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THE KADETS 1/4 THEOREM FOR POLYNOMIALS

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Abstract

We determine the maximal angular perturbation of the $(n+1)$th roots of unity permissible in the Marcinkiewicz-Zygmund theorem on L^p means of polynomials of degree at most n. For $p = 2$, the result is an analogue of the Kadets 1/4 theorem on perturbation of Riesz bases of holomorphic exponentials.

1. Introduction

A classical theorem of J. Marcinkiewicz and A. Zygmund generalizes the elementary mean value formula

$$(1)\qquad \frac{1}{n+1}\sum_{j=0}^{n}\big|P\big(e^{i\frac{2\pi j}{n+1}}\big)\big|^2 = \int_0^{2\pi} |P(e^{i\theta})|^2\,\frac{d\theta}{2\pi},$$

valid for holomorphic polynomials P of degree at most n, in the following way: For $1 < p < \infty$, there is a constant C_p independent of n such that

$$(2)\qquad \frac{C_p^{-1}}{n+1}\sum_{j=0}^{n}\big|P\big(e^{i\frac{2\pi j}{n+1}}\big)\big|^p \le \int_0^{2\pi} |P(e^{i\theta})|^p\,\frac{d\theta}{2\pi} \le \frac{C_p}{n+1}\sum_{j=0}^{n}\big|P\big(e^{i\frac{2\pi j}{n+1}}\big)\big|^p$$

for every complex polynomial P of degree at most n. (See [8] or Theorem 7.5 in Chapter X of [15].) It is natural to ask if the norm equivalence expressed by (2) remains valid if we replace the $(n+1)$th roots of unity $\omega_{nj} = e^{i\frac{2\pi j}{n+1}}$ by $n+1$ points z_{nj} on the unit circle with a less regular distribution. C. K. Chui, X.-C. Shen, and L. Zhong [2] considered this problem and found that the norm equivalence is stable under small perturbations of the points ω_{nj}. We will prove the following sharp version of their result:

THEOREM 1.1. *Suppose $1 < p < \infty$ and set $q = \max(p, p/(p-1))$. The following statement holds if and only if $\delta < 1/(2q)$: There is a constant C_p*

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independent of n such that if $|\arg(z_{nj}\overline{\omega_{nj}})| \le 2\pi\delta/(n+1)$ for $0 \le j \le n$, then

$$\frac{C_p^{-1}}{n+1}\sum_{j=0}^{n}|P(z_{nj})|^p \le \int_0^{2\pi}|P(e^{i\theta})|^p\,\frac{d\theta}{2\pi} \le \frac{C_p}{n+1}\sum_{j=0}^{n}|P(z_{nj})|^p \tag{3}$$

for every holomorphic polynomial P of degree at most n.

We will see that this theorem is a consequence of a general result of Chui and Zhong [3], characterizing the so-called L^p Marcinkiewicz-Zygmund families (to be defined below) in terms of Muckenhoupt (A_p) weights.

Readers familiar with Paley-Wiener spaces will see the analogy with the Kadets 1/4 theorem on perturbations of Riesz bases of complex exponentials in L^2 of an interval [4]. One may view polynomials as discrete versions of band-limited functions, with the degree of the polynomial being the counterpart to the notion of "bandwidth". The identity (1) is the discrete analogue of the Plancherel identity or – what amounts to the same – the Shannon formula for bandlimited functions. In the case when $p = 2$ and we require $\delta < 1/4$, our theorem corresponds precisely to the Kadets 1/4 theorem. The L^p version $(1 < p < \infty)$ of the Kadets theorem, analogous to our theorem, can be found in [7].

It is interesting to note that our problem as well as that of the classical Kadets theorem fits into a general theory of unconditional bases in so-called model spaces. (See [13], [10], and [6] for original work and [11] or [14] for more recent expositions.) In particular, the theorem of Chui and Zhong to be used in this note can be obtained from a theorem given in [6]. We refer to [9] for the details of this link and to [12], where the connection between Marcinkiewicz-Zygmund inequalities and model spaces was first mentioned explicitly.

For $p = 2$, the proof to be given below is an adaption of S. Khrushchev's proof of the classical Kadets 1/4 theorem [5], and, for general p, we act in a similar way as was done in [7]. Khrushchev also showed how to obtain other perturbation results, such as a theorem of S. Avdonin [1]. We will confine ourselves to proving the theorem stated above and refer to [9] for the counterpart of Avdonin's theorem as well as other analogues of results for Paley-Wiener spaces and families of complex exponentials.

2. Preliminaries

Suppose that for each nonnegative integer n we are given a set $\mathscr{Z}(n) = \{z_{nj}\}_{j=0}^{n}$ of $n+1$ distinct points on the unit circle. We denote by $\mathscr{Z} = \{\mathscr{Z}(n)\}_{n\ge 0}$ the corresponding triangular family of points. The family $\mathscr{Z}$ is declared to be uniformly separated if there exists a positive number ε such that

$$\inf_{j\neq k} |z_{nj} - z_{nk}| \geq \frac{\varepsilon}{n+1}$$

for every $n \geq 0$.

We will say that $\mathscr{Z}$ is an L^p Marcinkiewicz-Zygmund family if there exists a constant $C_p > 0$ such that for every $n \geq 0$ and complex polynomial P of degree at most n, we have

$$\frac{C_p^{-1}}{n+1}\sum_{j=0}^{n} |P(z_{nj})|^p \leq \int_0^{2\pi} |P(e^{i\theta})|^p d\theta \leq \frac{C_p}{n+1}\sum_{j=0}^{n} |P(z_{nj})|^p. \tag{4}$$

In order to describe such families, we associate with $\mathscr{Z}$ the following generating polynomials

$$F_n(z) = \prod_{j=0}^{n}\left(1 - \frac{n}{n+1}\overline{z_{jn}}z\right).$$

The theorem of Chui and Zhong reads as follows [3].

THEOREM 2.1. *Suppose* $1 < p < \infty$. *The family* $\mathscr{Z} = \{\mathscr{Z}(n)\}_{n\geq 0}$ *of points on the unit circle is an* L^p *Marcinkiewicz-Zygmund family if and only if it is uniformly separated and there exists a constant* K_p *such that*

$$\left(\frac{1}{|I|}\int_I |F_n(e^{i\theta})|^p d\theta\right)^{1/p}\left(\frac{1}{|I|}\int_I |F_n(e^{i\theta})|^{-p/(p-1)} d\theta\right)^{(p-1)/p} \leq K_p \tag{5}$$

for every subarc I *of the unit circle and every* $n \geq 0$.

In other words, the sequence $|F_n|^p$ satisfies a uniform (A_p) condition.

In the proof of the positive part of the $p = 2$ case of our theorem, we will make use of the equivalence between the (A_2) and Helson-Szegő conditions. We will derive the result for $p \neq 2$ from the $p = 2$ case using the following estimate.

LEMMA 2.2. *Let* $\alpha, \kappa > 0$ *be given, and set* $\rho_{\kappa n} = \max(1/2, 1-\kappa/(n+1))$. *If a given triangular family of real numbers* δ_{nj} *satisfies* $\sup_{nj} |\delta_{nj}| < 1/2$, *then*

$$\left|\prod_{j=0}^{n}\left(z - \rho_{\kappa n} e^{\frac{2\pi i(j+\alpha\delta_{nj})}{n+1}}\right)\right| = R_n(z)\left|\prod_{j=0}^{n}\left(z - \rho_{\kappa n} e^{\frac{2\pi i(j+\delta_{nj})}{n+1}}\right)\right|^{\alpha},$$

where $R_n(z)$ *is bounded from above and below by positive constants, independently of* $z \in \mathsf{T}$ *and* $n \geq 0$.

PROOF. Set

$$P_\beta(\theta) = \left|\prod_{j=0}^{n}(e^{i\theta} - \rho_{\kappa n} e^{i\lambda_j(\beta)})\right|, \quad \text{where } \lambda_j(\beta) = \frac{2\pi j}{n+1} + \frac{2\pi\beta\delta_{nj}}{n+1}.$$

We have

$$\log P_\beta(\theta) - \log P_0(\theta) = \operatorname{Re}\sum_{j=0}^{n}\int_{\rho_{\kappa n}e^{i\lambda_j(0)}}^{\rho_{\kappa n}e^{i\lambda_j(\beta)}} \frac{d\xi}{\xi - e^{i\theta}} = \sum_{j=0}^{n}\int_{\lambda_j(0)}^{\lambda_j(\beta)} h(\theta - t)\,dt,$$

where

$$h(t) = \frac{\rho_{\kappa n}\sin t}{1+\rho_{\kappa n}^2 - 2\rho_{\kappa n}\cos t}.$$

By the fundamental theorem of calculus,

$$\log P_\beta(\theta) - \log P_0(\theta)$$
$$= \sum_{j=0}^{n}(\lambda_j(\beta) - \lambda_j(0))h(\theta - \lambda_j(0)) + \sum_{j=0}^{n}\int_{\lambda_j(0)}^{\lambda_j(\beta)}\int_{\lambda_j(0)}^{t} h'(\theta - \tau)\,d\tau dt.$$

We compute $h'(t)$ and find that the absolute value of the latter sum is bounded independently of θ and n. Therefore,

$$\log P_\alpha(\theta) - \log P_0(\theta) = \alpha\sum_{j=0}^{n}\frac{2\pi\delta_{nj}}{n+1}h(\theta - \lambda_j(0)) + b_{n,\alpha}(z)$$
$$= \alpha(\log P_1(\theta) - \log P_0(\theta) - b_{n,1}(z)) + b_{n,\alpha}(z)$$

with uniform bounds on the L^∞ norms of $b_{n,\alpha}$. This gives the result because $P_0(\theta)$ is trivially bounded from above and below by positive constants, independently of $z \in \mathsf{T}$ and $n \geq 0$.

3. Proof of the theorem: Sufficiency

For each set $\mathscr{Z}(n)$, we define $C_p(\mathscr{Z}(n))$ as the minimum of all positive numbers C such that

$$\frac{C^{-1}}{n+1}\sum_{j=0}^{n}|P(z_{nj})|^p \leq \int_0^{2\pi}|P(e^{i\theta})|^p\frac{d\theta}{2\pi} \leq \frac{C}{n+1}\sum_{j=0}^{n}|P(z_{nj})|^p$$

for every complex polynomial P of degree at most n. Among all sets $\mathscr{Z}(n)$ satisfying $|\arg(z_{nj}\overline{\omega_{nj}})| \leq 2\pi\delta/(n+1)$ for $0 \leq j \leq n$, we may choose a set with maximal $C_p(\mathscr{Z}(n))$. From now on, we will assume that the points

$z_{nj} = \omega_{nj} e^{\frac{2\pi i \delta_{nj}}{n+1}}$ constitute a set of points with this extremal property. It suffices to show that the corresponding triangular family is an L^p Marcinkiewicz-Zygmund family. Clearly, this family is uniformly separated when $\delta < 1/(2q)$.

When $p = 2$, condition (5) is equivalent to the following uniform Helson-Szegő condition: There exist sequences u_n and v_n of real functions in $L^\infty(\mathsf{T})$ such that

$$(6) \quad |F_n|^2 = e^{u_n + \widetilde{v}_n} \qquad \text{with} \quad \sup_n \|u_n\|_\infty < \infty \ \text{ and } \ \sup_n \|v_n\|_\infty < \pi/2.$$

Here $v \mapsto \widetilde{v}$ denotes the conjugation operator.

We need two steps in order to identify the appropriate functions u_n and v_n. In the first step, we "pull" the points z_{nj} more deeply into the unit disc. For $\kappa > 0$, we set $\rho_{\kappa n} = \max(1/2, 1 - \kappa/(n+1))$. We define

$$F_{\kappa n}(z) = \prod_{j=0}^{n} (1 - \rho_{\kappa n} \overline{z_{nj}} z).$$

For fixed $\kappa > 0$, we find that

$$|F_n(e^{it})|^2 = e^{u_{\kappa n}(e^{it})} |F_{\kappa n}(e^{it})|^2,$$

with $\sup_n \|u_{\kappa n}\|_\infty < \infty$.

We now move to the second step. Writing

$$B_{\kappa n}(z) = \prod_{j=0}^{n} \frac{z - \rho_{\kappa n} z_{nj}}{1 - \rho_{\kappa n} \overline{z_{nj}} z},$$

we get

$$B_{\kappa n}(z) = z^{n+1} \frac{\overline{F_{\kappa n}(z)}}{F_{\kappa n}(z)} = z^{n+1} \frac{|F_{\kappa n}(z)|^2}{F_{\kappa n}^2(z)}$$

for $|z| = 1$. Since $F_{\kappa n}^2$ is an outer function with $F_{\kappa n}^2(0) = 1$, this means that $F_{\kappa n}^2 = e^{\widetilde{v_{\kappa n}}}$, where

$$v_{\kappa n}(e^{i\theta}) = \int_0^\theta \sum_{j=0}^{n} \frac{1 - \rho_{\kappa n}^2}{|e^{i\eta} - \rho_{\kappa n} z_{nj}|^2} \, d\eta - (n+1)\theta - c$$

and c is any suitable constant. If we set

$$c = \sum_{j=0}^{n} \int_{-2\pi\delta_j}^{0} \frac{1 - \rho_{\kappa n}^2}{|e^{i\eta} - \rho_{\kappa n} \omega_{nj}|^2} \, d\eta,$$

then we may write

$$v_{\kappa n}(e^{i\theta}) = \sum_{j=0}^{n} \int_{0}^{\theta - 2\pi\delta_j} \frac{1-\rho_{\kappa n}^2}{|e^{i\eta} - \rho_{\kappa n}\omega_{nj}|^2}\, d\eta - (n+1)\theta.$$

On the other hand, using that

$$\int_{\theta}^{\theta+2\pi/(n+1)} \sum_{j=0}^{n} \frac{1-\rho_{\kappa n}^2}{|e^{i\eta} - \rho_{\kappa n}\omega_{nj}|^2}\, d\eta = 2\pi$$

and

$$\left| \sum_{j=0}^{n} \frac{1-\rho_{\kappa n}^2}{|e^{i\eta} - \rho_{\kappa n}\omega_{nj}|^2} - (n+1) \right| \le \frac{C(n+1)}{\kappa},$$

we get

$$v_{\kappa n}(e^{i\theta}) = \sum_{j=0}^{n} \int_{\theta}^{\theta - 2\pi\delta_j} \frac{1-\rho_{\kappa n}^2}{|e^{i\eta} - \rho_{\kappa n}\omega_{nj}|^2}\, d\eta + O(\kappa^{-1})$$

when $\kappa \to \infty$. Consequently,

$$\begin{aligned} \|v_{\kappa n}\|_\infty &\le \sup_\theta \int_{\theta}^{\theta+2\pi\delta/(n+1)} \sum_{j=0}^{n} \frac{1-\rho_{\kappa n}^2}{|e^{i\eta} - \rho_{\kappa n}\omega_{nj}|^2}\, d\eta + O(\kappa^{-1}) \\ &= 2\pi\delta + O(\kappa^{-1}). \end{aligned}$$

Assuming $\delta < 1/4$, we now obtain (6) by choosing κ sufficiently large.

Finally, we consider the case $p \neq 2$. We introduce the triangular family given by the sets

$$\mathscr{Z}_{q/2}(n) = \{e^{i\lambda_{nj}(q/2)}\}_{j=0}^{n} \qquad \text{with} \quad \lambda_{nj}(q/2) = \frac{2\pi j}{n+1} + \frac{\pi q \delta_{nj}}{n+1}.$$

If $\delta < 1/(2q)$, then the $p = 2$ case applies. In other words, if we set

$$G_n(z) = \prod_{j=0}^{n} \left(1 - \rho_{\kappa n} e^{-i\lambda_{nj}(q)} z\right),$$

then the functions $|G_n|^2$ meet the uniform (A_2) condition. By Lemma 2.2 and Hölder's inequality, this implies that the functions $|F_n|^p$ satisfy the uniform (A_p) condition.

4. Proof of the theorem: Necessity

We will consider the sets

$$\mathscr{X}(2n) = \left\{e^{2\pi ij/(2n+1)}\right\}_{j=0}^{n} \bigcup \left\{e^{-2\pi i(j-2\delta)/(2n+1)}\right\}_{j=1}^{n},$$

which can be viewed as perturbations of the rotated $(2n+1)$th roots of unity $e^{2\pi\delta/(2n+1)}\omega_{(2n)j}$. Let F_{2n} be the generating polynomial for $\mathscr{X}(2n)$. We set $\phi_n(z) = F_{2n}(z)/(z^{2n+1} - \rho_{2n}^{2n+1})$ and observe that we may write

$$\phi_n(z) = \prod_{j=1}^{n} \frac{z - \rho_{2n} e^{\frac{-2\pi i(j-2\delta)}{2n+1}}}{z - \rho_{2n} e^{\frac{-2\pi ij}{2n+1}}}.$$

We have

$$\log|\phi_n(z)| = \mathrm{Re}(\log\phi_n(z)) = \mathrm{Re}\sum_{j=1}^{n}\int_{\Gamma_{nj}} \frac{d\xi}{\xi - z},$$

where Γ_{nj} is the arc with the parametrization $\Gamma_{nj}(t) = \rho_{2n} e^{\frac{-2\pi ij}{2n+1}} e^{\frac{it}{2n+1}}$, $0 \le t \le 4\delta\pi$. It follows that

$$|\phi_n(e^{it})| \longrightarrow \left|\frac{1-e^{it}}{1+e^{it}}\right|^{2\delta}$$

for $0 < t < \pi$. By Fatou's lemma,

$$\left(\int_0^\pi \left|\frac{1-e^{it}}{1+e^{it}}\right|^{2\delta p} dt\right)\left(\int_0^\pi \left|\frac{1-e^{it}}{1+e^{it}}\right|^{-\frac{2\delta p}{p-1}} dt\right)^{p-1} \le \liminf_n \int_0^\pi |\phi_n|^p \left(\int_0^\pi |\phi_n|^{-\frac{p}{p-1}}\right)^{p-1}.$$

Hence, when $\delta = 1/2q$, the weights $|\phi_n|^p$ do not meet the uniform (A_p) condition, and the same holds for the weights $|F_{2n}|^2$ since the polynomials $z^{2n+1} - \rho_{2n}^{2n+1}$ are uniformly bounded away from 0 for $|z| = 1$.

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