a^*. \tag{4}$$

PROOF. The first assertion holds from repeated applications of Proposition 2.4. Applying (3) of Definition 3.1 we have

$$\begin{aligned} p_{[v]_\ell} &= p_{X_\ell(v)} - p_{X_\ell(v) \cap r(Y_\ell(v))} \\ &= \sum_{a \in \mathscr{L}(X_\ell(v)E^1)} s_a p_{r(X_\ell(v),a)} s_a^* - \sum_{b \in \mathscr{L}(X_\ell(v) \cap r(Y_\ell(v))E^1)} s_b p_{r(X_\ell(v) \cap r(Y_\ell(v)), b)} s_b^*. \end{aligned}$$

In order to eliminate double counting of labels that are emitted by both $X_\ell(v)$ and $r(Y_\ell(v))$ we need to split $\mathscr{L}(X_\ell(v)E^1)$ into two disjoint parts (the labels

that come only out of $X_\ell(v)$ and those that come out of both $X_\ell(v)$ and $r(Y_\ell(v))$) to obtain

$$p_{[v]_\ell} = \sum_{a\in\mathscr{L}(X_\ell(v)E^1)\setminus\mathscr{L}(X_\ell(v)\setminus r(Y_\ell(v))E^1)} s_a p_{r(X_\ell(v),a)} s_a^* + \sum_{b\in\mathscr{L}(X_\ell(v)\cap r(Y_\ell(v))E^1)} s_b \left(p_{r(X_\ell(v),b)} - p_{r(X_\ell(v)\cap r(Y_\ell(v)),b)}\right) s_b^*.$$

We may replace $X_\ell(v)$ in the first sum by $[v]_\ell$ as the labels a are emitted only by the vertices in $[v]_\ell$ and not by the vertices in $X_\ell(v)\cap r(Y_\ell(v))$. In the second sum the labels b are emitted by both $[v]_\ell$ and $X_\ell(v)\cap r(Y_\ell(v))$, but we subtract the projections corresponding to the copies emitted by $X_\ell(v)\cap r(Y_\ell(v))$ and so we have equation (4) as required.

REMARK 3.5. Note that while proving Proposition 3.4 we have shown that for $[v]_\ell \in \Omega_\ell$ and $a \in \mathscr{A}$

$$r([v]_\ell, a) = r(X_\ell(v), a)\setminus r(r(Y_\ell(v)), a)$$

which is a difference of two elements of $\mathscr{E}^{0,-}_{\ell+1}$.

Recall from Proposition 2.6 that to a labelled graph $(E, \mathscr{L})$ over a finite alphabet $\mathscr{A}$ we may associate an essential symbolic matrix system $(M(E)_{\ell,\ell+1}, I(E)_{\ell,\ell+1})_{\ell\geq 1}$. By [16, Proposition 2.1] there is a unique (up to isomorphism) λ-graph system $\mathfrak{L}_{E,\mathscr{L}}$ associated to $(M(E)_{\ell,\ell+1}, I(E)_{\ell,\ell+1})_{\ell\geq 1}$. By [17, Theorem 3.6] one may associate a C^*-algebra $\mathcal{O}_{\mathfrak{L}_{E,\mathscr{L}}}$ to the λ-graph system $\mathfrak{L}_{E,\mathscr{L}}$ which is the universal C^*-algebra generated by partial isometries $\{t_a : a \in \mathscr{A}\}$ and projections $\{E_i^\ell : i = 1, \ldots, m(\ell)\}$ satisfying relations

$$\text{(5)}\quad \sum_{a\in\mathscr{A}} t_a t_a^* = 1$$

$$\text{(6)}\quad \sum_{i=1}^{m(\ell)} E_i^\ell = 1, \quad E_i^\ell = \sum_{j=1}^{m(\ell+1)} I(E)_{\ell,\ell+1}(i,j) E_j^{\ell+1} \text{ for } i = 1, \ldots, m(\ell)$$

$$\text{(7)}\quad t_a t_a^* E_i^\ell = E_i^\ell t_a t_a^* \quad \text{for } a \in \mathscr{A} \text{ and } i = 1, \ldots, m(\ell)$$

$$\text{(8)}\quad t_a^* E_i^\ell t_a = \sum_{j=1}^{m(\ell+1)} A_{\ell,\ell+1}(i,a,j) E_j^{\ell+1} \quad \text{for } a \in \mathscr{A} \text{ and } i = 1, \ldots, m(\ell)$$

where $A_{\ell,\ell+1}(i,a,j) = 1$ if a occurs in the formal sum $M(E)_{\ell,\ell+1}([v_i]_\ell, [v_j]_{\ell+1})$ and is 0 otherwise.

PROPOSITION 3.6. *Let $(E, \mathscr{L})$ be a left-resolving labelled graph over a finite alphabet. Then we have $C^*(E, \mathscr{L}, \mathscr{E}^{0,-}) \cong \mathcal{O}_{\mathfrak{L}_{E,\mathscr{L}}}$ where $\mathfrak{L}_{E,\mathscr{L}}$ is the λ-graph system associated to the symbolic matrix system $(M(E)_{\ell,\ell+1}, I(E)_{\ell,\ell+1})_{\ell\geq 1}$.*

PROOF. By Proposition 3.4 the elements $\{s_a : a \in \mathscr{A}\}$ and $\{p_{[v_i]_\ell} : i = 1, \ldots, m(\ell)\}$ form a generating set for $C^*(E, \mathscr{L}, \mathscr{E}^{0,-})$. Let $T_a = s_a$ and $F_i^\ell = p_{[v_i]_\ell}$ then $\{T_a, F_i^\ell\}$ satisfy relations (5) – (8) above. Hence by the universal property of $\mathcal{O}_{\mathfrak{L}_{E,\mathscr{L}}}$ there is a map $\pi_{T,F} : \mathcal{O}_{\mathfrak{L}_{E,\mathscr{L}}} \to C^*(E, \mathscr{L}, \mathscr{E}^{0,-})$ characterised by $\pi_{T,F}(t_a) = T_a$ and $\pi_{T,F}(E_i^\ell) = F_i^\ell$.

Let $\{t_a : a \in \mathscr{A}\}$ and $\{E_i^\ell : i = 1, \ldots, m(\ell)\}$ be generators for $\mathcal{O}_{\mathfrak{L}_{E,\mathscr{L}}}$. For $A \in \mathscr{E}_\ell^{0,-}$ and $a \in \mathscr{A}$ let $P_A = \sum_{i:[v_i]_\ell \subseteq A} E_i^\ell$ and $S_a = t_a$. One checks that $\{S_a, P_A\}$ is a representation of the labelled space $(E, \mathscr{L}, \mathscr{E}^{0,-})$. By universality of $C^*(E, \mathscr{L}, \mathscr{E}^{0,-})$ there is a map $\pi_{S,P} : C^*(E, \mathscr{L}, \mathscr{E}^{0,-}) \to \mathcal{O}_{\mathfrak{L}_{E,\mathscr{L}}}$ characterised by $\pi_{S,P}(s_a) = S_a$ and $\pi_{S,P}(p_A) = P_A$. In particular, we have $\pi_{S,P}(p_{[v_i]_\ell}) = P_{[v_i]_\ell}$ for all $i \in 1, \ldots, m(\ell)$. Our result follows since $\pi_{T,F}$ and $\pi_{S,P}$ are inverses of one another.

4. AF core

In this section we perform a detailed analysis of the AF core of $C^*(E, \mathscr{L}, \mathscr{E}^{0,-})$ which we will need to prove the main result of the following section.

DEFINITION 4.1. For $1 \leq k \leq \ell$ let

$$\mathscr{F}^k(\ell) = \overline{\operatorname{span}}\big\{s_\alpha p_A s_\beta^* : \alpha, \beta \in \mathscr{L}(E^k), A \in \mathscr{E}_\ell^{0,-}\big\}.$$

For $\ell \geq 1$ and $[v]_\ell \in \Omega_\ell$ we have $p_{[v]_\ell} \in \mathscr{F}^k(\ell)$ as $X_\ell(v), r(Y_\ell(v)) \in \mathscr{E}_\ell^{0,-}$ by Lemma 2.3(ii).

DEFINITION 4.2. For $1 \leq k \leq \ell$ and $[v]_\ell \in \Omega_\ell$ let

$$\mathscr{F}^k([v]_\ell) = \overline{\operatorname{span}}\big\{s_\alpha p_{[v]_\ell} s_\beta^* : \alpha, \beta \in \mathscr{L}(E^k)\big\}.$$

PROPOSITION 4.3. *For $1 \leq k \leq \ell$ we have*

(i) *$\mathscr{F}^k(\ell) \cong \bigoplus_{[v]_\ell} \mathscr{F}^k([v]_\ell)$, where each $\mathscr{F}^k([v]_\ell)$ is a finite-dimensional matrix algebra.*

(ii) *For each $v \in E^0$ there are $w_1, \ldots, w_n \in [v]_\ell$ such that $\mathscr{F}^k([v]_\ell) = \bigoplus_{i=1}^n \mathscr{F}^k([w_i]_{\ell+1})$. Hence $\mathscr{F}^k(\ell) \subseteq \mathscr{F}^k(\ell+1)$.*

(iii) *There is an embedding of $\mathscr{F}^k(\ell)$ into $\mathscr{F}^{k+1}(\ell+1)$.*

PROOF. For the first statement of (i), applying Proposition 2.4(ii) shows that every element $s_\alpha p_A s_\beta^* \in \mathscr{F}^k(\ell)$ can be written as a finite sum of elements of the form $s_\alpha p_{[v]_\ell} s_\beta^* \in \mathscr{F}^k([v]_\ell)$. The result follows as the summands

in the decomposition are mutually orthogonal since $|\alpha| = |\beta| = k$ and the equivalence classes $[v]_\ell$ are disjoint. For the second statement of (i) note that since $[v]_\ell$ can be written as the difference of two elements of $\mathscr{E}^{0,-}$ it receives only finitely many different labelled paths of length k and hence the set $\{s_\alpha p_{[v]_\ell} s_\beta^* : |\alpha| = |\beta| = k\}$ is finite. It is straightforward to show that the elements $s_\alpha p_{[v]_\ell} s_\beta^*$ form a system of matrix units in $\mathscr{F}^k([v]_\ell)$ and the result follows.

Part (ii) follows by Proposition 2.4(iii). Part (iii) follows from Definition 3.1(iv).

THEOREM 4.4. *Let $(E, \mathscr{L}, \mathscr{E}^{0,-})$ be a labelled space, then $\mathscr{F} = \overline{\bigcup_{k,\ell} \mathscr{F}^k(\ell)}$ is an AF algebra with $\mathscr{F} \cong C^*(E, \mathscr{L}, \mathscr{E}^{0,-})^\gamma$.*

PROOF. The first statement follows from Proposition 4.3. The second statement follows by an argument similar to that of [1, Lemma 2.2].

5. Cuntz-Krieger Uniqueness Theorem

Recall from [11, §3] that the directed graph E satisfies condition (L) if every loop has an exit; that is if $\lambda \in E^n$ is a loop, then there is some $1 \leq i \leq n$ such that the vertex $r(\lambda_i)$ emits more than one edge. Condition (L) is the key hypothesis for the Cuntz-Krieger uniqueness theorem for directed graphs (see [11, Theorem 3.7], [1, Theorem 3.1]). Since periodic paths in E^∞ arise from loops in E, condition (L) guarantees that there are lots of paths in E^∞ which are aperiodic.

In this section we seek an analogue for condition (L) in the context of labelled graphs which will allow us to prove a Cuntz-Krieger uniqueness theorem for labelled graph C^*-algebras. The correct analogue for condition (L) must ensure the existence of aperiodic paths in $\mathsf{X}^+_{E,\mathscr{L}}$. The two key difficulties to overcome in the context of labelled graphs are that we must accommodate the generalised vertices $[v]_\ell$ in a labelled graph and deal with the fact that a periodic path $x \in \mathsf{X}^+_{E,\mathscr{L}}$ need not arise from a loop in E.

The following definition is inspired by [20, Lemma 3.7].

DEFINITIONS 5.1. Let $(E, \mathscr{L}, \mathscr{E}^{0,-})$ be a labelled space, $[v]_\ell \in \Omega_\ell$ and $\alpha \in \mathscr{L}^*(E)$ be such that $|\alpha| > 1$ and $s(\alpha) \cap [v]_\ell \neq \emptyset$. We say that α is *agreeable* for $[v]_\ell$ if there are $\alpha', \beta, \gamma \in \mathscr{L}^*(E)$ with $|\beta| = |\gamma| \leq \ell$ and $\alpha = \beta\alpha' = \alpha'\gamma$. Otherwise we say that α is *disagreeable* for $[v]_\ell$.

We say that $[v]_\ell$ is *disagreeable* if there is an $N > 0$ such that for all $n > N$ there is an $\alpha \in \mathscr{L}^*(E)$ with $|\alpha| \geq n$ that is disagreeable for $[v]_\ell$.

The labelled space $(E, \mathscr{L}, \mathscr{E}^{0,-})$ is *disagreeable* if for every $v \in E^0$ there is an $L_v > 0$ such that $[v]_\ell$ is disagreeable for all $\ell > L_v$.

REMARK 5.2. Suppose that $[v]_p = \bigcup_{i=1}^m [w_i]_q$, where $q > p$. Then $L_v \geq L_{w_i}$ for all $i \in \{1, \ldots, m\}$ since each $w_i \in [v]_p$.

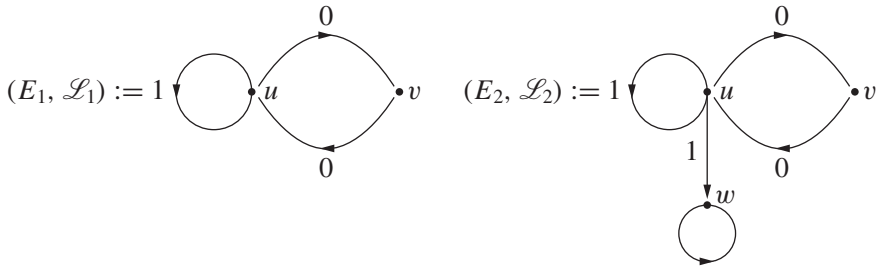
The following Lemma shows that the notion of disagreeability reduces to condition (L) for directed graphs and so is the appropriate condition for us to use in our Cuntz-Krieger uniqueness theorem and simplicity results.

LEMMA 5.3. *The directed graph E satisfies condition* (L) *if and only if the labelled space $(E, \mathscr{L}_t, \mathscr{E}^{0,-})$ is disagreeable.*

PROOF. Suppose that E satisfies condition (L). Observe that for all $\ell \geq 1$ and all $v \in E^0$, $[v]_\ell = \{v\}$. We show that every $v \in E^0$ is disagreeable. Let $L_v = 1$, $N = 1$, fix $n > N$ and $\ell > L_v$. If v does not lie on a loop, then any path α with $|\alpha| \geq n$ is disagreeable for $[v]_\ell = \{v\}$. If v does lie on a loop $\alpha = \alpha_1 \ldots \alpha_m$, without loss of generality we may assume that $s(\alpha) = v$. Since E satisfies condition (L) there is a path β with $s(\beta) = v$ and $\beta_{|\beta|} \notin \{\alpha_1, \ldots, \alpha_m\}$. The path $\alpha^n\beta$ has length $\geq n$ and is disagreeable for $[v]_\ell$.

Suppose E does not satisfy condition (L). Then there is a $v \in E^0$ and a simple loop α with $s(\alpha) = v$ that has no exit. Let $N > 0$. Then there is an n such that $|\alpha^n| > N$. Suppose $n \geq 2$. We claim that $\lambda = \alpha^n$ is agreeable for every $\ell > |\alpha|$. Set $\beta = \gamma = \alpha$ and $\lambda' = \alpha^{n-1}$. Since $\lambda = \beta\lambda' = \lambda'\gamma$ where $|\beta| = |\gamma| \leq \ell$ it follows that $[v]_\ell = \{v\}$ is agreeable for ℓ. Since α^n is the only path of length $n|\alpha|$ emitted by v, it follows that v is not disagreeable. Thus the labelled space $(E, \mathscr{L}, \mathscr{E}^{0,-})$ is not disagreeable.

EXAMPLES 5.4. (i) Recall from [3, Examples 3.3 (iii)] the labelled graphs



are set-finite, receiver set-finite, left-resolving presentations of the even shift.

Consider $(E_1, \mathscr{L}_1)$. We claim that $(E_1, \mathscr{L}_1, \mathscr{E}_1^{0,-})$ is disagreeable. Now for all $\ell \geq 1$ we have $[u]_\ell = \{u\}$. Let $L_u = 1$ and $N = 3$. Then for $n > N$ the labelled path $\alpha_n = 11^n0$ satisfies $|\alpha_n| = n + 2 \geq N$ and α_n is disagreeable for $[u]_\ell$ as its first and last symbols disagree. Also for all $\ell \geq 1$ we have $[v]_\ell = \{v\}$. If we let $N = 4$ and $L_v = 1$, then for each $n > N$ the path $\alpha_n = 0^{2n+1}1$ satisfies $|\alpha_n| = 2n + 2 \geq n$ and α_n is disagreeable for $[v]_\ell$ as

its first and last symbols disagree. Thus the labelled space $(E_1, \mathscr{L}_1, \mathscr{E}_1^{0,-})$ is disagreeable and our claim is established.

Consider $(E_2, \mathscr{L}_2)$. We claim that $[w]_\ell$ is agreeable for all $\ell \geq 2$. Now $[w]_\ell = \{w\}$ for all $\ell \geq 2$, and any labelled path α satisfying $s(\alpha) \cap [w]_\ell \neq \emptyset$ must have the form $\alpha = 0^n$ for some n. But $\alpha = 0^n$ is agreeable for $[w]_\ell$ for all $\ell \geq 2$ whenever $n \geq \ell + 1$: set $\alpha' = 0^{n-\ell}$, $\beta = \gamma = 0$. Thus $(E_2, \mathscr{L}_2, \mathscr{E}_2^{0,-})$ is not disagreeable.

(ii) Let G be a group with a finite set of generators $S = \{g_1, \ldots, g_m\}$, such that $g_i \neq g_j$ for $i \neq j$. The (right) *Cayley graph of G with respect to S* is the essential row-finite directed graph $E_{G,S}$ where $E^0_{G,S} = G$, $E^1_{G,S} = G \times S$ with range and source maps given by $r(h, g_i) = hg_i$ and $s(h, g_i) = h$ for $i = 1, \ldots, m$. The map $\mathscr{L}_{G,S}(h, g_i) = g_i$ gives us a set-finite, receiver set-finite, labelled graph $(E_{G,S}, \mathscr{L}_{G,S})$. Since G is cancelative it follows that $(E_{G,S}, \mathscr{L}_{G,S})$ is left resolving. As each vertex in $E_{G,S}$ receives the same labelled paths it follows that $[g]_\ell = G$ for all $g \in G$ and $\ell \geq 1$ and so $\mathscr{E}^{0,-}_{G,S} = \{\emptyset, G\}$. Each $g \in G$ emits the same m^ℓ labelled paths of length ℓ. So if $m = |S| > 1$, it follows that for all $[g]_\ell = G$ there is a disagreeable labelled path of length $n > 1$ beginning at $[g]_\ell = G$. Hence $(E_{G,S}, \mathscr{L}_{G,S}, \mathscr{E}^{0,-}_{G,S})$ is disagreeable.

THEOREM 5.5. *Let $(E, \mathscr{L}, \mathscr{E}^{0,-})$ be a disagreeable labelled space. If $\{T_\alpha, Q_A\}$ and $\{S_\alpha, P_A\}$ are two representations of $(E, \mathscr{L}, \mathscr{E}^{0,-})$ in which all the projections p_A, P_A are nonzero, then there is an isomorphism ϕ of $C^*(T_\alpha, Q_A)$ onto $C^*(S_\alpha, P_A)$ such that $\phi(T_\alpha) = S_\alpha$ and $\phi(Q_A) = P_A$.*

To prove this theorem we show that the representations $\pi_{T,Q}$ and $\pi_{S,P}$ of $C^*(E, \mathscr{L}, \mathscr{E}^{0,-})$ are faithful. The required isomorphism will then be $\phi = \pi_{S,P} \circ \pi^{-1}_{T,Q}$. The usual approach is to invoke symmetry and prove that

(a) $\pi_{S,P}$ is faithful on $C^*(E, \mathscr{L}, \mathscr{E}^{0,-})^\gamma$ and

(b) $\|\pi_{S,P}(\Phi(a))\| \leq \|\pi_{S,P}(a)\|$ for all $a \in C^*(E, \mathscr{L}, \mathscr{E}^{0,-})$.

Part (a) is proved in [3, Theorem 5.3]. To prove (b) we must do a little more work than is needed for graph C^*-algebras because of the more complicated structure of $C^*(E, \mathscr{L}, \mathscr{E}^{0,-})^\gamma$ as is discussed in section 4.

PROOF. By Proposition 3.4 every element of $C^*(E, \mathscr{L}, \mathscr{E}^{0,-})$ may be approximated by elements of the form

$$a = \sum_{(\alpha,[w]_\ell,\beta)\in F} c_{\alpha,[w]_\ell,\beta} s_\alpha p_{[w]_\ell} s^*_\beta$$

where F is finite, and so it is enough to prove (b) for such elements a.

Let $k = \max\{|\alpha|, |\beta| : (\alpha, [w]_\ell, \beta) \in F\}$. By Proposition 3.4 we may suppose (changing F if necessary), that every $(\alpha, [w]_\ell, \beta) \in F$ is such that

$\min\{|\alpha|, |\beta| : (\alpha, [w]_\ell, \beta) \in F\} = k$. Let $M = \max\{|\alpha|, |\beta| : (\alpha, [w]_\ell, \beta) \in F\}$ and $L = \max\{L_w : (\alpha, [w]_\ell, \beta) \in F\}$. By Remark 5.2 and Proposition 2.4(iii) we may suppose (again changing F if necessary, but not M or k) that $\ell \geq \max\{L, M-k\}$.

Since $|\alpha| = |\beta|$ implies that $|\alpha| = k$ we have

$$\Phi(a) = \sum_{(\alpha,[w]_\ell,\beta)\in F, |\alpha|=|\beta|} c_{\alpha,[w]_\ell,\beta} s_\alpha p_{[w]_\ell} s_\beta^* \in \mathscr{F}^k(\ell)$$

where Φ is the conditional expectation of $C^*(E, \mathscr{L}, \mathscr{E}^{0,-})$ onto $C^*(E, \mathscr{L}, \mathscr{E}^{0,-})^\gamma$. By Proposition 4.3(i) $\mathscr{F}^k(\ell)$ decomposes as the C^*-algebraic direct sum $\bigoplus_{[w]_\ell} \mathscr{F}^k([w]_\ell)$, so does its image under $\pi_{S,P}$, and there is a $[v]_\ell \in \Omega_\ell$ such that $\|\pi_{S,P}(\Phi(a))\|$ is attained on $\mathscr{F}^k([v]_\ell)$. Let $F_{[v]_\ell}$ denote the elements of F of the form $(\alpha, [v]_\ell, \beta)$, then we have

$$\|\pi_{S,P}(\Phi(a))\| = \left\| \sum_{(\alpha,[v]_\ell,\beta)\in F_{[v]_\ell}, |\alpha|=|\beta|} c_{\alpha,[v]_\ell,\beta} S_\alpha P_{[v]_\ell} S_\beta^* \right\|.$$

We write

$$b_v = \sum_{(\alpha,[v]_\ell,\beta)\in F_{[v]_\ell}, |\alpha|=|\beta|} c_{\alpha,[v]_\ell,\beta} S_\alpha P_{[v]_\ell} S_\beta^*$$

and let $G = \{\alpha : \text{either } (\alpha, [v]_\ell, \beta) \in F_{[v]_\ell} \text{ or } (\beta, [v]_\ell, \alpha) \in F_{[v]_\ell} \text{ with } |\alpha| = |\beta|\}$. Then $\operatorname{span}\{S_\alpha P_{[v]_\ell} S_\beta^* : \alpha, \beta \in G\}$ is a finite dimensional matrix algebra containing b_v.

Since $\ell > L$, $[v]_\ell$ is disagreeable. Hence there is an $n > M$ and a λ with $|\lambda| \geq n$ and $[v]_\ell \cap s(\lambda) \neq \emptyset$ which has no factorisation $\lambda = \lambda'\lambda'' = \lambda''\gamma$ for $\lambda', \gamma \in \mathscr{L}^{\leq(M-k)}(E)$ (as $M - k \leq \ell$). We claim that

$$Q = \sum_{\nu\in G} S_{\nu\lambda} P_{r([v]_\ell,\lambda)} S_{\nu\lambda}^*$$

is such that

$$\|Q\pi_{S,P}(\Phi(a))Q\| = \|\pi_{S,P}(\Phi(a))\|, \tag{9}$$

and

$$QS_\alpha P_{[v]_\ell} S_\beta^* Q = 0 \quad \text{when } (\alpha, [v]_\ell, \beta) \in F \text{ and } |\alpha| \neq |\beta|. \tag{10}$$

The formula for Q can be made sense of by a calculation similar to the one in Remark 3.4. A routine calculation verifies (9).

Now suppose that $(\alpha, [v]_\ell, \beta) \in F$ satisfies $|\alpha| \neq |\beta|$. Either α or β has length k, say $|\alpha| = k$. As before, $S^*_{\nu\lambda} S_\alpha$ is non-zero if and only if $\nu = \alpha$. Thus

$$\begin{aligned} Q S_\alpha P_{[v]_\ell} S^*_\beta Q &= \sum_{\nu \in G} S_{\alpha\lambda} P_{r([v]_\ell,\lambda)} S^*_{\alpha\lambda} S_\alpha P_{[v]_\ell} S^*_\beta S_{\nu\lambda} P_{r([v]_\ell,\lambda)} S^*_{\nu\lambda} \\ &= \sum_{\nu \in G} S_{\alpha\lambda} P_{r([v]_\ell,\lambda)} S^*_{\beta\lambda} S_{\nu\lambda} P_{r([v]_\ell,\lambda)} S^*_{\nu\lambda}. \end{aligned}$$

For $P_{r([v]_\ell,\lambda)} S^*_{\beta\lambda} S_{\nu\lambda} P_{r([v]_\ell,\lambda)}$ to be non-zero $\beta\lambda$ must extend $\nu\lambda$, which implies that $\beta\lambda = \nu\lambda\gamma$ for some γ. But then we have $\beta = \nu\lambda'$ for some initial segment λ' of λ as $|\beta| > |\nu|$. Hence $\lambda = \lambda'\lambda''$ which then implies that $\lambda = \lambda''\gamma$ as

$$\beta\lambda = \nu\lambda'\lambda'' = \nu\lambda\gamma = \nu\lambda'\lambda''\gamma$$

and that $|\lambda'| = |\gamma|$. Since $|\beta| \leq M$ and $|\nu| = k$ it follows that $|\lambda'| \leq M - k \leq \ell$. Thus λ is agreeable for $[v]_\ell$, a contradiction. Thus $Q S_\alpha P_{[v]_\ell} S^*_\beta Q = 0$, and we have verified (10).

The rest of the proof is now standard (see, for example, [20, p. 31]).

6. Simplicity and Pure Infiniteness

Recall from [10, Corollary 6.8] that a directed graph E is *cofinal* if for all $x \in E^\infty$ and $v \in E^0$ there is a path $\lambda \in E^*$ and $N \geq 1$ such that $s(\lambda) = v$ and $r(\lambda) = r(x_N)$. Along with condition (L), cofinality is the key hypothesis in the simplicity results for directed graphs (see [10, Corollary 6.8], [1, Proposition 5.1]).

In this section we seek an analogue for cofinality in the context of labelled graphs which will allow us to prove a simplicity theorem for labelled graph C^*-algebras. The two key difficulties to overcome in the context of labelled graphs are that we must accommodate the generalised vertices $[v]_\ell$ in a labelled graph and the fact that there may be many representatives of a given infinite labelled path $x \in \mathsf{X}^+_{E,\mathscr{L}}$.

DEFINITIONS 6.1. Let $(E, \mathscr{L}, \mathscr{E}^{0,-})$ be a labelled space and $\ell \geq 1$. We say that $(E, \mathscr{L}, \mathscr{E}^{0,-})$ is *ℓ-cofinal* if for all $x \in \mathsf{X}^+_{E,\mathscr{L}}$, $[v]_\ell \in \Omega_\ell$, and $w \in s(x)$ there is an $R(w) \geq \ell$, an $N \geq 1$ and a finite number of labelled paths $\lambda_1, \ldots, \lambda_m$ such that for all $d \geq R(w)$ we have $\bigcup_{i=1}^m r([v]_\ell, \lambda_i) \supseteq r([w]_d, x_1 \ldots x_N)$.

We say that $(E, \mathscr{L}, \mathscr{E}^{0,-})$ is *cofinal* if there is an $L > 0$ such that $(E, \mathscr{L}, \mathscr{E}^{0,-})$ is *ℓ-cofinal* for all $\ell > L$.

EXAMPLES 6.2. (i) Recall from Example 2.5 that a directed graph E may be considered to be a labelled graph with the trivial labelling $\mathscr{L}_t$. Let E be a

cofinal directed graph and fix $v \in E^0$, $x \in E^\infty$. Since $w = s(x)$ is the only vertex with $r(w, x_1 \dots x_n) \neq \emptyset$ for all n, we may put $R(w) = 1$ and invoke cofinality of E to get the required N and λ so that $(E, \mathscr{L}_t, \mathscr{E}^{0,-})$ is cofinal with $L = 1$. Thus the definition of cofinality for labelled graphs reduces to the usual definition of cofinality for directed graphs.

(ii) The labelled space $(E_2, \mathscr{L}_2, \mathscr{E}_2^{0,-})$ of Example 5.4(i) is not ℓ-cofinal for $\ell \geq 2$, and so not cofinal. To see this, observe that $[w]_\ell = \{w\}$ for $\ell \geq 2$ and there is no labelled path joining w to the infinite path $(100)^\infty$.

The following result will allow us to prove cofinality for many interesting examples.

LEMMA 6.3. *Let $(E, \mathscr{L}, \mathscr{E}^{0,-})$ be a labelled space. If E is row-finite, transitive and $\mathscr{E}^{0,-}$ contains $\{v\}$ for all $v \in E^0$ then $(E, \mathscr{L}, \mathscr{E}^{0,-})$ is cofinal with $L = 1$.*

PROOF. Let $w \in E^0$. Since $\{w\} \in \mathscr{E}^{0,-}$ there must be an $R(w) \geq 1$ such that $[w]_d = \{w\}$ for all $d \geq R(w)$.

Let $\ell \geq 1$ and choose $[v]_\ell \in \Omega_\ell$. Let $w \in E^0$, and choose $R(w)$ as in the first paragraph. Let $x \in \mathsf{X}^+_{E,\mathscr{L}}$ be such that $w \in s(x)$. Let $N \geq 1$. Then as E is row-finite there are only finitely many paths $\mu_1, \dots, \mu_m$ in E with $s(\mu_i) = w$ and $\mathscr{L}(\mu_i) = x_1 \dots x_N$. By transitivity of E there are paths $\lambda_1, \dots, \lambda_m \in E^*$ with $s(\lambda_i) = v$ and $r(\lambda_i) = r(\mu_i)$. Then

$$\bigcup_{i=1}^{m} r([v]_\ell, \mathscr{L}(\lambda_i)) \supseteq r([w]_d, x_1 \dots x_N)$$

as required. Thus $(E, \mathscr{L}, \mathscr{E}^{0,-})$ is cofinal with $L = 1$.

THEOREM 6.4. *Let $(E, \mathscr{L}, \mathscr{E}^{0,-})$ be cofinal and disagreeable. Then $C^*(E, \mathscr{L}, \mathscr{E}^{0,-})$ is simple.*

PROOF. Since every ideal in a C^*-algebra is the kernel of a representation, it suffices to prove that every non-zero representation $\pi_{S,P}$ of $C^*(E, \mathscr{L}, \mathscr{E}^{0,-})$ is faithful. Suppose $\pi_{S,P}$ is a non-zero representation of $C^*(E, \mathscr{L}, \mathscr{E}^{0,-})$. If we have $P_{[v]_\ell} = 0$ for all $v \in E^0$ and $\ell \geq 1$ then $\pi_{S,P} = 0$. Thus there is a $w \in E^0$ and a $d \geq 1$ with $P_{[w]_d} \neq 0$. Fix $[v]_\ell \in \Omega_\ell$. We aim to prove that $P_{[v]_\ell} \neq 0$. Since $[w]_d$ is the disjoint union of finitely many equivalence classes $[w_i]_k$ whenever $k \geq d$, for each k there is an i such that $P_{[w_i]_k} \neq 0$. So without loss of generality, for a given $[v]_\ell \in \Omega_\ell$, we may assume that $d \geq R(w)$.

Since $(E, \mathscr{L}, \mathscr{E}^{0,-})$ is set-finite we apply (4) of Proposition 3.4 to obtain

$$P_{[w]_d} = \sum_{x_1 \in \mathscr{L}([w]_d E^1)} S_{x_1} P_{r([w]_d, x_1)} S_{x_1}^*.$$

Since the left-hand side is nonzero it follows that $S_{x_1} P_{r([w]_d,x_1)} S_{x_1}^* \neq 0$ for some $x_1 \in \mathscr{L}([w]_d E^1)$ which implies that $P_{r([w]_d,x_1)} \neq 0$. Arguing as in the proof of Proposition 3.4 we have

$$P_{r([w]_d,x_1)} = \sum_{x_2 \in \mathscr{L}(r([w]_d,x_1)E^1)} S_{x_2} P_{r(r([w]_d,x_1),x_2)} S_{x_2}^*$$

and so we may deduce that there is an x_2 with $P_{r(r([w]_d,x_1),x_2)} = P_{r([w]_d,x_1x_2)} \neq 0$. Continuing in this way we produce $x = x_1x_2 \ldots \in \mathsf{X}_{E,\mathscr{L}}^+$ such that $P_{r([w]_d,x_1\ldots x_n)} \neq 0$ for all $n \geq 1$.

Let $\ell \geq 1$ and $[v]_\ell \in \Omega_\ell$. Since $d > R(w)$, by cofinality, there are finitely many labelled paths $\lambda_1, \ldots, \lambda_m$ and an $N \geq 1$ such that $\bigcup_{i=1}^m r([v]_\ell, \lambda_i) \supseteq r([w]_d, x_1 \ldots x_N)$. Since $P_{r([w]_d,x_1,\ldots,x_N)} \neq 0$ we must have $P_{r([v]_\ell,\lambda_i)} \neq 0$ for some $i \in \{1, \ldots, m\}$. Since $r([v]_\ell, \lambda_i) \subseteq r(\lambda_i)$ it then follows that $P_{r(\lambda_i)} \neq 0$ and hence $S_{\lambda_i} \neq 0$. Since $P_{[v]_\ell} = \sum_{\lambda \in \mathscr{L}([v]_\ell E^{|\lambda_i|})} S_\lambda P_{r([v]_\ell,\lambda)} S_\lambda^*$ it then follows that $P_{[v]_\ell} \neq 0$ as required.

Thus all the projections $P_{[v]_\ell}$ are non-zero and Theorem 5.5 implies that $\pi_{S,P}$ is faithful, completing our proof.

EXAMPLES 6.5. (i) The labelled space $(E_1, \mathscr{L}_1, \mathscr{E}_1^{0,-})$, shown to be disagreeable in Examples 5.4(i) is cofinal with $L = 1$. This follows by Lemma 6.3(i) since E_1 is row-finite, transitive and $\{v\} \in \mathscr{E}_1^{0,-}$ for all $v \in E_1^0$. Hence $C^*(E_1, \mathscr{L}_1, \mathscr{E}_1^{0,-})$ is simple by Theorem 6.4.

(ii) The labelled space $(E_{G,S}, \mathscr{L}_{G,S}, \mathscr{E}_{G,S}^{0,-})$ of Examples 5.4(ii) is cofinal with $L = 1$. To see this recall that $[g]_\ell = E_{G,S}^0 = G$ for all $\ell \geq 1$. Fix $[g]_\ell \in \Omega_\ell$ and $x \in \mathsf{X}_{E_{G,S},\mathscr{L}_{G,S}}^+$. For $h \in G$, $d \geq R(h) = 1$ and $n = 1$ we have $r([h]_d, x_1) = G$. Let λ_1 be any element of S, then $r([g]_\ell, \lambda_1) = G = r([h]_d, x_1)$. Hence $C^*(E_{G,S}, \mathscr{L}_{G,S}, \mathscr{E}_{G,S}^{0,-})$ is simple by Theorem 6.4.

We now turn our attention to the question of pure infiniteness for simple labelled graph C^*-algebras. For graph C^*-algebras the key hypotheses are condition (L) and every vertex connects to a loop (see [11, Theorem 3.9], [1, Proposition 5.4]). As we already have an analogue of condition (L), we must now seek to find a suitable replacement for the requirement that every vertex connects to a loop in the context of labelled graphs. Again, there are two difficulties to overcome: we must accommodate the generalised vertices $[v]_\ell$ in a labelled graph and find the correct analogue of a loop.

DEFINITIONS 6.6. The labelled path α is *repeatable* if $\alpha^n \in \mathscr{L}^*(E)$ for all $n \geq 1$. We say that *every vertex connects to a repeatable labelled path* if for every $[v]_m \in \Omega_m$ there is a $w \in E^0$, $L(w) \geq 1$ and labelled paths $\alpha, \delta \in \mathscr{L}^*(E)$ with $w \in r([v]_m, \delta\alpha)$ such that $[w]_\ell \subseteq r([w]_\ell, \alpha)$ for all $\ell \geq L(w)$.

REMARK 6.7. The requirement that $[w]_\ell \subseteq r([w]_\ell, \alpha)$ for all $\ell \geq L(w)$ ensures that α is repeatable, $\delta\alpha^i \in \mathscr{L}^*(E)$ for all $i \geq 1$ and that $r([w]_\ell, \alpha^i) \neq \emptyset$ for all sufficiently large ℓ.

Our proof of the pure infiniteness result requires the following lemma whose proof follows along similar lines to that of [1, Lemma 5.4].

LEMMA 6.8. *Let $(E, \mathscr{L}, \mathscr{E}^{0,-})$ be a labelled space, $v \in E^0$ and $\ell \geq 1$. Let t be a positive element of $\mathscr{F}^k([v]_\ell)$. Then there is a projection r in the C^*-subalgebra of $\mathscr{F}^k([v]_\ell)$ generated by t such that $rtr = \|t\|r$.*

THEOREM 6.9. *Let $(E, \mathscr{L}, \mathscr{E}^{0,-})$ be cofinal and disagreeable. If every vertex connects to a repeatable labelled path then $C^*(E, \mathscr{L}, \mathscr{E}^{0,-})$ is simple and purely infinite.*

PROOF. We know that $C^*(E, \mathscr{L}, \mathscr{E}^{0,-})$ is simple by Theorem 6.4. We show that every hereditary subalgebra A of $C^*(E, \mathscr{L}, \mathscr{E}^{0,-})$ contains an infinite projection; indeed we shall produce one which is dominated by a fixed positive element $a \in A$ with $\|\Phi(a)\| = 1$.

By Proposition 3.4 we may choose a positive element $b \in \operatorname{span}\{s_\alpha p_{[v]_\ell} s_\beta^* : \alpha, \beta \in \mathscr{L}^*(E), [v]_\ell \in \Omega_\ell\}$ such that $\|a - b\| < \frac{1}{4}$. Suppose $b = \sum_{(\alpha,[w]_\ell,\beta)\in F} c_{\alpha,[w]_\ell,\beta} s_\alpha p_{[w]_\ell} s_\beta^*$ where F is a finite subset of $\mathscr{L}^*(E) \times \Omega \times \mathscr{L}^*(E)$. The element $b_0 := \Phi(b)$ is positive and satisfies $\|b_0\| \geq \frac{3}{4}$.

Let $k = \max\{|\alpha|, |\beta| : (\alpha, [w]_\ell, \beta) \in F\}$. By repeatedly applying (4) we may suppose (changing F if necessary) that $\min\{|\alpha|, |\beta| : (\alpha, [w]_\ell, \beta) \in F\} = k$. Let $M = \max\{|\alpha|, |\beta| : (\alpha, [w]_\ell, \beta) \in F\}$, $L_F = \max\{L_w : (\alpha, [w]_\ell, \beta) \in F\}$ and let L be the smallest number such that $(E, \mathscr{L}, \mathscr{E}^{0,-})$ is ℓ-cofinal for all $\ell \geq L$. Then from Proposition 2.4 and Remark 5.2 we may assume that $b_0 \in \bigoplus_{w:(\alpha,[w]_\ell,\beta)\in F} \mathscr{F}^k([w]_m)$ for some $m \geq \max\{L, L_F, M\}$. In fact, $\|b_0\|$ must be attained in some summand $\mathscr{F}^k([v]_m)$. Let b_1 be the component of b_0 in $\mathscr{F}^k([v]_m)$, and note that $b_1 \geq 0$ and $\|b_1\| = \|b_0\|$. By Lemma 6.8 there is a projection $r \in C^*(b_1) \subseteq \mathscr{F}^k([v]_m)$ such that $rb_1r = \|b_1\|r$. Since b_1 is a finite sum of $s_\alpha p_{[v]_m} s_\beta^*$ we can write r as a sum $\sum c_{\alpha\beta} s_\alpha p_{[v]_m} s_\beta^*$ over all pairs of paths in

$$S = \{\alpha \in \mathscr{L}(E^k) : \text{either } (\alpha, [w]_\ell, \beta) \in F \text{ or } (\beta, [w]_\ell, \alpha) \in F \text{ and } [w]_\ell \subseteq r(\alpha)\}.$$

As $m \geq L_v$, $[v]_m$ is disagreeable and there is an $n > M$ and a $\lambda \in \mathscr{L}^*(E)$ with $|\lambda| \geq n$ which is disagreeable for $[v]_m$. Since $m \geq M \geq M - k$ as well we may employ the same argument as in the proof of Theorem 5.5 to produce a projection $Q := \sum_{\gamma\in S} s_{\gamma\lambda} p_{r([v]_m,\lambda)} s_{\gamma\lambda}^*$ such that $Qs_\alpha p_{[v]_m} s_\beta^* Q = 0$ unless

$|\alpha| = |\beta| = k$ and $[v]_m \subseteq r(\alpha) \cap r(\beta)$. Since $r \in C^*(b_1)$ we have

$$r = \sum c_{\alpha\beta} s_\alpha p_{[v]_m} s_\beta^* = \sum c_{\alpha\beta} s_\alpha (s_\lambda p_{r([v]_m,\lambda)} s_\lambda^* + (p_{[v]_m} - s_\lambda p_{r([v]_m,\lambda)} s_\lambda^*)) s_\beta^*$$
$$\geq Q$$

so that

$$QbQ = Qb_0Q = Qrb_1rQ = \|b_1\| rQ = \|b_0\| Q \geq \frac{3}{4} Q.$$

Since $\|a - b\| \leq \frac{1}{4}$ we have $QaQ \geq QbQ - \frac{1}{4}Q \geq \frac{1}{2}Q$ and so QaQ is invertible in $QC^*(E, \mathscr{L}, \mathscr{E}^{0,-})Q$. Let c denote its inverse and put $v = c^{1/2} Q a^{1/2}$. Then $vv^* = c^{1/2} QaQc^{1/2} = Q$, and $v^*v = a^{1/2} Qc Q a^{1/2} \leq \|c\| a$ and so v^*v belongs to the hereditary subalgebra A. To finish, we must show that v^*v is an infinite projection.

We wish to find a labelled path β with $r([v]_m, \beta) \neq \emptyset$ whose initial segment is λ and whose terminal segment is a repeatable labelled path. We choose $x \in r([v]_m, \lambda)$. Then $[x]_{m+|\lambda|} \subseteq r([v]_m, \lambda)$ and by hypothesis $[x]_{m+|\lambda|}$ connects to a repeatable path: That is, there is a $w \in E^0$, $L(w) \geq 1$ and paths $\alpha, \delta \in \mathscr{L}^*(E)$ such that $w \in r([x]_{m+|\lambda|}, \delta\alpha)$, and $[w]_n \subseteq r([w]_n, \alpha)$ for all $n \geq L(w)$. The required path is $\beta = \lambda\delta\alpha$. Let $N = \max\{L_w, L(w)\}$. We claim that $p_{[w]_n}$ is an infinite projection for all $n \geq N$. As $n \geq L(w)$, we know from Remark 6.7 that we have $r([w]_n, \alpha^i) \neq \emptyset$, for $i \geq 1$. Moreover, as $n \geq L_w$ we know that $[w]_n$ is disagreeable. Hence there must be a labelled path γ with $[w]_n \cap s(\gamma) \neq \emptyset$ and $i \geq 1$ with $|\gamma| = |\alpha^i|$, and $\gamma \neq \alpha^i$. We compute

$$p_{[w]_n} \leq s_{\alpha^i} p_{r([w]_n,\alpha^i)} s_{\alpha^i}^* < s_{\alpha^i} p_{r([w]_n,\alpha^i)} s_{\alpha^i}^* + s_\gamma p_{r([w]_n,\gamma)} s_\gamma^* \leq p_{[w]_n}$$

and our claim is established.

We now demonstrate the existence of an infinite subprojection of Q. If μ is any labelled path with $|\mu| = k \leq M \leq m$ and $r(\mu) \cap s(\lambda) \cap [v]_m \neq \emptyset$ then for $n \geq N$ such that $[w]_n \subseteq r([v]_m, \lambda\delta\alpha)$ (note that such an n exists as $[w]_n \subseteq r([v]_m, \lambda\delta\alpha)$ for all sufficiently large n) we have

$$p_{[w]_n} = p_{[w]_n} s_{\mu\lambda\delta\alpha}^* s_{\mu\lambda\delta\alpha} \sim s_{\mu\lambda\delta\alpha} p_{[w]_n} s_{\mu\lambda\delta\alpha}^* \leq s_{\mu\lambda} p_{r([v]_m,\lambda)} s_{\mu\lambda}^*.$$

Because the projection $s_{\mu\lambda} p_{r([v]_m,\lambda)} s_{\mu\lambda}^*$ is a minimal projection in the matrix algebra $\operatorname{span}\{s_{\mu\lambda} p_{r([v]_m,\lambda)} s_{\nu\lambda}^* : \mu, \nu \in S\}$, it is equivalent to a subprojection of Q. It follows that Q is infinite, and, since $Q = vv^* \sim v^*v$ this completes the proof.

EXAMPLES 6.10. (i) In the labelled space $(E_1, \mathscr{L}_1, \mathscr{E}_1^{0,-})$ of Examples 5.4(i) every vertex connects to the repeatable path 0. Since $(E_1, \mathscr{L}_1, \mathscr{E}_1^{0,-})$ is cofi-

nal and disagreeable, $C^*(E_1, \mathscr{L}_1, \mathscr{E}_1^{0,-})$ is simple and purely infinite by Theorem 6.9.

(ii) Suppose that, for a group G, the set S contains (not necessarily distinct) elements $g_1, \ldots, g_n$ such that $g_1 \ldots g_n = 1_G$. Then every vertex in the labelled graph $(E_{G,S}, \mathscr{L}_{G,S})$ of Examples 5.4(ii) connects to the repeatable labelled path $g_1 \ldots g_n$. If in addition we have $|S| > 1$, then by Examples 5.4(ii) and Examples 6.5(ii) $(E_{G,S}, \mathscr{L}_{G,S}, \mathscr{E}_{G,S}^{0,-})$ is cofinal and disagreeable and so $C^*(E_{G,S}, \mathscr{L}_{G,S}, \mathscr{E}_{G,S}^{0,-})$ is simple and purely infinite by Theorem 6.9.

7. Some labelled graph presentations of non-sofic shift spaces

7.1. Dyck Shifts

In this section we associate a labelled graph to a Dyck shift in such a way that the resulting labelled space C^*-algebra is simple and purely infinite.

First we recall the definition of the Dyck shift (see, for example, [19], [18]). Let $N \geq 1$ be a fixed positive integer. The Dyck shift D_N has alphabet $\mathscr{A} = \{\alpha_1, \ldots, \alpha_N, \beta_1, \ldots, \beta_N\}$ where the symbols α_i correspond to opening brackets of type i and the symbols β_i are their respective closing brackets. We say that a word $\gamma_1 \ldots \gamma_n \in \mathscr{A}^*$ is admissible if $\gamma_1 \ldots \gamma_n$ does not contain any substring $\alpha_i \beta_j$ with $i \neq j$. Thus the language of the Dyck shift consists of all strings of properly matched brackets of types $\alpha_1, \ldots \alpha_N$.

The following algorithm gives a labelled graph presentation of a Dyck shift.

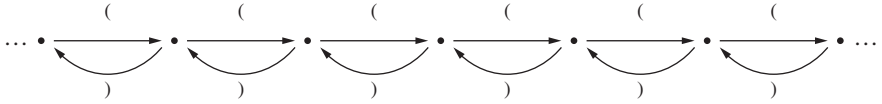
(1) Fix $N \geq 1$ and an alphabet $\{\alpha_1, \ldots, \alpha_N, \beta_1, \ldots, \beta_N\}$.

(2) Draw an unrooted, infinite, directed tree in which every vertex receives one edge and emits N edges (i.e. an N-ary tree). Label the N branches from each node, working from left to right, by $\alpha_1, \ldots, \alpha_N$.

(3) For each $i \in \{1, \ldots, N\}$ and each edge e labelled α_i, draw an edge from $r(e)$ to $s(e)$ with label β_i.

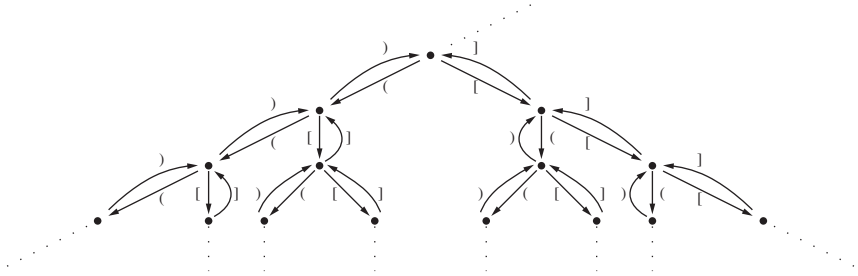
The resulting labelled graph $(E_N, \mathscr{L}_N)$ is a left-resolving labelled graph which presents the Dyck shift D_N.

EXAMPLES 7.1. (1) Let $N = 1$ and $\mathscr{A} = \{(\ ,\ )\}$. The above algorithm gives the following labelled graph presentation of D_1.



Of course, the above labelled graph is not the optimal presentation of D_1, as D_1 has no constraints and so is the full shift on the symbols (and).

(2) Let $N = 2$ and let $\mathscr{A} = \{ (,\ [,\ ),\ ] \}$. The above algorithm gives the following labelled graph presentation of D_2.



We thank W. Krieger for pointing out that the above labelled graph is an a-synchronizing Shannon graph of the Dyck shift (see [8]).

PROPOSITION 7.2. *Let $N \geq 1$ and $\mathscr{A} = \{\alpha_1, \ldots, \alpha_N, \beta_1, \ldots, \beta_N\}$. Then $C^*(E_N, \mathscr{L}_N, \mathscr{E}_N^{0,-})$ is simple and purely infinite.*

PROOF. Let $\Lambda_\ell = \{\lambda_1^\ell, \ldots, \lambda_{N^\ell}^\ell\}$ be the labelled paths of length ℓ which consist of only α_i's (opening braces), and let $\Xi_\ell = \{\mu_1^\ell, \ldots, \mu_{N^\ell}^\ell\}$ be the labelled paths of length ℓ which consist of only β_i's (closing braces), organised in such a way that for all i the word $\lambda_i^\ell \mu_i^\ell$ belongs to the language of D_N. Since every vertex $v \in E_N^0$ receives one opening brace and N closing braces, it follows that v receives a unique $\lambda_i^\ell \in \Lambda_\ell$ one sees that $\Omega_\ell = \{[v_i^\ell]_\ell : i = 1, \ldots, N^\ell\}$ where v_i^ℓ is some vertex in $r(\lambda_i^\ell)$. Moreover, every vertex $v \in E_N^0$ emits exactly one closing brace (the closing version of the one it receives) and N opening braces, so every v which receives λ_i^ℓ also emits μ_i^ℓ.

For $1 \leq i, j \leq N^\ell$ let $\mu_{ij}^\ell = \mu_i^\ell \lambda_j^\ell$ then $s(\mu_{ij}^\ell) = [v_i^\ell]_\ell$ as the only vertices which emit μ_i^ℓ are those which receive λ_i^ℓ. Moreover, we have $r(\mu_{ij}^\ell) = r(\lambda_j^\ell) = [v_j^\ell]_\ell$ since every vertex in E_N^0 (emits the labelled path λ_i^ℓ and hence) receives a labelled path μ_i^ℓ which originates from a vertex in $[v_i^\ell]_\ell$, that it $r([v_i^\ell]_\ell, \mu_i^\ell) = E_N^0$.

Fix $\ell \geq 1$, $[v]_\ell \in \Omega_\ell$ and $x \in \mathsf{X}^+_{E_N, \mathscr{L}_N}$. Without loss of generality suppose that $[v]_\ell = [v_1^\ell]_\ell$. Then by definition of the μ_{ij}^ℓ we have

$$\bigcup_{j=1}^{N^\ell} r([v_1^\ell]_\ell, \mu_{1j}^\ell) = E_N^0$$

and hence the labelled space $(E_N, \mathscr{L}_N, \mathscr{E}_N^{0,-})$ is cofinal with $L = 1$.

We now show that $(E_N, \mathscr{L}_N, \mathscr{E}_N^{0,-})$ is disagreeable. For $n \geq 1$, every vertex v emits the labelled path $\alpha_1^n \beta_1$, which is disagreeable for $[v]_\ell$. Hence $[v]_\ell$ is

disagreeable for all $\ell \geq 1$. It follows that $C^*(E_N, \mathscr{L}_N, \mathscr{E}_N^{0,-})$ is simple by Theorem 6.4.

Since every vertex $v \in E_N^0$ emits the repeatable labelled path $\alpha_1\beta_1$ it follows that every generalised vertex in $(E_N, \mathscr{L}_N, \mathscr{E}_N^{0,-})$ connects to a repeatable labelled path. Thus $C^*(E_N, \mathscr{L}_N, \mathscr{E}_N^{0,-})$ is purely infinite by Theorem 6.9.

REMARK 7.3. The essential symbolic matrix system $(M(E_N)_{\ell,\ell+1}, I(E_N)_{\ell,\ell+1})_{\ell\geq 1}$ associated to $(E_N, \mathscr{L}_N, \mathscr{E}_N^{0,-})$ gives rise to the λ-graph system $\mathfrak{L}^{Ch(D_N)}$ described on [18, p. 5] (for example). Hence by Proposition 3.6 it follows that $C^*(E_N, \mathscr{L}_N, \mathscr{E}_N^{0,-}) \cong \mathcal{O}_{\mathfrak{L}^{Ch(D_N)}}$. Moreover by [18, Proposition 5.1] we know that

$$K_0(C^*(E_N, \mathscr{L}_N, \mathscr{E}_N^{0,-})) \cong \mathsf{Z}/N\mathsf{Z} \oplus C(\mathfrak{K}, \mathsf{Z}) \text{ and } K_1(C^*(E_N, \mathscr{L}_N, \mathscr{E}_N^{0,-})) \cong 0$$

where $C(\mathfrak{K}, \mathsf{Z})$ denotes the abelian group of all Z-valued continuous functions on the Cantor set $\mathfrak{K}$. Since the K-theory of $C^*(E_N, \mathscr{L}_N, \mathscr{E}_N^{0,-})$ is not finitely generated it follows that $C^*(E_N, \mathscr{L}_N, \mathscr{E}_N^{0,-})$ cannot be isomorphic to a unital graph algebra (indeed $C^*(E_N, \mathscr{L}_N, \mathscr{E}_N^{0,-})$ is not semiprojective).

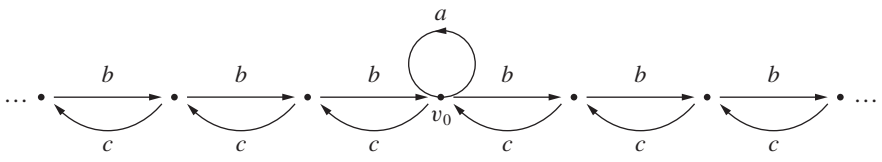
Note that the essential symbolic matrix system $(M(E_N)_{\ell,\ell+1}, I(E_N)_{\ell,\ell+1})_{\ell\geq 1}$ associated to $(E_N, \mathscr{L}_N, \mathscr{E}_N^{0,-})$ is not the same as the one described in [9, Proposition 2.1]. In [9] the λ-graphs associated to symbolic matrix systems are "upward directed" whereas in [16] they are "downward directed". This results from the change of time direction mentioned on [9, p. 81]. Hence to form the appropriate "upward directed" versions for $(E_N, \mathscr{L}_N, \mathscr{E}_N^{0,-})$, it would seem natural to reverse the arrows in E_N.

7.2. A Further Example

Consider the shift space X over the alphabet $\mathscr{A} = \{a, b, c\}$ whose language consists of all words in $\{a, b, c\}$ such that the numbers of b's and c's occurring between any pair of consecutive a's are equal.

Note that the shift X is not sofic: suppose otherwise. Then there is a finite labelled graph $(E_X, \mathscr{L}_X)$ with $|E_X^0| = n$ which presents X. Let α be a path in $(E_X, \mathscr{L}_X)$ which presents $ab^{2n}c^{2n}a$. Then since the number of c's in $\mathscr{L}_X(\alpha)$ is greater than n, α must contain a cycle τ such that $\mathscr{L}_X(\tau) = c^m$ for some $m \leq n$. Write $\alpha = \alpha'\tau\alpha''$. Then $\beta = \alpha'\tau^2\alpha''$ is a path in E_X^* which presents the forbidden word $ab^{2n}c^{2n+m}a$.

The shift X has the following labelled graph presentation $(E_X, \mathscr{L}_X)$:



Since the graph E_X is transitive, it is straightforward to check from the above presentation that X is irreducible.

Since each vertex in E_X to the right (resp. left) of v_0 receives a unique labelled path of the form ab^n (resp. ac^n) it follows that $\{v\} \in \mathscr{E}^{0,-}$ for all $v \in E_X^0$. Since E_X is row-finite it follows that $(E_X, \mathscr{L}_X, \mathscr{E}_X^{0,-})$ is cofinal by Lemma 6.3.

For $n \geq 1$ every $v \in E_X^0$ emits the labelled path $b^n c$, which is disagreeable for $[v]_\ell$. Hence $[v]_\ell$ is disagreeable for all $\ell \geq 1$ and so $C^*(E_X, \mathscr{L}_X, \mathscr{E}_X^{0,-})$ is simple by Theorem 6.4.

Every $v \in E_X^0$ emits the repeatable path bc and since E_X is transitive, it follows that every generalised vertex connects to a repeatable path. Thus $C^*(E_X, \mathscr{L}_X, \mathscr{E}_X^{0,-})$ is purely infinite by Theorem 6.9.

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