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ON FIXED POINTS OF INVOLUTIONS OF COMPACT RIEMANN SURFACES

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Abstract

We find a bound for the total number of fixed points of k commuting involutions of compact Riemann surfaces and we study its attainment. We also find a bound for such number for a pair of non-commuting involutions in terms of the order of their product and the genus of the surface. Finally, we study its attainment, topological type of the action of such pair and the nature of the locus of corresponding surfaces in Teichmüller space.

1. Introduction

The study of the fixed points of automorphisms of Riemann surfaces allow to better understanding the topology of the action and is related to studies of the Weierstrass points of such surfaces [1], [2], [4], [5], [7], [8], [11].

It is well known that an involution of a compact Riemann surface of genus g have at most $2g+2$ fixed points. Here we show that k commuting involutions of such surface have at most $2g-2+2^{k+1}$ fixed points in total and this bound is achieved for arbitrary k and g for which $g \equiv 1 \mod 2^{k-2}$ and $k \leq (g-1)/2^{k-2}+3$. Then, we prove that two involutions whose product has order n have at most $4(g-1)/n+8$ fixed points in common. Finally, we study the attainment of the last bound, the topological type of the action of such pair and the nature of the locus of corresponding surfaces in Teichmüller space.

2. Preliminaries

Here and in the sequel a Riemann surface will have the genus $g \geq 2$ and will be represented as the orbit space $X = \mathscr{H}/\Gamma$ of the hyperbolic plane $\mathscr{H}$ with respect to the action of some surface Fuchsian group Γ. A group G of its holomorphic automorphisms will be given by an epimorphism $\theta : \Lambda \to G$ with $\ker\theta = \Gamma$, for some Fuchsian group Λ, say with signature $(h; m_1, \ldots, m_r)$. Such group has the presentation:

$$\langle x_1, \ldots, x_r, a_1, b_1, \ldots, a_h, b_h \mid x_1^{m_1}, \ldots, x_r^{m_r}, x_1 \ldots x_r[a_1, b_1] \ldots [a_h, b_h]\rangle.$$

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The integers $m_1, \ldots, m_r$ are the periods and they correspond to the ramification data of the action on $\mathscr{H}$. The generators x_i are elliptic and arbitrary set of elliptic elements satisfying the above relations will be called a set of *canonical elliptic generators* of Λ. The Fuchsian surface group Γ has signature $(g; -)$.

Now each Fuchsian Λ group has a fundamental region whose area $\mu(\Lambda)$ for a group with the above signature is equal to

$$2\pi\left(2h-2+\sum_{i=1}^{r}\left(1-\frac{1}{m_i}\right)\right)$$

and for a subgroup Λ' of Λ we have the following Riemann-Hurwitz index formula

$$[\Lambda : \Lambda'] = \mu(\Lambda')/\mu(\Lambda).$$

Now Macbeath [9] showed that given a group G of automorphisms of a surface so represented the number $\mathrm{F}(\varphi)$ of points of X fixed by $\varphi \in G$ is given by the formula

$$\mathrm{F}(\varphi) = |\mathrm{N}_G(\langle\varphi\rangle)| \sum 1/m_i, \tag{1}$$

where N denotes the normalizer and the sum is taken over those i for which φ is conjugate in G to a power of $\theta(x_i)$.

It is worth to mention that similar formulas exist also for the number of isolated fixed points of automorphisms of non-orientable unbordered Klein surfaces [3], [6].

3. On Fixed points of involutions of Riemann surfaces

We start the section studying the total number of fixed points of k commuting involutions of a Riemann surface of genus g.

THEOREM 3.1. *k commuting involutions of a Riemann surface of genus g have at most $2g-2+2^{k+1}$ fixed points in total and this bound is attained for arbitrary k and g for which $g \equiv 1 \mod 2^{k-1}$ and $k \leq (g-1)/2^{k-2}+3$.*

PROOF. Let $X = \mathscr{H}/\Gamma$, let $\sigma_1, \ldots, \sigma_k$ be commuting involutions and let F be the total number of their fixed points. Let $G = \langle\sigma_1, \ldots, \sigma_k\rangle = \Lambda/\Gamma$ for a Fuchsian group Λ with signature $(h; m_1, \ldots, m_r)$. Then $\mu(\Lambda) \geq 2\pi(-2+r/2)$ and therefore by the Riemann-Hurwitz formula

$$r \leq 4(g-1)/|G|+4. \tag{2}$$

Now $|G| \leq 2^k$ and so (1) and (2) give $F \leq r|G|/2 \leq 2(g-1)+2|G| \leq 2g-2+2^{k+1}$.

To show the attainment of our bound, let Λ be a Fuchsian group with signature $(0; 2, \overset{r}{\ldots}, 2)$, where $r = (g-1)/2^{k-2} + 4$ and $k \leq (g-1)/2^{k-2} + 3$. Let

$$\theta : \Lambda \to G = Z_2^k = \langle \sigma_1, \ldots, \sigma_k \rangle$$

be an epimorphism which, according to k being even or odd, send consecutive canonical elliptic generators into

$$\sigma_1 \ldots \sigma_{k/2}, \sigma_{k/2+1} \ldots \sigma_k, \sigma_1, \ldots, \sigma_{k-1}, \underbrace{\sigma_k, \ldots, \sigma_k}_{r-k-1}$$

or

$$\sigma_1 \ldots \sigma_k, \sigma_1, \ldots, \sigma_{k-1}, \underbrace{\sigma_k, \ldots, \sigma_k}_{r-k}.$$

Then, by (1), for $\Gamma = \ker\theta$, $X = \mathscr{H}/\Gamma$ is a Riemann surface of genus g having k commuting involutions with $2g - 2 + 2^{k+1}$ fixed points in total.

REMARK. We conjecture that the bound from the above theorem hold for non-commuting involutions also.

The remainder of the section will be devoted to pairs of involutions. From the above theorem it follows that two commuting involutions of a Riemann surface of genus g have at most $2g + 6$ fixed points in total. Now we shall deal with two, not necessarily commuting, involutions.

THEOREM 3.2. *Let σ, σ' be two involutions of a Riemann surface X of genus g whose product has order n. Then σ and σ' have at most $4(g-1)/n + 8$ fixed points in total.*

PROOF. Let $X = \mathscr{H}/\Gamma$ and $\langle \sigma, \sigma' \rangle = D_n = \Lambda/\Gamma$, where Λ is a Fuchsian group with signature $(h; m_1, \ldots, m_r)$. Then $r \leq 2(g-1)/n + 4$ by the Riemann-Hurwitz formula. Now for n even, σ' and σ are not conjugate and they have normalizers of order 4. So by (1) they have at most $4(g-1)/n + 8$ fixed points in total. For n odd, σ' and σ are conjugate and they have normalizers of order 2. So by (1) each of them has at most $2(g-1)/n + 4$ fixed points and therefore our result follows.

From the above theorem we obtain the following generalization of a result of Farkas and Kra [2].

COROLLARY 3.3. *Let σ and σ' be p- and q-hyperelliptic involutions of a Riemann surface of genus g and let n be the order of their product. Then*

$$g \leq \frac{n}{n-1}(p+q) + 1.$$

THEOREM 3.4. *The bound from Theorem 3.2 is attained if and only if n divides $g-1$.*

PROOF. Let $\langle\sigma,\sigma'\rangle = \mathrm{D}_n = \Lambda/\Gamma$, where as in the proof of Theorem 3.2, Λ is a Fuchsian group with signature $(h; m_1,\dots,m_r)$, with $m_1,\dots,m_s = 2$ and $m_{s+1},\dots,m_r > 2$.

First let n be even and assume that the bound is attained. If $h \neq 0$, then by the Riemann-Hurwitz formula $s \leq 2(g-1)/n$ and so our involutions have at most $4(g-1)/n$ fixed points in total. Now if $s < r$, then again by the Riemann-Hurwitz formula $s \leq 2(g-1)/n + 8/3$ and so our involutions have at most $4(g-1)/n + 16/3$ fixed points in total, which again is less than the bound. So the bound may be attained only for Λ with signature $(0; 2, \overset{r}{\dots}, 2)$. However now, the product of the odd number of conjugates of x or y is a conjugate of x or y itself and so we see that r is even. But by the Riemann-Hurwitz formula $n(r-4) = 2(g-1)$ which give n to divide $g-1$. Conversely for n dividing $g-1$, let Λ be a Fuchsian group with signature $(0; 2, \overset{r}{\dots}, 2)$, where $r = 2(g-1)/n + 4$ and let $\theta : \Lambda \to \mathrm{D}_n$ be an epimorphism which sends consecutive canonical elliptic generators into

$$\underbrace{\sigma,\dots,\sigma}_{r-2},\sigma',\sigma'.$$

Then σ and σ' are involutions acting on a Riemann surface of genus g, whose product has order n and which have $4(g-1)/n+4$ and 4 fixed points respectively.

Now let n be odd. In the same way as for even n we argue that for the bound to be attained, Λ must have signature $(0; 2, \overset{r}{\dots}, 2)$, where $r = 2(g-1)/n+4$ must be even since the product of an odd number of conjugates of σ can not be trivial. So n divides $g-1$. Conversely for n dividing $g-1$, let Λ be a Fuchsian group with signature $(0; 2, \overset{r}{\dots}, 2)$, where $r = 2(g-1)/n+4$ and let $\theta : \Lambda \to \mathrm{D}_n$ be an epimorphism which sends consecutive canonical elliptic generators into

$$\underbrace{\sigma,\dots,\sigma}_{r/2},\underbrace{\sigma',\dots,\sigma'}_{r/2}.$$

Then σ and σ' are involutions acting on a Riemann surface of genus g, whose product has order n and which have $2(g-1)/n + 4$ fixed points each.

4. Topological type of the action

Recall that two actions given by epimorphisms $\theta, \theta' : \Lambda \to G$ are said to be topologically equivalent if $\theta'\alpha = \beta\theta$ for some automorphisms $\alpha : \Lambda \to \Lambda$ and $\beta : G \to G$.

THEOREM 4.1. *Let k and l be even integers such that $k+l = 4(g-1)/n+8$, where n divides $g-1$ and in addition $k = l$ for odd n. Then up to topological equivalence there is only one action of two involutions on a Riemann surface of genus g, which have k and l fixed points and whose product has order n.*

PROOF. To define such an action it is enough to take a Fuchsian group Λ with signature $(0; 2, \overset{r}{\ldots}, 2)$, where $r = 2(g-1)/n + 4$ and an epimorphism $\theta : \Lambda \to \mathrm{D}_n = \langle a, b\rangle$ sending the canonical elliptic generators into

$$\underbrace{a, \ldots, a}_{k/2}, \underbrace{b, \ldots, b}_{l/2}. \tag{3}$$

Conversely, any action of two involutions a and b on a Riemann surface of genus g whose product has order n and which have the maximum number of fixed points in total is given by an epimorphism $\theta : \Lambda \to \mathrm{D}_n$, for a Fuchsian group Λ, such that each $\theta(x_i)$ is conjugate to a or b. So actually such an action is given by a generating vector $(\theta(x_1), \ldots, \theta(x_r))$, for which each $\theta(x_i)$ is conjugated to a or b and $\theta(x_1)\ldots\theta(x_r) = 1$. Now the actions corresponding to two generating vectors $(\theta(x_1), \ldots, \theta(x_r))$ and $(\theta'(x_1), \ldots, \theta'(x_r))$ are equivalent if and only if

$$\varphi\theta(x_i) = \theta'\psi(x_i)$$

for some automorphisms $\psi : \Lambda \to \Lambda$ and $\varphi : \mathrm{D}_n \to \mathrm{D}_n$ for $i = 1, \ldots, r$.

Using a conjugation in D_n and id_Λ, we see that our vector is equivalent to one that has a coordinate equal to a. Consider a generating vector

$$(a_1, \ldots, a_{i-1}, c, a_{i+1}, \ldots, a_r).$$

By appropriate modification we can obtain an equivalent vector at which c is moved on arbitrary position $1 \le j \le r$. For $i < j$, let us consider an automorphism ψ_{ij} of Λ which send consecutive canonical generators to

$$x_1, \ldots, x_{i-1}, x_i x_{i+1} x_i, \ldots, x_i x_j x_i, x_i, x_{j+1}, \ldots, x_r.$$

Then the pair $(\psi_{i,j}, \mathrm{id}_{\mathrm{D}_n})$ induces the equivalence of our vector to the vector

$$(a_1, \ldots, a_{i-1}, a_{i+1}^c, a_{i+2}^c, \ldots, a_j^c, c, a_{j+1}, \ldots, a_r)$$

Similarly for $j < i$, let an automorphism φ_{ji} of Λ send the consecutive canonical generators onto

$$x_1, \ldots, x_{j-1}, x_i, x_i x_j x_i, \ldots, x_i x_{i-1} x_i, x_{i+1}, \ldots, x_r.$$

Then the pair $(\varphi_{j,i}, \mathrm{id}_{\mathrm{D}_n})$ induces the equivalence of our vector to the vector

$$(a_1, \ldots, a_{j-1}, c, a_j^c, a_{j+1}^c, \ldots a_{i-1}^c, a_{i+1}, \ldots, a_r). \tag{4}$$

Now we shall show that composing the above transformations in an appropriate way each generating vector is equivalent to

$$(5)\qquad (\underbrace{a, \dots, a}_{k_0}, \underbrace{(ab)a, \dots, (ab)a}_{k_1}, \dots, \underbrace{(ab)^{n-1}a, \dots, (ab)^{n-1}a}_{k_{n-1}})$$

for some k_i.

For, given a generating vector $(a_1, a_2, \dots, a_r)$ we define first its n-adic counter $\sum_{i=1}^{r} \ell_i n^{r-i}$, where $a_i = (ab)^{\ell_i} a$. Now, using (4) we move all entries a to the front starting from the right to the left. Then we move entries $(ab)a$ in the resulting vector to the positions behind the entries a, then all entries of the form $(ab)^2 a$ to the positions behind $(ab)a$ etc. But, at some stage, moving $(ab)^k a$, some $(ab)^m a$ for $m < k$ could appear behind. In such case we move them to their right positions behind of $(ab)^m a$ already sorted and then we start the process with $(ab)^{m+1}$, $(ab)^{m+2} a$ etc. Observe that each moving decreases the counter and so finally one ends up with a sorted vector, as required.

Observe now that

$$a^{\varepsilon_0}\,((ab)a)^{\varepsilon_1} \dots ((ab)^{n-1}a)^{\varepsilon_{n-1}}$$

is nontrivial for arbitrary nontrivial choices $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{n-1} \in \{0, 1\}$. Indeed if the number of nontrivial ε_i is odd then this expression is conjugate to a or b, while if $\varepsilon_{i_1}, \varepsilon_{i_2}, \dots, \varepsilon_{i_{2k}}$ are all nonzero exponents then it is equal to $(ba)^m$, where $m = i_{2k} - i_{2k-1} + i_{2k-2} - i_{2k-3} + \dots + i_2 - i_1$ is an integer in the range between 1 and $i_{2k} < n$. As a result all k_i in (5) are even.

Let us notice that composing automorphisms $\varphi_{i,i+1}$ with $\varphi_{i+1,i+2}$ we obtain the equivalence

$$(a_1, \dots, a_{i-1}, c, d, d, a_{i+3}, \dots, a_r) \sim (a_1, \dots, a_{i-1}, d, d, c, a_{i+3}, \dots, a_r)$$

and conversely composing $\psi_{i+1,i+2}$ and $\psi_{i,i+1}$ we have the equivalence

$$(a_1, \dots, a_{i-1}, d, d, c, a_{i+3}, \dots, a_r) \sim (a_1, \dots, a_{i-1}, c, d, d, a_{i+3}, \dots, a_r).$$

So denoting blocks of the same letters of even length by S, T, U, we obtain the following equivalences:

$$(6)\qquad (S, \underbrace{x, \dots, x}_{n}, \underbrace{y, \dots, y}_{2m}, T) \sim (S, \underbrace{y, \dots, y}_{2m}, \underbrace{x, \dots, x}_{n}, T)$$

$$(S, \underbrace{y, \dots, y}_{2m}, T, U) \sim (S, \underbrace{y, \dots, y}_{2m-1}, T^y, y, U)$$

$$(S, \underbrace{y, \ldots, y}_{2m}, T, U) \sim (S, \underbrace{y, \ldots, y}_{2m}, T^y, U) \tag{7}$$

Recall that we already have $k_0 \neq 0$ in (5) and observe that

$$(ab)^\beta (ab)^\alpha a (ab)^{-\beta} = (ab)^{\alpha+2\beta} a, \ (ba)^\beta (ab)^\alpha a (ba)^{-\beta} = (ab)^{\alpha-2\beta} a. \tag{8}$$

Let in (5), $k_0, k_{i_1}, \ldots, k_{i_s}$ be all elements different than 0. Observe that, since (5) is a generating vector $\gcd(i_1, \ldots, i_s)$ is coprime with n and so in particular if n is even, some of i_j, say i_1 is odd. Then, by (6), (7) and (8), given integers $\gamma_{i_2}, \ldots, \gamma_{i_s}$, our vector is equivalent to

$$(\underbrace{a, \ldots, a}_{k_0}, \underbrace{(ab)^k a, \ldots, (ab)^k a}_{k_{i_1}}, \ldots),$$

where $k = i_1 + 2(i_2\gamma_{i_2} + \cdots + i_s\gamma_{i_s})$. Let

$$u = \gcd(i_2, \ldots, i_s), \ v = \gcd(i_1, 2u).$$

So k can be of the form $i_1 + 2tu$ for arbitrary t. But, by the Dirichlet theorem on arithmetic progression, between elements of this form there are infinitely many elements of the form vp, where p is prime and so since $\gcd(v, n) = 1$ we see that some k is relatively prime to n. But then the pair $(\mathrm{id}_\Lambda, \psi^{-1})$, where $\psi(a) = a$, $\psi(b) = (ab)^k a$, shows that our sequence is equivalent to

$$(\underbrace{a, \ldots, a}_{k_0}, \underbrace{b, \ldots, b}_{k_{i_1}}, U_1, U_2, \ldots, U_m)$$

where U_i denote a segment of an even length of the same elements. By applying (6) and (7) we can conjugate elements of U_1 simultaneously by a or b and then (6) yields a vector of the same form, just with smaller number of U_i, and hence the assertion follows by induction.

5. The Teichmüller space of the corresponding surfaces

We start the section reminding briefly the principal concepts and facts concerning the Teichmüller spaces of Riemann surfaces of genus g in the context of Fuchsian groups. For, let $\mathscr{L}$ be the group of conformal automorphisms of $\mathscr{H}$ and given a Fuchsian group Λ, let $\mathrm{R}(\Lambda)$ be the set of group monomorphisms $\Lambda \to \mathscr{L}$. Now two elements τ, τ' of $\mathrm{R}(\Lambda)$ are said to be equivalent if for some $t \in \mathscr{L}$, $\tau(\lambda) = t\tau'(\lambda)t^{-1}$ for arbitrary $\lambda \in \Lambda$. Then the corresponding orbit space $\mathrm{T}(\Lambda)$ is a Teichmüller space of Λ. For Λ having signature $(g; m_1, \ldots, m_r)$ it is a real cell of dimension $\mathrm{d}(\Lambda) = 6g - 6 + 2r$ and for Γ

being a surface Fuchsian group of the orbit genus $g \geq 2$, $\mathrm{T}(\Gamma)$ is the Teichmüller space of Riemann surfaces of genus g. Now given a monomorphism $i : \Lambda' \to \Lambda$, we have isometric embedding $m : \mathrm{T}(\Lambda) \to \mathrm{T}(\Lambda')$. Finally we have the modular group $\mathrm{M}(\Lambda) = \mathrm{Aut}(\Lambda)/\mathrm{Inn}(\Lambda)$ which acts on $\mathrm{T}(\Lambda)$ by $\bar{\alpha}[\tau] = [\alpha \circ \tau]$. Consider the following subset of $\mathrm{T}(\Gamma)$ for a Fuchsian surface group Γ of the orbit genus g.

$$\mathrm{T}(g,n) = \left\{ [\tau] \in \mathrm{T}(\Gamma) \;\middle|\; \begin{array}{l} \mathscr{H}/\tau(\Gamma) \text{ has two involutions } \sigma, \tau \text{ whose} \\ \text{product has order } n \text{ and they have} \\ 4(g-1)/n + 8 \text{ fixed points in total} \end{array} \right\}.$$

THEOREM 5.1. *The space* $\mathrm{T}(g,n)$ *is a union of submanifolds of dimension* $4(g-1)/n + 2$.

PROOF. For even integers k and l for which $k+l = 4(g-1)/n+8$ consider the following subset of $\mathrm{T}(g,n)$:

$$\mathrm{T}(g,n;k,l) = \left\{ [\tau] \in \mathrm{T}(\Gamma) \;\middle|\; \begin{array}{l} \mathscr{H}/\tau(\Gamma) \text{ has two involutions } \sigma, \tau \text{ whose} \\ \text{product has order } n \text{ and they have } k \text{ and} \\ l \text{ fixed points respectively} \end{array} \right\}.$$

Then $\mathrm{T}(g,n)$ is a union of $\mathrm{T}(g,n;k,l)$ and we shall show that each of the last is a union of submanifolds of the dimension specified in the theorem. The proof of this actually repeats, with slight modifications, the arguments from the Lemma 3 in [10]. Let $\theta : \Lambda \to \mathrm{D}_n$ be an epimorphism defined in (3). For $\Gamma = \ker\theta$, let $i : \Gamma \hookrightarrow \Lambda$ be the embedding. Let $[\tau] \in \mathrm{T}(g,n;k,l)$ and let $\tau' : \Lambda \to \mathscr{L}$ be a monomorphism such that $\mathrm{D}_n \cong \tau'(\Lambda)/\tau(\Gamma)$. Then for $\Gamma' = \tau'^{-1}(\tau(\Gamma))$, the projection $\pi : \Lambda \to \Lambda/\Gamma' = \mathrm{D}_n$ is equivalent to θ and so, by Theorem 4.1, we have the commutative diagram

$$\begin{tikzcd}
\Gamma \arrow[r, "\tau"] & \mathscr{L} & \\
\Gamma' \arrow[u, "\beta"] \arrow[r, hook] & \Lambda \arrow[u, "\tau'"'] \arrow[r, two heads, "\pi"] & \mathrm{D}_n \\
\Gamma \arrow[u, "\varphi'"] \arrow[r, hook, "i"] & \Lambda \arrow[u, "\varphi"'] \arrow[r, "\theta"] & \mathrm{D}_n \arrow[u, "\psi"']
\end{tikzcd}$$

where $\beta, \varphi, \varphi', \psi$ are isomorphisms. Hence, for $\alpha = (\beta\varphi')^{-1}$, $[\tau] \in \bar{\alpha}m\mathrm{T}(\Lambda)$ and therefore

$$\mathrm{T}(g,n;k,l) \subseteq \bigcup_{\bar{\alpha}\in\mathrm{M}(\Gamma_g)} \bar{\alpha}m\mathrm{T}(\Lambda).$$

The above inclusion can be obviously reversed and so $\mathrm{T}(g,n;k,l)$ is a union of submanifolds of dimension $\mathrm{d}(\Lambda)$ which is equal to $4(g-1)/n + 2$ indeed.

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