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MAXIMAL OPERATORS OF SCHRÖDINGER TYPE WITH A COMPLEX PARAMETER

PER SJÖLIN

Abstract

Maximal operators of Schrödinger type but with a complex parameter are considered. For these operators we obtain results which in a certain sense lie between the results for the corresponding maximal operators for solutions to the Schrödinger equation and for solutions to the heat equation.

1. Introduction

Letting f belong to the Schwartz space $\mathscr{S}(\mathsf{R})$ we set

$$S_t f(x) = \int_{\mathsf{R}} e^{ix\xi} e^{it\xi^2} \widehat{f}(\xi)\, d\xi, \qquad x \in \mathsf{R}.$$

Here t is a complex number with $\operatorname{Im} t \geq 0$ and $\widehat{f}$ denotes the Fourier transform of f, defined by

$$\widehat{f}(\xi) = \int_{\mathsf{R}} e^{-i\xi x} f(x)\, dx.$$

We also set $U(x,t) = (2\pi)^{-1} S_t f(x)$ for $x \in \mathsf{R}$ and $t \in \mathsf{R}$. It then follows that $U(x,0) = f(x)$ and U satisfies the Schrödinger equation $i\,\partial U/\partial t = \partial^2 U/\partial x^2$.

We also introduce the maximal function $S^* f$ defined by

$$S^* f(x) = \sup_{0<t<1} |S_t f(x)|, \qquad x \in \mathsf{R},$$

and define Sobolev spaces H_s by setting

$$H_s = \{f \in \mathscr{S}';\ \|f\|_{H_s} < \infty\}, \qquad s \in \mathsf{R},$$

where

$$\|f\|_{H_s} = \left(\int_{\mathsf{R}} (1+\xi^2)^s |\widehat{f}(\xi)|^2 d\xi \right)^{1/2}.$$

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It is well-known that the estimate

$$\|S^*f\|_2 \le C\|f\|_{H_s}$$

holds for $s > 1/2$ and does not hold for $s < 1/2$ (see [2]). Here $\|S^*f\|_2$ denotes the norm of S^*f in the space $L^2(\mathsf{R})$. We then set

$$H_u f(x) = S_{iu} f(x) = \int_{\mathsf{R}} e^{ix\xi} e^{-u\xi^2} \widehat{f}(\xi)\, d\xi, \qquad x \in \mathsf{R},$$

for $u \ge 0$. If we set $V(x,u) = (2\pi)^{-1} H_u f(x)$ for $x \in \mathsf{R}$ and $u \ge 0$, then $V(x,0) = f(x)$ and V satisfies the heat equation $\partial V/\partial u = \partial^2 V/\partial x^2$. We also introduce the maximal function H^*f defined by

$$H^*f(x) = \sup_{0<u<1} |H_u f(x)|, \qquad x \in \mathsf{R}.$$

It is then well-known that the estimate $\|H^*f\|_2 \le C\|f\|_{H_s}$ holds if and only if $s \ge 0$.

We shall then introduce a class of maximal operators for which one has results lying between the above results for S^* and H^*. For $0 < \gamma < \infty$ we set

$$P_u f(x) = S_{u+iu^\gamma} f(x) = \int_{\mathsf{R}} e^{ix\xi} e^{iu\xi^2} e^{-u^\gamma \xi^2} \widehat{f}(\xi)\, d\xi, \qquad x \in \mathsf{R}, \;\; 0 < u < 1,$$

and

$$P^*f(x) = \sup_{0<u<1} |P_u f(x)|, \qquad x \in \mathsf{R}.$$

We shall here study the inequality

$$\|P^*f\|_2 \le C\|f\|_{H_s} \tag{1}$$

for various values of γ. We have the following results.

THEOREM 1.
(i) *For* $0 < \gamma \le 1$ (1) *holds if and only if* $s \ge 0$.
(ii) *For* $\gamma = 2$ (1) *holds if and only if* $s \ge 1/4$.
(iii) *If* $\gamma \ge 4$ *and* (1) *holds then* $s \ge 1/2 - 1/\gamma$.

For $\gamma > 0$ we let E_γ denote the set of all s with the property that (1) holds. Also set $s(\gamma) = \inf E_\gamma$ for $\gamma > 0$. Using the fact that $\lim_{u\to 0} P_u f(x) = 2\pi f(x)$ it is easy to see that $s(\gamma) \ge 0$. We shall use the following lemma.

LEMMA 1. *Assume that g and h are continuous functions on the interval* $(0,1)$ *and that* $0 \le g(u) \le h(u)$ *for* $0 < u < 1$. *Set*

$$P_g^* f(x) = \sup_{0<u<1} |S_{u+ig(u)} f(x)|$$

and

$$P_h^* f(x) = \sup_{0<u<1} |S_{u+ih(u)} f(x)|.$$

Then

$$\|P_h^* f\|_2 \le C \|P_g^* f\|_2.$$

It follows from the lemma that $s(\gamma)$ is an increasing function of γ on the interval $(0, \infty)$. Also the result mentioned above for the operator S^* implies that $s(\gamma) \le 1/2$ (take $g(u) \equiv 0$ and $h(u) = u^\gamma$ in the lemma). The results in Theorem 1 can then be stated in the following way.

THEOREM 2.

(i) *For* $0 < \gamma \le 1$ *one has* $s(\gamma) = 0$.

(ii) $s(2) = 1/4$.

(iii) *For* $\gamma > 4$ *one has* $1/2 - 1/\gamma \le s(\gamma) \le 1/2$ *and hence* $\lim_{\gamma\to\infty} s(\gamma) = 1/2$.

In Section 2 we shall prove Lemma 1 and state and prove a second lemma. In Section 3 we shall prove Theorem 1.

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2. Lemmas

PROOF OF LEMMA 1. We have

$$\begin{aligned} S_{u+ih(u)} f(x) &= \int e^{ix\xi} e^{iu\xi^2} e^{-h(u)\xi^2} \widehat{f}(\xi)\, d\xi \\ &= \int e^{ix\xi} e^{iu\xi^2} e^{-g(u)\xi^2} e^{-(h(u)-g(u))\xi^2} \widehat{f}(\xi)\, d\xi \\ &= \int e^{ix\xi} e^{iu\xi^2} e^{-g(u)\xi^2} e^{-v\xi^2} \widehat{f}(\xi)\, d\xi \end{aligned}$$

where $v = h(u) - g(u) \ge 0$.

For $v = 0$ one has

$$S_{u+ih(u)} f(x) = S_{u+ig(u)} f(x).$$

We then assume $v > 0$. It is well-known that

$$e^{-v\xi^2} = \widehat{K_v}(\xi) = \int e^{-i\xi y} K_v(y)\, dy,$$

where
$$K_v(y) = \frac{1}{v^{1/2}} \frac{1}{2\sqrt{\pi}} e^{-y^2/(4v)} = \frac{1}{v^{1/2}} K(y/v^{1/2})$$

with $K(y) = e^{-y^2/4}/(2\sqrt{\pi}\,)$. It follows that

$$\begin{aligned} S_{u+ih(u)} f(x) &= \int e^{ix\xi} e^{iu\xi^2} e^{-g(u)\xi^2} \left(\int e^{-i\xi y} K_v(y)\, dy \right) \widehat{f}(\xi)\, d\xi \\ &= \int \left(\int e^{i(x-y)\xi} e^{iu\xi^2} e^{-g(u)\xi^2}\, \widehat{f}(\xi)\, d\xi \right) K_v(y)\, dy \\ &= \int S_{u+ig(u)} f(x-y) K_v(y)\, dy = K_v * S_{u+ig(u)} f(x) \end{aligned}$$

and hence
$$|S_{u+ih(u)} f(x)| \le K_v * P_g^* f(x) \le C M P_g^* f(x)$$

where M denotes the Hardy-Littlewood maximal operator.

We conclude that

$$P_h^* f(x) \le P_g^* f(x) + C M P_g^* f(x)$$

and since M is a bounded operator on $L^2(\mathsf{R})$ we obtain $\| P_h^* f \|_2 \le C \| P_g^* f \|_2$.

The following lemma was proved in [3].

LEMMA 2. *Assume that* $a > 1$, $1/2 \le s < 1$ *and* $\mu \in C_0^\infty(\mathsf{R})$. *Then*

$$\left| \int_{\mathsf{R}} e^{ix\xi + it|\xi|^a} |\xi|^{-s} \mu(\xi/N)\, d\xi \right| \le C \frac{1}{|x|^{1-s}}$$

for $x \in \mathsf{R}$, $t \in \mathsf{R}$ *and* $N = 1, 2, 3, \dots$. *Here the constant* C *may depend on* s *and* a *but not on* x, t *or* N.

The next lemma will be used to prove that $s(2) \le 1/4$.

LEMMA 3. *Assume that* $1/2 \le \alpha < 1$, $0 < d_1 < 1$, $0 < d_2 < 1$, *and* $\mu \in C_0^\infty(\mathsf{R})$ *and* μ *even and real-valued. Then*

$$\left| \int_{\mathsf{R}} \frac{e^{i((d_1-d_2)\xi^2 - x\xi)}}{(1+\xi^2)^{\alpha/2}} e^{-(d_1^2+d_2^2)\xi^2} \mu(\xi/N)\, d\xi \right| \le K(x)$$

for $x \in \mathsf{R}$ *and* $N = 1, 2, 3, \dots$, *where* $K \in L^1(\mathsf{R})$. *Here* K *is independent of* d_1, d_2 *and* N, *and one may take* $K(x) = Cx^{-2}$ *for* $|x| \ge C_0$ *and* $K(x) = C|x|^{\alpha-1}$ *for* $|x| < C_0$. *Here* C_0 *denotes a large constant.*

PROOF OF LEMMA 3. The structure in the proof of the lemma will be the same as in the proof of Lemma 3 in [4]. Without loss of generality we may assume $d_2 < d_1$. We set $d = d_1 - d_2$ and $\varepsilon = d_1^2 + d_2^2$ so that $0 < d < 1$ and $0 < \varepsilon < 2$. First assume $|x| \geq C_0$ where C_0 denotes a large constant.

We choose an even function $\varphi_0 \in C_0^\infty(\mathsf{R})$ such that $\varphi_0(\xi) = 1$ for $|\xi| \leq 1/2$ and $\varphi_0(\xi) = 0$ for $|\xi| \geq 1$.

Set

$$\psi(\xi) = (1+\xi^2)^{-\alpha/2} e^{-\varepsilon\xi^2} \mu(\xi/N)$$

and $\psi_0 = \psi\varphi_0$ so that $\operatorname{supp}\psi_0 \subset [-1, 1]$.

We also set $\rho = |x|/(2d)$ and let K denote a large constant. Then choose $\varphi_2 \in C_0^\infty(\mathsf{R})$ so that $\operatorname{supp}\varphi_2 \subset [\rho/4,\ 2K\rho]$ and $\varphi_2(\xi) = 1$ for $\rho/2 \leq \xi \leq K\rho$. We may also assume that $|\varphi_2'(\xi)| \leq C\xi^{-1}$ and $|\varphi_2''(\xi)| \leq C\xi^{-2}$ for $\xi > 0$. We then set $\varphi_3 = (1-\varphi_2)\chi_{[K\rho,\infty)}$ and $\varphi_1 = (1-\varphi_2-\varphi_0)\chi_{[0,\rho/2]}$.

Let $\check{\varphi}$ be defined by $\check{\varphi}(\xi) = \varphi(-\xi)$ and set $\varphi_{-1} = \check{\varphi}_1$, $\varphi_{-2} = \check{\varphi}_2$ and $\varphi_{-3} = \check{\varphi}_3$. Setting $F(\xi) = d\xi^2 - x\xi$ we then have

$$\int_{-\infty}^{\infty} e^{iF}\psi\, d\xi = \sum_{j=0}^{3} \int e^{iF}\psi\varphi_j\, d\xi + \sum_{j=1}^{3} \int e^{iF}\psi\varphi_{-j}\, d\xi.$$

The integrals $\int e^{iF}\psi\varphi_{-j}$ can be reduced to $\int e^{iF}\psi\varphi_j$ for $j = 1, 2, 3$. Setting $\psi_j = \psi\varphi_j$, $j = 0, 1, 2, 3$, it is therefore sufficient to estimate the integrals

$$J_j = \int e^{iF}\psi_j\, d\xi, \qquad j = 0, 1, 2, 3.$$

We claim that one has the following estimates for $j = 1, 2, 3$ and $\xi \geq 1/2$:

$$|\psi_j(\xi)| \leq C\frac{1}{(1+\xi^2)^{\alpha/2}}, \tag{2}$$

$$|\psi_j'(\xi)| \leq C\frac{1}{(1+\xi^2)^{\alpha/2}\xi}, \tag{3}$$

and

$$|\psi_j''(\xi)| \leq C\frac{1}{(1+\xi^2)^{\alpha/2}\xi^2}. \tag{4}$$

We set $h(\xi) = h_\varepsilon(\xi) = e^{-\varepsilon\xi^2}$ for $\xi \geq 1/2$ and $0 < \varepsilon < 2$. The above estimates for ψ_j, ψ_j' and ψ_j'' will follow if we can prove that

$$|h'(\xi)| \leq C\frac{1}{\xi}, \qquad \xi \geq 1/2, \tag{5}$$

and

$$|h''(\xi)| \le C\frac{1}{\xi^2}, \qquad \xi \ge 1/2, \tag{6}$$

with C independent of ε.

We have

$$h'(\xi) = -e^{-\varepsilon\xi^2} 2\,\varepsilon\,\xi$$

and

$$h''(\xi) = -e^{-\varepsilon\xi^2} 2\,\varepsilon + e^{-\varepsilon\xi^2} 4\,\varepsilon^2\,\xi^2.$$

It follows that for $\xi \ge 1/2$ one has

$$|h'(\xi)| \le e^{-\varepsilon\xi^2} 2\,\varepsilon\,\xi = e^{-\varepsilon\xi^2} 2\,\varepsilon\,\xi^2 \frac{1}{\xi} \le 2(\max_{t>0} te^{-t})\frac{1}{\xi} = 2A\frac{1}{\xi},$$

where $A = \max_{t\ge 0} te^{-t}$. Also

$$|h''(\xi)| \le e^{-\varepsilon\xi^2} 2\,\varepsilon + 4e^{-\varepsilon\xi^2}\varepsilon^2\xi^2$$
$$= e^{-\varepsilon\xi^2} 2\,\varepsilon\,\xi^2 \frac{1}{\xi^2} + 4e^{-\varepsilon\xi^2}\varepsilon^2\xi^4\frac{1}{\xi^2} \le 2A\frac{1}{\xi^2} + 4B\frac{1}{\xi^2},$$

where $B = \max_{t\ge 0} t^2 e^{-t}$. Hence (5) and (6) are proved and (2), (3) and (4) follow.

We shall first estimate J_0. We have

$$J_0 = \int e^{-ix\xi} e^{id\xi^2} \psi_0(\xi)\,d\xi$$

where $\operatorname{supp}\psi_0 \subset [-1, 1]$ and two integrations by parts give the estimate $|J_0| \le Cx^{-2}$.

We shall then estimate J_2. One has

$$J_2 = \int e^{iF}\psi_2\,d\xi$$

where

$$\psi_2(\xi) = (1+\xi^2)^{-\alpha/2} e^{-\varepsilon\xi^2} \mu(\xi/N)\varphi_2(\xi)$$

and $\operatorname{supp}\psi_2 \subset [\rho/4,\ 2K\rho]$.

We have $F''(\xi) = 2d$ and van der Corput's Lemma with the second derivative (see Stein [5], p. 334) gives

$$[J_2| \le Cd^{-1/2}\left(\max|\psi_2| + \int |\psi_2'|\,d\xi\right). \tag{7}$$

It is clear that

(8) $$\max|\psi_2| \le C\rho^{-\alpha}e^{-c\varepsilon\rho^2}$$

where c denotes a positive constant.

We also set
$$\nu(\xi) = (1+\xi^2)^{-\alpha/2}\mu(\xi/N)\varphi_2(\xi)$$
so that
$$\psi_2(\xi) = \nu(\xi)e^{-\varepsilon\xi^2}$$
and
$$\psi_2'(\xi) = \nu(\xi)\left(\frac{d}{d\xi}e^{-\varepsilon\xi^2}\right) + \nu'(\xi)e^{-\varepsilon\xi^2}.$$
It follows that
$$\begin{aligned}\int|\psi_2'(\xi)|\,d\xi &\le \int\left|\nu(\xi)\left(\frac{d}{d\xi}e^{-\varepsilon\xi^2}\right)\right|d\xi + \int|\nu'(\xi)e^{-\varepsilon\xi^2}|\,d\xi\\ &\le C\rho^{-\alpha}\int_{\rho/4}^{2K\rho}\left|\frac{d}{d\xi}e^{-\varepsilon\xi^2}\right|d\xi + \int_{\rho/4}^{2K\rho}\rho^{-\alpha-1}\,d\xi\,e^{-c\varepsilon\rho^2}\\ &\le C\rho^{-\alpha}\int_{\rho/4}^{2K\rho} -\left(\frac{d}{d\xi}e^{-\varepsilon\xi^2}\right)d\xi + C\rho^{-\alpha}e^{-c\varepsilon\xi^2}\\ &= -C\rho^{-\alpha}\big[e^{-\varepsilon\xi^2}\big]_{\rho/4}^{2K\rho} + C\rho^{-\alpha}e^{-c\varepsilon\rho^2} \le C\rho^{-\alpha}e^{-c\varepsilon\rho^2}.\end{aligned}$$
Combining this estimate with (7) and (8) we obtain
$$\begin{aligned}|J_2| \le Cd^{-1/2}\rho^{-\alpha}e^{-c\varepsilon\rho^2} &= Cd^{-1/2}\left(\frac{|x|}{d}\right)^{-\alpha}e^{-c\varepsilon x^2/d^2}\\ &\le Cd^{\alpha-1/2}|x|^{-\alpha}e^{-cx^2} \le Ce^{-cx^2},\end{aligned}$$
since $\alpha \ge 1/2$ and $d^2 \le \varepsilon$. In fact
$$d^2 = (d_1-d_2)^2 = d_1^2+d_2^2-2\,d_1d_2 \le d_1^2+d_2^2 = \varepsilon.$$

This concludes the estimate of J_2 and we shall then estimate J_1. One has
$$J_1 = \int e^{iF}\psi_1\,d\xi$$
where $\operatorname{supp}\psi_1 \subset [1/2, \rho/2]$ and $F' = 2\,d\xi - x$.

On the interval $[1/2, \rho/2]$ one has
$$2\,d\xi \le 2\,d\rho/2 = d\,\rho = d\,\frac{|x|}{2d} = \frac{|x|}{2} \qquad \text{and} \qquad |F'| \ge |x|/2 \ge 2\,d\xi.$$

Also $F'' = 2d$ and $F^{(3)} = 0$. It follows that $|F''|/|F'| \le C\xi^{-1}$ for $1/2 \le \xi \le \rho/2$. Integrating by parts twice we obtain

$$
\begin{aligned}
J_1 = \int e^{iF}\psi_1\,d\xi &= \int iF'e^{iF}\frac{\psi_1}{iF'}\,d\xi \\
&= -\int e^{iF}\left(\frac{1}{iF'}\psi_1' - \frac{1}{i}\frac{F''}{(F')^2}\psi_1\right)d\xi \\
&= -\int iF'e^{iF}\left(\frac{1}{(iF')^2}\psi_1' + \frac{F''}{(F')^3}\psi_1\right)d\xi \\
&= \int e^{iF}\left(\frac{1}{i^2}\frac{1}{(F')^2}\psi_1'' - \frac{1}{i^2}\frac{2F''}{(F')^3}\psi_1'\right. \\
&\qquad \left. + \frac{F''}{(F')^3}\psi_1' + \frac{F^{(3)}}{(F')^3}\psi_1 - \frac{3(F'')^2}{(F')^4}\psi_1\right)d\xi
\end{aligned}
$$

and hence

$$
\begin{aligned}
|J_1| &\le C\int \frac{1}{|F'|^2}\left(|\psi_1''| + \frac{|F''|}{|F'|}|\psi_1'| + \frac{|F''|^2}{|F'|^2}|\psi_1|\right)d\xi \\
&\le C\frac{1}{x^2}\int_{1/2}^{\infty}\xi^{-\alpha-2}d\xi = C\frac{1}{x^2}.
\end{aligned}
\tag{9}
$$

It remains to estimate $J_3 = \int e^{iF}\psi_3\,d\xi$. Here $\operatorname{supp}\psi_3 \subset [K\rho, \infty)$. For

$$\xi \ge K\rho = K\frac{|x|}{2d}$$

we have

$$2\,d\xi \ge 2\,dK\frac{|x|}{2d} = K|x|$$

and hence $|F'| \ge c\,|x|$ and $|F'| \ge c\,d\,\xi$.

We can estimate J_3 in the same way as we estimated J_1. We can use the inequality (9) with ψ_1 replaces by ψ_3 and J_1 replaced by J_3. One obtains $|J_3| \le C|x|^{-2}$.

We have proved Lemma 3 in the case $|x| \ge C_0$. It remains to study the case $|x| < C_0$. The estimate in this case follows from the proof in [3] of our Lemma 2. The proof of Lemma 3 is complete.

3. Proof of Theorem 1

PROOF OF THEOREM 1. We shall first study the case $0 < \gamma \leq 1$. It is well-known that e^{-ax^2} has Fourier transform $\sqrt{\pi}a^{-1/2}e^{-\xi^2/4a}$ for $a > 0$. For $\operatorname{Re} z > 0$ we then set $z^{1/2} = |z|^{1/2}e^{i(\arg z)/2}$ where $-\pi/2 < \arg z < \pi/2$. By use of elementary properties of analytic functions it is then easy to prove that e^{-zx^2} has Fourier transform

$$\frac{\sqrt{\pi}}{z^{1/2}}e^{-\xi^2/4z}$$

for $\operatorname{Re} z > 0$. It also follows that $e^{-z\xi^2}$ is the Fourier transform of

$$\frac{1}{2\sqrt{\pi}z^{1/2}}e^{-x^2/4z} \tag{10}$$

for $\operatorname{Re} z > 0$.

Setting $t = u + iv$ with $u > 0$, $v > 0$ we then have $e^{it\xi^2} = e^{i(u+iv)\xi^2} = e^{-(v-iu)\xi^2}$. Taking $z = v-iu$ in (10) it then follows that $e^{-(v-iu)\xi^2}$ is the Fourier transform of

$$K(x) = \frac{1}{2\sqrt{\pi}(v-iu)^{1/2}}e^{-x^2/(4(v-iu))}.$$

It is clear that

$$|K(x)| \leq \frac{1}{(v^2+u^2)^{1/4}}\left|e^{-x^2/(4(v-iu))}\right|.$$

Since

$$\frac{1}{v-iu} = \frac{v}{v^2+u^2} + i\frac{u}{v^2+u^2}$$

we conclude that

$$|K(x)| \leq \frac{1}{(v^2+u^2)^{1/4}}e^{-x^2v/(4(v^2+u^2))}.$$

Letting $0 < u < 1$ and $v = u^\gamma$ with $0 < \gamma \leq 1$ we then obtain

$$\begin{aligned}|K(x)| &\leq \frac{1}{(u^{2\gamma}+u^2)^{1/4}}\exp\left(-\frac{x^2}{4}\frac{u^\gamma}{u^{2\gamma}+u^2}\right)\\ &\leq \frac{1}{u^{\gamma/2}}e^{-cx^2/u^\gamma} = \frac{1}{u^{\gamma/2}}L(x/u^{\gamma/2}),\end{aligned}$$

where $c > 0$ and $L(x) = e^{-cx^2}$. It follows that $P^*f(x) \leq CMf(x)$ and hence the first part of Theorem 1 follows.

We shall then study the case $\gamma = 2$. We let $u(x)$ denote a measurable function on R with $0 < u(x) < 1$ and set

$$Pf(x) = \int_{\mathsf{R}} e^{ix\xi} e^{iu(x)\xi^2} e^{-u(x)^2\xi^2} \widehat{f}(\xi)\, d\xi, \qquad x \in \mathsf{R}.$$

We have to prove that

$$\|Pf\|_2 \le C\|f\|_{H_s} = C\left(\int_{\mathsf{R}} |\widehat{f}(\xi)|^2 (1+\xi^2)^s d\xi\right)^{1/2} \tag{11}$$

for $s \ge 1/4$. We may also assume $s < 1/2$.

Setting $g(\xi) = \widehat{f}(\xi)\,(1+\xi^2)^{s/2}$ and defining T by

$$Tg(x) = \int_{\mathsf{R}} e^{ix\xi} e^{iu(x)\xi^2} e^{-u(x)^2\xi^2} (1+\xi^2)^{-s/2} g(\xi)\, d\xi,$$

we have $Pf(x) = Tg(x)$. It is therefore sufficient to prove that $\|Tg\|_2 \le C\|g\|_2$. For $N = 1, 2, 3, \dots$ set

$$T_N g(x) = \chi_N(x) \int_{\mathsf{R}} e^{ix\xi} e^{iu(x)\xi^2} e^{-u(x)^2\xi^2} (1+\xi^2)^{-s/2} \rho_N(\xi) g(\xi)\, d\xi.$$

Here $\chi_N(x) = \chi(x/N)$ and $\rho_N(\xi) = \rho(\xi/N)$ and χ and $\rho \in C_0^\infty(\mathsf{R})$ and have the property that $\chi(x) = \rho(x) = 1$ for $|x| \le 1$ and $\chi(x) = \rho(x) = 0$ for $|x| \ge 2$. We also assume that χ and ρ are even and real-valued. It is sufficient to prove that

$$\|T_N g\|_2 \le C\|g\|_2, \qquad N = 1, 2, 3, \dots.$$

It is clear that

$$T_N^* h(\xi) = \rho_N(\xi)(1+\xi^2)^{-s/2} \int_{\mathsf{R}} e^{-ix\xi} e^{-iu(x)\xi^2} e^{-u(x)^2\xi^2} \chi_N(x) h(x)\, dx$$

and it is sufficient to prove that

$$\|T_N^* h\|_2 \le C\|h\|_2, \qquad N = 1, 2, 3, \dots. \tag{12}$$

Invoking Lemma 3 we now have

$$\begin{aligned}
&\|T_N^* h\|_2^2 \\
&\quad = \int T_N^* h(\xi)\overline{T_N^* h(\xi)}\, d\xi \\
&\quad = \int \rho_N(\xi)^2(1+\xi^2)^{-s}\left(\int e^{-ix\xi}e^{-iu(x)\xi^2}e^{-u(x)^2\xi^2}\chi_N(x)h(x)\,dx\right) \\
&\qquad\quad \times \left(\int e^{iy\xi}e^{iu(y)\xi^2}e^{-u(y)^2\xi^2}\chi_N(y)\overline{h(y)}\,dy\right) \\
&\quad = \iint\left(\int (1+\xi^2)^{-s}e^{i(y-x)\xi}e^{i(u(y)-u(x))\xi^2}e^{-(u(x)^2+u(y)^2)\xi^2}\mu(\xi/N)\,d\xi\right) \\
&\qquad\quad \times\ \chi_N(x)\chi_N(y)h(x)\overline{h(y)}\,dxdy \\
&\quad \le C\iint K(x-y)|h(x)|\,|h(y)|\,dxdy \le C\|h\|_2^2.
\end{aligned}$$

Here we have set $\mu = \rho^2$ and according to Lemma 3 we have $K \in L^1(\mathsf{R})$ since $1/4 \le s < 1/2$. Hence (12) and (11) are proved.

We shall then prove that if $\gamma = 2$ and $s < 1/4$ then (1) does not hold. First choose $g \in C_0^\infty(\mathsf{R})$ such that $\operatorname{supp} g \subset (-1, 1)$, $g(\xi) \ge 0$ and $g(\xi) = 1$ for $|\xi| \le 1/2$. Then let $v > 0$ denote a small number and define a function f_v by setting

$$\widehat{f_v}(\xi) = v\, g(v\xi + 1/v).$$

It is well-known and easy to prove that $\|f_v\|_{H_s} \to 0$ as $v \to 0$ if $s < 1/4$ (see [1]). Setting $u = u(x) = xv^2/2$ and assuming $0 < x < 1/100$ we have

$$P_u f_v(x) = \int v\, g(v\xi + 1/v)e^{ix\xi}e^{ixv^2\xi^2/2}e^{-x^2v^4\xi^2/4}d\xi.$$

In this integral we make a change of variable $\eta = v\xi + 1/v$, so that $d\eta = v\,d\xi$ and $\xi = \eta/v - 1/v^2$. One obtains

$$P_u f_v(x) = \int g(\eta)e^{iF(\eta)}e^{G(\eta)}d\eta$$

where

$$F(\eta) = x\left(\frac{\eta}{v} - \frac{1}{v^2}\right) + \frac{xv^2}{2}\left(\frac{\eta}{v} - \frac{1}{v^2}\right)^2$$

and

$$G(\eta) = -\frac{x^2v^4}{4}\left(\frac{\eta}{v} - \frac{1}{v^2}\right)^2.$$

Hence

$$F(\eta) = \frac{x\eta}{v} - \frac{x}{v^2} + \frac{xv^2}{2}\left(\frac{\eta^2}{v^2} + \frac{1}{v^4} - \frac{2\eta}{v^3}\right)$$

$$= \frac{x\eta}{v} - \frac{x}{v^2} + \frac{x\eta^2}{2} + \frac{x}{2v^2} - \frac{x\eta}{v} = \frac{x\eta^2}{2} - \frac{x}{2v^2}$$

and it follows that

$$|P_u f_v(x)| = \left|\int_{-1}^{1} g(\eta) e^{ix\eta^2/2} e^{G(\eta)} d\eta\right|.$$

We have

$$|G(\eta)| \leq \frac{x^2 v^4}{4}\left(\frac{2}{v^2}\right)^2 = x^2 \leq 1$$

for $|\eta| \leq 1$ and we conclude that

$$|P_u f_v(x)| \geq \int_{-1}^{1} g(\eta)\cos(x\eta^2/2)\, e^{G(\eta)} d\eta \geq \int_{-1}^{1} g(\eta)\frac{1}{2}e^{-1}\, d\eta \geq \frac{1}{2e}.$$

Hence $P^* f_v(x) \geq 1/(2e)$ for $0 < x < 1/100$ and $\|P^* f_v\|_2 \geq c > 0$ for small v. It follows that the estimate $\|P^* f_v\|_2 \leq C\|f_v\|_{H_s}$ does not hold for $s < 1/4$. Hence the statement in Theorem 1 in the case $\gamma = 2$ has been proved.

It remains to study the case $\gamma \geq 4$. Take g as above and choose f so that $\widehat{f}(\xi) = g(\xi + N)$ where N denotes a large positive number. Also set $u = u(x) = x/(2N)$ and assume $2^{-1}N^{1-2/\gamma} \leq x \leq N^{1-2/\gamma}$.

Setting $\eta = \xi + N$ we obtain

$$P_u f(x) = \int e^{ix\xi} e^{iu\xi^2} e^{-u^\gamma \xi^2} g(\xi + N)\, d\xi$$

$$= \int e^{ix(\eta - N)} e^{iu(\eta - N)^2} e^{-u^\gamma(\eta - N)^2} g(\eta)\, d\eta$$

$$= \int g(\eta) e^{iF(\eta)} e^{G(\eta)} d\eta$$

where $F(\eta) = x(\eta - N) + u(\eta - N)^2$ and $G(\eta) = -u^\gamma(\eta - N)^2$. Hence

$$F(\eta) = x\eta - xN + u\eta^2 + uN^2 - u\,2\eta\, N$$

$$= x\eta - xN + \frac{x}{2N}\eta^2 + \frac{x}{2N}N^2 - 2\eta\frac{x}{2N}N$$

$$= x\eta - xN + \frac{x}{2N}\eta^2 + \frac{xN}{2} - x\eta$$

$$= \frac{x}{2N}\eta^2 - \frac{xN}{2}$$

and it follows that

$$|P_u f(x)| = \left| \int_{-1}^{1} g(\eta) e^{ix\eta^2/(2N)} e^{G(\eta)} d\eta \right|.$$

We have

$$|G(\eta)| \leq u^\gamma (2N)^2 = \frac{x^\gamma}{2^\gamma N^\gamma} 4N^2 \leq N^{(1-2/\gamma)\gamma} N^{2-\gamma} = 1$$

for $|\eta| \leq 1$ and we conclude that

$$|P_u f(x)| \geq \int_{-1}^{1} g(\eta) \cos(x\eta^2/(2N)) e^{G(\eta)} d\eta \geq \int_{-1}^{1} g(\eta) \frac{1}{2} e^{-1}\, d\eta \geq \frac{1}{2e}.$$

Hence $P^* f(x) \geq 1/(2e)$ for $2^{-1} N^{1-2/\gamma} \leq x \leq N^{1-2/\gamma}$ and it follows that

$$\|P^* f\|_2 \geq c\, N^{(1-2/\gamma)/2} = c\, N^{1/2-1/\gamma}.$$

On the other hand it is easy to see that $\|f\|_{H_s} \leq CN^s$, and if (1) holds, one obtains $N^{1/2-1/\gamma} \leq CN^s$. We conclude that $s \geq 1/2 - 1/\gamma$ and the proof of Theorem 1 is complete.

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DEPARTMENT OF MATHEMATICS
ROYAL INSTITUTE OF TECHNOLOGY
S-100 44 STOCKHOLM
SWEDEN
E-mail: pers@math.kth.se