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ON THE SPECTRUM OF THE GENERALIZED GELFAND PAIR $(U(p,q), H_n)$, $p+q=n$

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Abstract

It is known that the spectrum of the Gelfand pair $(U(n), H_n)$ is homeomorphic to the Heisenberg fan.

In this paper after defining a suitable notion of spectrum, we prove an analogous result for the generalized Gelfand pair $(U(p,q), H_n)$, $p+q=n$.

1. Introduction

Let $n \in \mathsf{N}$ and let p, q nonnegative integers such that $p+q=n$. Let H_n be the Heisenberg group defined by $H_n = \mathsf{C}^n \times \mathsf{R}$ with group law $(z,t)(z',t') = \big(z+z', t+t'-\frac{1}{2}\operatorname{Im} B(z,z')\big)$ where $B(z,w) = \sum_{j=1}^{p} z_j\overline{w}_j - \sum_{j=p+1}^{n} z_j\overline{w}_j$. For $x=(x_1,\dots,x_n)\in \mathsf{R}^n$, we write $x=(x',x'')$ with $x'\in\mathsf{R}^p$, $x''\in\mathsf{R}^q$. So, R^{2n} can be identified with C^n via the map $\varphi(x',x'',y',y'') = (x'+iy', x''-iy'')$, $x', y' \in \mathsf{R}^p$, $x'', y''\in\mathsf{R}^q$. In this setting, the form $-\operatorname{Im} B(z,w)$ agrees with the standard symplectic form on $\mathsf{R}^{2(p+q)}$, and the vector fields $X_j = -\frac{1}{2}y_j\frac{\partial}{\partial t} + \frac{\partial}{\partial x_j}$, $Y_j = \frac{1}{2}x_j\frac{\partial}{\partial t} + \frac{\partial}{\partial y_j}$, $j=1,\dots,n$ and $U=\frac{\partial}{\partial t}$ form a standard basis for the Lie algebra h_n of H_n. Thus H_n can be viewed as $\mathsf{R}^n\times\mathsf{R}^n\times\mathsf{R}$ via the map $(x,y,t)\to(\varphi(x,y),t)$. From now on, we will use freely this identification.

Let $\mathscr{S}(H_n)$ be the Schwartz space on H_n and let $\mathscr{S}'(H_n)$ be the space of corresponding tempered distributions. Consider the action, by automorphism, of $U(p,q)$ on H_n given by $g.(z,t) = (gz,t)$. So $U(p,q)$ acts on $L^2(H_n)$, $\mathscr{S}(H_n)$ and $\mathscr{S}'(H_n)$ in the canonical way.

Let $U(p,q)H_n$ denote the semidirect product of $U(p,q)$ and H_n. It is well known that the pair $(U(p,q)H_n, U(p,q))$ is a generalized Gelfand pair, that is, for each irreducible unitary representation π of $U(p,q)H_n$, the space of distribution vectors fixed by $U(p,q)$ is at most one dimensional. This definition extends the notion of Gelfand pair, which in our case happens when $p=0$ or $q=0$. As usual we will write $(U(p,q), H_n)$ to refer to the generalized Gelfand pair $(U(p,q)H_n, U(p,q))$. A consequence of being a generalized Gelfand pair is that the subalgebra $\mathscr{U}_{U(p,q)}(h_n)$ of the left invariant and $U(p,q)$ invariant

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differential operators is commutative. We refer to [13] for a detailed study of the theory of generalized Gelfand pairs. By another way, it is easy to see that this subalgebra is generated by L and U where $L = \sum_{j=1}^{p}(X_j^2 + Y_j^2) - \sum_{j=p+1}^{n}(X_j^2 + Y_j^2)$ and U is as above (cf. [7]).

The description of the unitary dual of $U(p,q)H_n$ is given in [15]. Let $\mathscr{P}$ be the cone of the bi-$U(p,q)$-invariant, positive-definite distributions on $U(p,q)H_n$. We say that $T \in \mathscr{P}$ is extremal in $\mathscr{P}$ if and only if $S \in \mathscr{P}$ and $T - S \in \mathscr{P}$ imply $S = \alpha T$ for some $\alpha \in \mathrm{R}$. For $S, S' \in \mathscr{P}$ we write $S \sim S'$ if and only if $S = \alpha S'$ for some $\alpha > 0$. Thus $\sim$ is an equivalence relation on $\mathscr{P}$. For $S \in \mathscr{P}$ we put $[S]$ for its equivalence class.

By general theory (see [5], [13]) one knows that there exists a one to one correspondence between the set of unitary representations π of $U(p,q)H_n$ admitting a cyclic distribution vector ξ_π fixed by $U(p,q)$ (spherical representations), and the set of the equivalence class of bi-$U(p,q)$-invariant, positive-definite distributions. More precisely, for such π and ξ_π, and for $\varphi \in C^\infty(U(p,q)H_n)$, it is easy to see that $\pi(\varphi)\xi_\pi$ is a C^∞-vector for π. Define $T_\pi \in D'(U(p,q)H_n)$ by

$$T_\pi(\varphi) = \langle \xi_\pi, \pi(\varphi)\xi_\pi \rangle$$

(T_π is called a reproducing distribution for π.) With these notations, the quoted correspondence is given by $\pi \to [T_\pi]$. We recall also that π is irreducible if and only if T_π is extremal in $\mathscr{P}$. As usual, we will identify the bi-$U(p,q)$-invariant distributions on $U(p,q)H_n$ with the $U(p,q)$-invariant distributions on H_n.

Let us recall some facts concerning the compact case $p = n$, $q = 0$, i.e., when $U(p,q) = U(n)$. Since $(U(n), H_n)$ is a Gelfand pair, the convolution algebra of the $U(n)$-invariant integrable functions on H_n is commutative. Its spectrum, denoted by $\Delta(U(n), H_n)$ can be identified, via integration, with the set of bounded spherical functions of the pair $(U(n), H_n)$. Moreover, for this Gelfand pair (as remarked in [2]), the set of bounded spherical functions is precisely the set of positive definite spherical functions, and so $\Delta(U(n), H_n)$ *is the set of extremal points in the cone of $U(n)$-invariant, positive definite functions on H_n*. These spherical functions can be classified (see [1]) as:

a) The spherical functions of type I, i.e., those that restricted to the center of H_n are nontrivial characters. These are given by

$$\Phi_{\lambda,k}(z,t) = e^{-i\lambda t}\mathscr{L}_k^{n-1}\left(\frac{|\lambda|}{2}|z|^2\right)e^{-\frac{|\lambda|}{4}|z|^2}, \qquad \lambda \neq 0,\ k \geq 0$$

where $\mathscr{L}_k^{n-1}$ is the Laguerre polynomial of order $n-1$ and degree k normalized by $\mathscr{L}_k^{n-1}(0) = 1$.

b) The spherical functions η_w of type II , i.e., those that are constant on the center. They are given, for $w \in \mathbb{C}^n - \{0\}$, by

$$\eta_w(z,t) = \frac{2^{n-1}(n-1)!}{(|z||w|)^{n-1}} J_{n-1}(|z||w|)$$

where J_{n-1} is the Bessel function of order $n-1$ of the first kind, and by

$$\eta_0(z,t) = 1.$$

In [3] is defined a map $\mathscr{E} : \Delta(U(n), H_n) \to [0,\infty) \times \mathbb{R}$ by $\mathscr{E}(\Psi) = (-\widehat{L}(\Psi), i\widehat{U}(\Psi))$, where $\widehat{L}(\Psi)$ and $\widehat{U}(\Psi)$ denote the eigenvalues of L and U respectively, associated to Ψ. The image of $\mathscr{E}$ is the so called Heisenberg fan $\mathscr{A}(U(n), H_n)$ and it is the set

$$\left\{(|\lambda|(2k+n), \lambda) : \lambda \neq 0,\ k \in \mathbb{N} \cup \{0\}\right\} \cup \left\{[0,\infty) \times \{0\}\right\}.$$

There, it is proved that $\mathscr{E}$ is a homeomorphism from $\Delta(U(n), H_n)$ (equipped with the Gelfand topology) onto the Heisenberg fan (provided with the topology induced by $\mathbb{R}^2$).

We assume from now on that $n \geq 2$, $p \geq 1$, $q \geq 1$ and we turn now to the generalized Gelfand pair $(U(p,q), H_n)$, $p+q=n$. Let E be the set of extremal points of $\mathscr{P}$. Motivated by the quoted results in the compact case, we define

DEFINITION 1.1. $\Delta(U(p,q), H_n) = E/\sim$, equipped with the quotient topology of the pointwise convergence topology of $\mathscr{S}'(H_n)$.

In order to describe $\Delta(U(p,q), H_n)$ we need to recall some facts. For $\lambda \neq 0$, let π_λ denote the Schroedinger representation of H_n, realized on $L^2(\mathbb{R}^n)$. According to [10], this representation can be extended to a representation $\widetilde{\pi}_\lambda$ of $U(p,q)H_n$ by the rule $\widetilde{\pi}_\lambda(k,z,t) = W_\lambda(k)\pi_\lambda(z,t)$, for $k \in U(p,q)$, $(z,t) \in H_n$, where W_λ denotes the metaplectic representation of $U(p,q)$ (defined there) acting on $L^2(\mathbb{R}^n)$. For $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_i \in \mathbb{N} \cup \{0\}$, let $\|\alpha\| = \sum_{i=1}^{p} \alpha_i - \sum_{i=p+1}^{n} \alpha_i$ and, for $x = (x_1, \ldots, x_n)$, let $h_\alpha(x) = h_{\alpha_1}(x_1) \ldots h_{\alpha_n}(x_n)$ where h_l denotes the l-th Hermite function. For $k \in Z$, let H_k be the closed subspace of $L^2(\mathbb{R}^n)$ generated by $\{h_\alpha : \|\alpha\| = k\}$. Then (see e.g. [4]) W_λ decomposes in irreducible representations of $U(p,q)$ as

$$L^2(\mathbb{R}^n) = \bigoplus_{k \in \mathbb{Z}} H_k.$$

Let γ_k denote the restriction of W_λ to H_k and let γ_k^* be its adjoint representation.

For $v \in \mathbb{C}^n$ let also $\chi_v(z,t) = e^{i \operatorname{Re} B(z,v)}$ and let K_v be the stabilizer of v in $U(p,q)$, that is $K_v = \{g \in U(p,q) : gv = v\}$, and extends χ_v to $K_v H_n$ by $\widetilde{\chi}_v(k,z,t) = \chi_v(z,t)$. Then $\Xi_v := \operatorname{Ind}_{K_v H_n}^{U(p,q)H_n}(\widetilde{\chi}_v)$ is an irreducible representation of $U(p,q)H_n$, and for $v, v' \in \mathbb{C}^n$ it holds that Ξ_v is equivalent to $\Xi_{v'}$ if and only if $B(v) = B(v')$ (see [15]).

The spherical representations of $U(p,q)H_n$ are given in [15]. They are

i) Those of the form $\gamma_k^* \otimes \widetilde{\pi_\lambda}$. For them, a reproducing distribution $S_{\lambda,k}$, found in [14], is given by

$$\langle S_{\lambda,k}, \varphi \rangle = \operatorname{tr} \pi_\lambda(\varphi)_{|H_k} \tag{1.1}$$

ii) Those of the form Ξ_v. A corresponding reproducing distribution is given by

$$\langle S_\sigma, \varphi \rangle = \int_{B(u,u)=-\sigma} \int_{H_n} e^{i \operatorname{Re} B(u,z)} \varphi(z,t)\, dz\, dt\, d\mu_\sigma(u) \tag{1.2}$$

where $\sigma = B(v,v)$ and $d\mu_\sigma$ denotes the surface measure on $B(u,u) = \sigma$. In other words S_σ is a sort of Fourier Transform of the measure $d\mu_\sigma$.

iii) The trivial representation, with reproducing distribution 1.

The above list shows that for each $[T_\pi] \in \Delta(U(p,q), H_n)$, T_π is a tempered distribution on H_n.

Observe that if Ψ is an extremal point of $\mathscr{P}$, then Ψ is a joint eigendistribution of $-L$ and iU (cf. [5]). Indeed, $-L(S_{\lambda,k}) = |\lambda|(2k+p-q)S_{\lambda,k}$, $iU(S_{\lambda,k}) = \lambda S_{\lambda,k}$ and $-L(S_\sigma) = \sigma S_\sigma$, $iU(S_\sigma) = 0$ (cf. [14], [7]). Following [3], we define the map $\mathscr{E} : \Delta(U(p,q), H_n) \to \mathbb{R}^2$ by

$$\mathscr{E}([\Psi]) = (-\widehat{L}(\Psi), i\widehat{U}(\Psi)),$$

where $\widehat{L}(\Psi)$ and $\widehat{U}(\Psi)$ denote the eigenvalues of L and U respectively, associated to Ψ. Let $\mathscr{A}(U(p,q), H_n)$ denote the image of $\mathscr{E}$. Equipped with the relative topology of $\mathbb{R}^2$ it is called the Heisenberg fan of the generalized Gelfand pair $(U(p,q), H_n)$ and it is given by

$$\mathscr{A}(U(p,q), H_n) = \left\{(|\lambda|(2k+p-q), \lambda) : \lambda \neq 0, k \in \mathbb{Z}\right\} \cup \left\{(\sigma, 0) : \sigma \in \mathbb{R}\right\}$$

Our main result is the following

THEOREM 1.2. *The map $\mathscr{E} : \Delta(U(p,q), H_n) - \{[1]\} \to \mathscr{A}(U(p,q), H_n)$ is a homeomorphism.*

REMARK 1.3. As observed by J. Faraut in [5], and in contrast with the compact case, in the case of a generalized Gelfand pair a spherical distribution is not necessarily an extremal point of $\mathscr{P}$. For example, in our case, the solution space of $-L(S)=0, iU(S)=0$ is two dimensional and a basis is given by $\{1, S_0\}$. After the proof of the above Theorem, it is easy to see that [1] is an isolate point of $\Delta(U(p,q), H_n)$.

2. The joint eigendistributions of L and iT

We begin this section by describing the space $\mathscr{S}'(\mathsf{C}^n)^{U(p,q)}$ of tempered distributions which are $U(p,q)$ invariant. We adapt the results by A. Tengstrand detailed in [12], for the passage from the real to the complex case.

To this end, we take bipolar coordinates on C^n: for $(x_1, y_1, \ldots, x_n, y_n)$, we set $\sigma = \sum_{j=1}^{p}(x_j^2+y_j^2) - \sum_{j=p+1}^{n}(x_j^2+y_j^2)$, $\rho = \sum_{j=1}^{n}(x_j^2+y_j^2)$, $u = (x_1, y_1, \ldots, x_p, y_p)$ and $v = (x_{p+1}, y_{p+1}, \ldots, x_n, y_n)$. So $u = \left(\frac{\rho+\sigma}{2}\right)^{\frac{1}{2}}\omega_u$, $v = \left(\frac{\rho-\sigma}{2}\right)^{\frac{1}{2}}\omega_v$, where ω_u belongs to the $2p-1$ dimensional sphere S^{2p-1} and $\omega_v \in S^{2q-1}$.

By the change of variables theorem, we have that

$$\int_{\mathsf{C}^n} f(z)\,dz = \int_{-\infty}^{\infty}\int_{|\sigma|<\rho}\int_{S^{2p-1}\times S^{2q-1}} f\left(\left(\frac{\rho+\sigma}{2}\right)^{\frac{1}{2}}\omega_u, \left(\frac{\rho-\sigma}{2}\right)^{\frac{1}{2}}\omega_v\right) d\omega_u\, d\omega_v (\rho+\sigma)^{p-1}(\rho-\sigma)^{q-1}\, d\rho\, d\sigma$$

We define the map M on $\mathscr{S}(\mathsf{R}^{2n})$ by

$$Mf(\rho,\tau) = \int_{S^{2p-1}\times S^{2q-1}} f\left(\left(\frac{\rho+\tau}{2}\right)^{\frac{1}{2}}\omega_u, \left(\frac{\rho-\tau}{2}\right)^{\frac{1}{2}}\omega_v\right) d\omega_u\, d\omega_v,$$

and

$$Nf(\tau) = \int_{|\tau|}^{\infty} Mf(\rho,\sigma)(\rho+\tau)^{p-1}(\rho-\tau)^{q-1} d\rho.$$

In other words, Nf is the integral of f on the surface $B(z,z)=\tau$ provided with a suitable surface measure.

Let H denote the Heaviside function (i.e., $H(\tau)=\chi_{(0,\infty)}(\tau)$) and let $\mathscr{H}$ be the space of the functions $\varphi : \mathsf{R} \to \mathsf{C}$ such that $\varphi(\tau) = \varphi_1(\tau) + \tau^{n-1}\varphi_2(\tau)H(\tau)$, $\varphi_1, \varphi_2 \in \mathscr{S}(\mathsf{R})$. It is proved in [12] that $\mathscr{H}$, with an adequate topology, is a Fréchet space. Moreover, following straighforward the proof of Lemma 4.2 and Lemma 4.3 there, we obtain that

$$N : \mathscr{S}(\mathsf{R}^{2n} - \{0\}) \to \mathscr{S}(\mathsf{R}), \qquad \text{and} \qquad N : \mathscr{S}(\mathsf{R}^{2n}) \to \mathscr{H}$$

are (linear) continuous, surjective maps. Now, let $\mu \in \mathscr{S}'(\mathsf{R}^{2n})^{U(p,q)}$. Then, there exists a unique $T \in \mathscr{S}'(\mathsf{R})$ such that

$$\langle \mu, f\rangle = \langle T, Nf\rangle \qquad \text{for every} \quad f \in \mathscr{S}(\mathsf{R}^{2n} - \{0\}).$$

Indeed, let $\Phi(x_1, y_1, \dots, x_n, y_n) = (\rho, \tau, \omega_u, \omega_v)$ the change of coordinates and let $J(\Phi^{-1})$ be the Jacobi determinant. If $\mu \circ \Phi$ is the distribution defined by $\langle \mu \circ \Phi, f\rangle = \langle \mu, (f \circ \Phi^{-1}) J(\Phi^{-1})\rangle$, then as $U(p, q)$ acts transitively on the surface $B(z, z) = \tau$, $\mu \circ \Phi$ is independent of ρ, ω_u and ω_v. So, T is well defined and the uniqueness of T follows from the surjectivity of N.

Moreover, the adjoint map of N, $N' : \mathscr{H}' \to \mathscr{S}'(\mathsf{R}^{2n})^{U(p,q)}$, is injective and the same lines of Theorem 5.1 in [12] prove that N' is a homeomorphism.

For $f \in \mathscr{S}(H_n)$, we will write $Nf(\tau, t)$ for $N(f(., t))(\tau)$. We have that for all $\varphi \in \mathscr{S}(\mathsf{R}^2)$

$$\int_{-\infty}^{\infty} \int_{C^n} \varphi(B(z), t) f(z, t)\, dz\, dt = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Nf(\tau, t)\varphi(\tau, t)\, d\tau\, dt.$$

Our next step is to compute, for $\sigma \in \mathsf{R}$, the solutions $S \in S'(H_n)^{U(p,q)}$ of the problem

$$\begin{cases} -L(S) = \sigma S, \\ iU(S) = 0 \end{cases} \tag{2.1}$$

i.e., the $U(p, q)$ invariant tempered joint eigendistributions of $-L$ and iU corresponding to a pair $(\sigma, \lambda) \in \mathscr{A}(U(p, q), H_n)$ with $\lambda = 0$. For such a solution S, $U(S) = 0$ gives $S = F \otimes 1$ with $F \in S'(\mathsf{R}^{2n})$. Since

$$L = \Box + \left(\sum_{j=1}^{p}(x_j^2 + y_j^2) - \sum_{j=p+1}^{n}(x_j^2 + y_j^2)\right)\frac{\partial^2}{\partial t^2} + \frac{\partial}{\partial t}\sum_{j=1}^{n}\left(x_j\frac{\partial}{\partial y_j} - y_j\frac{\partial}{\partial x_j}\right),$$

where

$$\Box = \sum_{j=1}^{p}\left(\frac{\partial^2}{\partial {x_j}^2} + \frac{\partial^2}{\partial {y_j}^2}\right) - \sum_{j=p+1}^{n}\left(\frac{\partial^2}{\partial {x_j}^2} + \frac{\partial^2}{\partial {y_j}^2}\right),$$

and S is $U(p, q)$ invariant, from (2.1) we get

$$-\Box(F) = \sigma F. \tag{2.2}$$

Conversely, for each solution $F \in S'(\mathsf{R}^{2n})$ of this equation, $S = F \otimes 1$ solves (2.1). It is proved in [12] that $N(\Box f) = D(Nf)$ for $f \in S(\mathsf{R}^{2n})$, where D is the differential operator

$$D = 4\left(\tau\frac{\partial^2}{\partial \tau^2} + (2 - n)\frac{\partial}{\partial \tau}\right). \tag{2.3}$$

Writing $F = N'(T)$ with $T \in \mathscr{H}'$, (2.2) becomes $D'T = -\sigma T$, where D' is the adjoint of D given by $D'T = 4(\tau T'' + nT')$, i.e., (2.2) is equivalent to

$$D'T + \sigma T = 0. \tag{2.4}$$

If $T \in \mathscr{H}'$ is a solution of (2.4) then (since D is elliptic) its restrictions $T_{|\mathscr{D}(0,\infty)}$ and $T_{|\mathscr{D}(-\infty,0)}$) are functions belonging to $C^\infty(0,\infty)$ and $C^\infty(-\infty,0)$, respectively. They are solutions, on the respective semiaxis, of the equation

$$4(\tau v''(\tau) + nv'(\tau)) + \sigma v(\tau) = 0. \tag{2.5}$$

Consider the case $\sigma > 0$. A computation shows that a function $y : (0,\infty) \to \mathsf{R}$ is a solution of (2.5) if and only if

$$y(\tau) = \frac{w((\sigma\tau)^{\frac{1}{2}})}{(\sigma\tau)^{\frac{n-1}{2}}}$$

for some w that solves, on $(0,\infty)$, the Bessel equation of order $n-1$

$$\tau^2 w''(\tau) + \tau w'(\tau) + (\tau^2 - (n-1)^2)w(\tau) = 0, \qquad \tau > 0. \tag{2.6}$$

For $m \in \mathsf{N} \cup \{0\}$, let J_m be the Bessel function of first kind of order m,

$$J_m(\tau) = \left(\frac{\tau}{2}\right)^m \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+m)!} \left(\frac{\tau}{2}\right)^{2k}. \tag{2.7}$$

and let $N_m : (0,\infty) \to \mathsf{R}$ be the Neumann function defined by

$$N_m(\tau) = \frac{2}{\pi} J_m(\tau) \log\left(\frac{\tau}{2}\right) - \frac{1}{\pi} \sum_{k=0}^{m-1} \frac{(m-k-1)!}{k!} \left(\frac{\tau}{2}\right)^{2k-m} \tag{2.8}$$

$$- \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+m)!} [\psi(k+1) + \psi(k+m+1)] \left(\frac{\tau}{2}\right)^{m+2k} \tag{2.9}$$

where $\psi(m+1) := -\gamma + \sum_{j=1}^{m} \frac{1}{j}$ and γ is the Euler constant.

For $\tau > 0$, $\sigma > 0$ and $m \in \mathsf{N} \cup \{0\}$, let

$$y_m(\tau) = m! \frac{J_m((\sigma\tau)^{\frac{1}{2}})}{(\sigma\tau)^{\frac{m}{2}}}, \qquad z_m(\tau) = \frac{N_m((\sigma\tau)^{\frac{1}{2}})}{(\sigma\tau)^{\frac{m}{2}}}. \tag{2.10}$$

We observe that, for $\tau > 0$,

$$y_m(\tau) = m! \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+m)!} \left(\frac{\sigma\tau}{4}\right)^k \tag{2.11}$$

and so y_m has an analytic extension to R, still denoted by y_m, given by (2.11). Note that $\{J_{n-1}, N_{n-1}\}$ is a basis of the space of the solutions of (2.6) on $(0,\infty)$ and so $\{y_{n-1}, z_{n-1}\}$ is a basis of the solution space of (2.5) on $(0,\infty)$. Moreover, since y_{n-1} is analytic on R, it solves (2.5) on the whole line. A suitable (for our purposes) solution $\widetilde{y}_{n-1}$, linearly independent with y_{n-1}, of the equation (2.5) on $(-\infty, 0)$ can be chosen as follows. We propose $\widetilde{y}_{n-1}(\tau) = c(\tau)y_{n-1}(\tau)$ which gives the equation $\tau y_{n-1}(\tau)c''(\tau) + (ny_{n-1}(\tau) + 2\tau y'_{n.1}(\tau))c'(\tau) = 0$ that solved for c gives $c(\tau) = A\int_{-\infty}^{\tau} \frac{1}{y_{n-1}^2(s)|s|^n}\,ds + B$ with A and B arbitrary constants ($c(\tau)$ is well defined by Lemma 2.1 below). We pick $A = 1$, $B = 0$ to obtain

$$\widetilde{y}_{n-1}(\tau) = y_{n-1}(\tau)\int_{-\infty}^{\tau} \frac{1}{y_{n-1}^2(s)|s|^n}\,ds.$$

So a basis of the solution space of (2.5) on $(-\infty, 0)$ is given by $\{y_{n-1}, \widetilde{y}_{n-1}\}$.

LEMMA 2.1. *Assume that $\sigma > 0$. Then for $\tau < 0$ it holds that $y_{n-1}(\tau) > 0$ and $y'_{n-1}(\tau) < 0$. Moreover, there exist positive constants A, B such that for τ negative with absolute value large enough*

$$y_{n-1}(\tau) \geq Ae^{B|\tau|^{\frac{1}{2}}}, \qquad y'_{n-1}(\tau) \leq -Ae^{B|\tau|^{\frac{1}{2}}}. \tag{2.12}$$

PROOF. For $\tau < 0$, from (2.11),

$$\begin{aligned} y_{n-1}(\tau) &= (n-1)!\sum_{k=0}^{\infty} \frac{1}{k!(k+n-1)!}\left(\frac{\sigma|\tau|}{4}\right)^k \\ &\geq (n-1)!\sum_{k=0}^{\infty} \frac{1}{(k+n-1)!^2}\left(\frac{\sigma|\tau|}{4}\right)^k \\ &\geq (n-1)!\left(\frac{\sigma|\tau|}{4}\right)^{-(n-1)} \sum_{k=0}^{\infty} \frac{1}{(2(k+n-1))!}\left(\sqrt{\frac{\sigma|\tau|}{4}}\right)^{2(k+n-1)}. \end{aligned}$$

So, $y_{n-1}(\tau) > 0$ and

$$y_{n-1}(\tau) \geq (n-1)!\left(\frac{\sigma|\tau|}{4}\right)^{-(n-1)}\left(\cosh\sqrt{\frac{\sigma|\tau|}{4}} - P_{n-2}\left(\sqrt{\frac{\sigma|\tau|}{4}}\right)\right),$$

where P_{n-2} is the Taylor polynomial, around the origin, and of degree $n-2$, of cosh. This gives the first inequality in (2.12). Similarly, for $\tau < 0$,

$$\begin{aligned} y'_{n-1}(\tau) &= -(n-1)!\frac{\sigma}{4}\sum_{k=1}^{\infty}\frac{1}{(k-1!)(k+n-1)!}\left(\frac{\sigma|\tau|}{4}\right)^{k-1} \\ &\leq -(n-1)!\sum_{k=1}^{\infty}\frac{1}{(k+n-1)!^2}\left(\frac{\sigma|\tau|}{4}\right)^{k-1} \\ &= -(n-1)!\sum_{k=0}^{\infty}\frac{1}{(k+n)!^2}\left(\frac{\sigma|\tau|}{4}\right)^{k}. \end{aligned}$$

In particular, $y'_{n-1}(\tau) < 0$. Proceeding as before we obtain that

$$y'_{n-1}(\tau) \leq -(n-1)!\left(\frac{\sigma|\tau|}{4}\right)^{-n}\left(\cosh\sqrt{\frac{\sigma|\tau|}{4}} - Q_{n-1}\left(\sqrt{\frac{\sigma|\tau|}{4}}\right)\right)$$

for a polynomial Q_{n-1} of degree $n-1$, which implies the remaining assertion of the Lemma.

The following provides information about the asymptotic behavior of z_{n-1} and $\widetilde{y}_{n-1}$ at the origin and at $-\infty$.

LEMMA 2.2. *Assume that* $\sigma > 0$. *Then*

i) $\lim_{\tau\to 0^+} \tau^{n-1} z_{n-1}(\tau) = -\frac{2^{n-1}\Gamma(n-1)}{\pi\sigma^{n-1}}$, $\lim_{\tau\to+\infty} z_{n-1}(\tau) = 0$

ii) $\lim_{\tau\to 0^-} \tau^{n-1}\widetilde{y}_{n-1}(\tau) = -\frac{1}{n-1}$, $\lim_{\tau\to-\infty} \widetilde{y}_{n-1}(\tau) = 0$

iii) $\widetilde{y}_{n-1}$ *is integrable on* $(-\infty, -1)$.

PROOF. The assertions in i) are a direct consequence of the asymptotic behavior of N_{n-1} at the origin and at $+\infty$. The first assertion of ii) follows from the definition of $\widetilde{y}_{n-1}$ and the L'Hôpital rule. To see the second one, we note that from Lemma 2.1 we have $\widetilde{y}_{n-1}(\tau) > 0$ for $\tau < 0$. Also, for τ negative with absolute value large enough,

$$\begin{aligned} 0 < \widetilde{y}_{n-1}(\tau) = y_{n-1}(\tau)\int_{-\infty}^{\tau}\frac{1}{y_{n-1}^2(s)|s|^n}\,ds \\ \leq \int_{-\infty}^{\tau}\frac{1}{y_{n-1}(s)|s|^n}\,ds \leq A'e^{-B|\tau|^{\frac{1}{2}}} \end{aligned}$$

for some positive constants A' and B. Thus iii) holds and also $\lim_{\tau\to-\infty}\widetilde{y}_{n-1}(\tau) = 0$.

LEMMA 2.3. *Assume that* $\sigma > 0$. *Then*

i) *there exist the limits* $\lim_{\tau\to 0^+} \tau^n z'_{n-1}(\tau)$, $\lim_{\tau\to 0^-} \tau^n \widetilde{y}'_{n-1}(\tau)$ *and they are finite and different from zero.*

ii) $\lim_{\tau\to\infty} z'_{n-1}(\tau) = 0$ *and* $\lim_{\tau\to-\infty} \widetilde{y}'_{n-1}(\tau) = 0$.

PROOF. A computation using the definition of z_{n-1}, that $2N'_{n-1} = N_{n-2} - N_n$ and the asymptotic behavior of then Neumann functions (see [8], p. 134 and 135) gives the assertions of the lemma about z_{n-1}. On other hand, from the definition of $\widetilde{y}_{n-1}$,

$$\widetilde{y}'_{n-1}(\tau) = y'_{n-1}(\tau)\int_{-\infty}^{\tau} \frac{1}{y^2_{n-1}(s)|s|^n}ds + \frac{1}{y_{n-1}(\tau)|\tau|^n}. \tag{2.13}$$

Since y_{n-1} is continuous and $y_{n-1}(0) = 1$, from (2.13) it follows that the limit $\lim_{\tau\to 0^-} \tau^n\widetilde{y}'_{n-1}(\tau)$ exists, is finite and different from zero. To prove the remaining assertion of the lemma we rewrite (2.5) as

$$4\frac{d}{d\tau}(\tau\widetilde{y}'_{n-1}(\tau)) = -4(n-1)\widetilde{y}'_{n-1}(\tau) - \sigma\widetilde{y}'_{n-1}(\tau).$$

Now, for $\tau < -1$, an integration on $(\tau, -1)$ gives

$$\begin{aligned}-4(\widetilde{y}'_{n-1}(-1) + \tau\widetilde{y}'_{n-1}(\tau))\\ = -4(n-1)(\widetilde{y}_{n-1}(-1) - \widetilde{y}_{n-1}(\tau)) - \sigma\int_{\tau}^{-1}\widetilde{y}_{n-1}(s)\,ds\end{aligned}$$

and so $\widetilde{y}'_{n-1}(\tau) = A\tau^{-1}\widetilde{y}_{n-1}(\tau) + B\tau^{-1} - \sigma\tau^{-1}\int_\tau^{-1}\widetilde{y}_{n-1}(s)\,ds$ with A and B independent of τ. Thus, by Lemma 2.2, $\lim_{\tau\to-\infty}\widetilde{y}'_{n-1}(\tau) = 0$.

From the asymptotic behavior of J_{n-1} at $+\infty$ (cf. [8], p. 134–135), we have that $\lim_{\tau\to+\infty} y_{n-1}(\tau) = 0$. In particular, $y_{n-1}H \in \mathscr{H}'$.

PROPOSITION 2.4. *For* $\sigma > 0$ *the distribution* $T = (y_0H)^{(n-1)}$ *is a solution in* $\mathscr{H}'$ *of* $D'T + \sigma T = 0$.

PROOF. We first look for distributions $\widetilde{T} = y_{n-1}H + \sum_{j=0}^{n-2} c_j\delta^{(j)}$ with $c_0, \dots, c_{n-2} \in \mathsf{R}$ such that $D'\widetilde{T} + \sigma\widetilde{T} = 0$. Let $S = y_{n-1}H$. A computation shows that $D'S = 4(n-1)y_{n-1}(0)\delta = 4(n-1)\delta$. Also, for $0 \le j \le n-2$, $D'(\delta^{(j)}) + \sigma\delta^{(j)} = 4(n-2-j)\delta^{(j+1)} + \sigma\delta^{(j)}$. So $D'\widetilde{T} + \sigma\widetilde{T} = 0$ if and only if $c_0 = -\frac{4(n-1)}{\sigma}$ and $c_j = -\frac{4(n-1-j)}{\sigma}c_{j-1}$ for $j = 1, \dots, n-2$, i.e., if and only if $c_j = \left(-\frac{4}{\sigma}\right)^{j+1}\frac{(n-1)!}{(n-2-j)!}$. Let $\widetilde{T}$ be defined with these constants and observe that

$$(y_0H)^{(n-1)} = y_0^{(n-1)} + \sum_{j=0}^{n-2} y_0^{(j)}(0)\delta^{(n-2-j)},$$

so the proposition will follow if we show that for some constant $A \neq 0$,

$$y_0^{(n-1)} = Ay_{n-1} \quad \text{and} \quad y_0^{(j)}(0) = Ac_{n-2-j} \qquad \text{for } 0 \leq j \leq n-2,$$

but, taking the corresponding derivatives in the series expansion for y_0, it is easy to see that these conditions are fulfilled by $A = \left(-\frac{\sigma}{4}\right)^{n-1}$.

For $g \in C(0, +\infty)$ with growth at most polynomial at $+\infty$ and such that $\lim_{\tau \to 0^+} \tau^{n-1} g(\tau)$ exists and is finite we define $Pf^+(g) \in \mathscr{H}'$ by

$$\langle Pf^+(g), \varphi \rangle =: \int_0^1 g(\tau)\left(\varphi(\tau) - \sum_{j=0}^{n-2} \frac{\varphi^{(j)}(0)}{j!}\tau^j\right) d\tau + \int_1^\infty g(\tau)\varphi(\tau)\, d\tau$$

Similarly, for $g \in C(-\infty, 0)$ satisfying the analogous conditions at $-\infty$ and at the origin, let $Pf^-(g) \in \mathscr{H}'$ given by

$$\langle Pf^-(g), \varphi \rangle =: \int_{-1}^0 g(\tau)\left(\varphi(\tau) - \sum_{j=0}^{n-2} \frac{\varphi^{(j)}(0)}{j!}\tau^j\right) dt + \int_{-\infty}^{-1} g(\tau)\varphi(\tau)\, d\tau.$$

We recall that for $\varphi \in \mathscr{H}$, since $\varphi(\tau) = \varphi_1(\tau) + \tau^{n-1} H(\tau)\varphi_2(\tau)$ with $\varphi_1, \varphi_2 \in S(\mathsf{R})$, φ has an asymptotic development, near the origin, of the form

$$\varphi(\tau) \simeq \sum_{j \geq 0} B_j(\varphi)\tau^j + \sum_{j \geq n-1} A_j(\varphi)\tau^j H(\tau) \tag{2.14}$$

with $A_j(\varphi) = 0$ for $0 \leq j \leq n-2$. It is proved in [12] that if $S \in \mathscr{H}'$ is supported at the origin then $S = \sum_{j=0}^m \alpha_j A_j + \sum_{j=0}^m \beta_j B_j$ for some $m \in \mathsf{N} \cup \{0\}$ and $\alpha_1, \ldots, \alpha_m; \beta_1, \ldots, \beta_m \in \mathsf{C}$.

If $v \in C^2(0, \infty)$ (respectively $v \in C^2(-\infty, 0)$) is a solution of $D'v + \sigma v = 0$ on $(0, \infty)$ (respectively on $(-\infty, 0)$) satisfying that $\lim_{\tau \to 0^*}(\tau^{n-1} v(\tau))$ exists and is finite (resp. $\lim_{\tau \to 0^-}(\tau^{n-1} v(\tau))$ exists and is finite), an integration by parts shows that, for $0 < a < b \leq +\infty$ (resp. $-\infty \leq a < b < 0$),

$$\int_a^b v(\tau)(D + \sigma I)(\varphi)(\tau)\, d\tau = R(v, b, \varphi) - R(v, a, \varphi) \tag{2.15}$$

where, for $\xi \in \mathsf{R} - \{0\}$,

$$R(v, \xi, \varphi) := 4\xi(v(\xi)\varphi'(\xi) - v'(\xi)\varphi(\xi)) + 4(1-n)v(\xi)\varphi(\xi) \tag{2.16}$$

and $R(v, \pm\infty, \varphi) := \lim_{\xi \to \pm\infty} R(v, \xi, \varphi)$.

LEMMA 2.5. i) *Assume* $\sigma > 0$. *Then there exist constants* $c_0, \dots, c_{n-1}$ *and* $d_0, \dots, d_{n-1}$ *with* $c_{n-1} \neq 0$ *and* $d_{n-1} \neq 0$ *such that*

$$(D' + \sigma I)Pf^{+}(z_{n-1}) = \sum_{j=0}^{n-2} c_j B_j + c_{n-1}(A_{n-1} + B_{n-1}), \tag{2.17}$$

$$(D' + \sigma I)Pf^{-}(\widetilde{y}_{n-1}) = \sum_{j=0}^{n-2} d_j B_j + d_{n-1}B_{n-1}. \tag{2.18}$$

ii) *For* $\sigma = 0$ *and* $a, b \in \mathsf{R}$, *the assertions in* i) *hold with* z_{n-1} *and* $\widetilde{y}_{n-1}$ *replaced by* $a + b\tau^{1-n}$.

PROOF. A computation shows that for $\varphi \in \mathscr{H}$, $P_{n-2}(D\varphi) = D(P_{n-2}\varphi)$, where $P_{n-2}(\varphi)$ denotes the Taylor polynomial of φ of degree $n-2$ around the origin. Then, from (2.15),

$$\begin{aligned}
&\langle Pf^{+}(z_{n-1}), (D + \sigma I)\varphi\rangle \\
&\quad = \int_0^1 z_{n-1}((D + \sigma I)\varphi - P_{n-2}(D + \sigma I)\varphi) + \int_1^\infty z_{n-1}(D + \sigma I)\varphi \\
&\quad = \int_0^1 z_{n-1}(D + \sigma I)(\varphi - P_{n-2}\varphi) + \int_1^\infty z_{n-1}(D + \sigma I)\varphi \\
&\quad = R(z_{n-1}, 1, \varphi - P_{n-2}\varphi) - \lim_{\varepsilon\to 0} R(z_{n-1}, \varepsilon, \varphi - P_{n-2}\varphi) \\
&\qquad\quad + \lim_{b\to+\infty} R(z_{n-1}, b, \varphi) - R(z_{n-1}, 1, \varphi).
\end{aligned}$$

By Lemmas 2.2 and 2.3, $\lim_{b\to\infty} R(z_{n-1}, b, \varphi) = 0$. Thus

$$\langle (D' + \sigma I)Pf^{+}(z_{n-1}), \varphi\rangle = -R(1, P_{n-2}(\varphi)) - \lim_{\varepsilon\to 0^+} R(\varepsilon, \varphi - P_{n-2}(\varphi)).$$

A computation using the asymptotic development (2.14) gives that

$$R(z_{n-1}, \varepsilon, \varphi - P_{n-2}\varphi) = -4z'_{n-1}(\varepsilon)\varepsilon^{n}(A_{n-1}(\varphi) + B_{n-1}(\varphi)) + o(\varepsilon)$$

with $\lim_{\varepsilon\to 0^+} o(\varepsilon) = 0$. Then, by Lemma 2.3,

$$\langle (D' + \sigma I)Pf^{+}(z_{n-1}), \varphi\rangle = -R(1, P_{n-2}\varphi) - c_{n-1}(A_{n-1}(\varphi) + B_{n-1}(\varphi))$$

for some constant with $c_{n-1} \neq 0$. This gives (2.17) and the proof of (2.18) is similar using that $R(z_{n-1}, -\varepsilon, \varphi - P_{n-2}\varphi) = -4\widetilde{y}'_{n-1}(-\varepsilon)(-\varepsilon)^{n}B_{n-1}(\varphi) + o(\varepsilon)$. The same arguments give also ii).

For $S \in \mathscr{H}'$ supported at the origin we have

$$S = \sum_{j=0}^{\infty} \alpha_j A_j + \sum_{j=0}^{\infty} \beta_j B_j$$

with each $\alpha_j, \beta_j \in C$ and $\alpha_j = \beta_j = 0$ for j large enough. So, from (2.14) a computation gives that, for $\sigma \in \mathsf{R}$,

$$\begin{aligned}(2.19)\quad D'S + \sigma S &= \sigma\alpha_0 A_0 + \sigma\beta_0 B_0 \\ &+ \sum_{j=1}^{\infty}(4j(j+1-n)\alpha_{j-1} + \sigma\alpha_j)A_j + \sum_{j=0}^{\infty}(4j(j+1-n)\beta_{j-1} + \sigma\beta_j)B_j\end{aligned}$$

LEMMA 2.6. *Let $S \in \mathscr{H}'$ supported at the origin and let $\sigma \neq 0$.*

i) *If*

$$(2.20)\qquad (D' + \sigma I)S = \sum_{j=0}^{n-2} c_j B_j + c_{n-1}A_{n-1} + d_{n-1}B_{n-1}$$

with $c_0, c_1, \ldots, c_{n-1}, d_{n-1} \in C$, then $c_{n-1} = d_{n-1} = 0$.

ii) *If $D'S + \sigma S = 0$, then $S = 0$.*

PROOF. i) From (2.20) and (2.19) we get, for $j \geq n$,

$$(2.21)\qquad 4j(j+1-n)\alpha_{j-1} + \sigma\alpha_j = 0$$

and also $\sigma\alpha_{n-1} = c_{n-1}$. So $c_{n-1} \neq 0$ implies $\alpha_{n-1} \neq 0$ and thus $\alpha_j \neq 0$ for $j \geq n$ which is a contradiction. Then $c_{n-1} = 0$ and similarly $d_{n-1} = 0$.

ii) If $D'S + \sigma S = 0$ then, from (2.19), $\alpha_0 = 0$ and also $4j(j+1-n)\alpha_{j-1} + \sigma\alpha_j = 0$ for $j \geq 1$ Thus $\alpha_j = 0$ for all j and similarly $\beta_j = 0$ for each j.

The following lemma is a direct consequence of (2.19) and the fact that $A_{n-2} = 0$

LEMMA 2.7. *Let $S \in \mathscr{H}'$ supported at the origin,*

i) *If $D'S = 0$ then $S = cB_{n-2}$ for some $c \in \mathsf{C}$.*

ii) *If $D'S = cB_0$ and $S \neq 0$ then $c = 0$.*

For $T \in \mathscr{H}'$, let $T^{\vee}$ given by $\langle T^{\vee}, \varphi\rangle = \langle T, \varphi^{\vee}\rangle$ where $\varphi^{\vee}(\tau) = \varphi(-\tau)$.

THEOREM 2.8. i) *For $\sigma > 0$, $T \in \mathscr{H}'$ is a solution of $D'T + \sigma T = 0$ if and only if $T = c(y_0 H)^{(n-1)}$ for some $c \in \mathsf{R}$.*

ii) *For $\sigma = 0$, $T \in \mathscr{H}'$ is a solution of $D'T = 0$ if and only if $T = c1 + dB_{n-2}$ for some $c, d \in \mathsf{R}$.*

iii) *For* $\sigma < 0$, $T \in \mathcal{H}'$ *solves* $(D' + \sigma I)T = 0$ *if and only if* $T^\vee$ *solves* $(D' - \sigma I)T^\vee = 0$.

PROOF. iii) is immediate. To see i) consider a solution $T \in \mathcal{H}'$ of $D'T + \sigma T = 0$. Then $T_{|D(0,+\infty)} = ay_{n-1} + bz_{n-1}$ and $T_{|D(-\infty,0)} = \alpha y_{n-1} + \beta \widetilde{y}_{n-1}$ for some constants a, b, α, β. From Lemma 2.3 and Proposition 2.4, and since T is a tempered distribution we get $\alpha = 0$. Thus

$$S := T - a(y_0 H)^{(n-1)} - bPf^+(z_{n-1}) - \beta Pf^-(\widetilde{y}_{n-1})$$

is a distribution supported at the origin and, by Lemma 2.5, it satisfies

$$D'S + \sigma S = \sum_{j=0}^{n-2} \mu_j B_j - bc_{n-1}(A_{n-1} + B_{n-1}) - \beta d_{n-1} B_{n-1},$$

with $c_{n-1} \neq 0$ and $d_{n-1} \neq 0$. Thus Lemma 2.6 gives $bc_{n-1} = 0$ and $bc_{n-1} + \beta d_{n-1} = 0$. So $b = \beta = 0$, $S = T - a(y_0 H)^{(n-1)}$ and $D'S + \sigma S = 0$. Now, Lemma 2.6 implies $S = 0$, i.e., $T = a(y_0 H)^{(n-1)}$. Reciprocally, by Proposition 2.4, each distribution T of this form is a solution of $D'T + \sigma T = 0$.

To see ii) observe that the solutions of $\tau v''(\tau) + nv'(\tau) = 0$ on $(0, +\infty)$ (resp. on $(-\infty, 0)$) are generated by 1 and τ^{1-n}. If $\tau T'' + nT = 0$, then $T_{|D(0,+\infty)} = a + b\tau^{1-n}$ and $T_{|D(-\infty,0)} = \alpha + \beta\tau^{1-n}$ for some constants a, b, α, β. Consider $S = T - Pf^+(a + b\tau^{1-n}) - Pf^-(\alpha + \beta\tau^{1-n})$. Proceeding as in the proof of i) we get $b = \beta = 0$. Then $T_{|D(0,+\infty)} = a1$ and $T_{|D(-\infty,0)} = \alpha 1$. Let $\widetilde{S} = T - aH - \alpha(1 - H)$. Since $D'H = B_0$, we have $D'\widetilde{S} = (\alpha - a)B_0$ and so, by Lemma 2.7, $a = \alpha$ and $\widetilde{S} = dB_{n-2}$ for some $d \in \mathrm{R}$. Then $T = a1 + dB_{n-2}$. On the other hand it is clear that 1 and B_{n-2} are solutions of $D'T = 0$.

For $\sigma \in \mathrm{R}$ let $S^\#_\sigma \in \mathcal{S}'(H_n)$ be defined by

$$\langle S^\#_\sigma, f\rangle = (-1)^{n-1} \int_{-\infty}^{\infty} \int_0^{\infty} J_0((\sigma\tau)^{\frac{1}{2}})(Nf(.,t))^{(n-1)}(\tau)\, d\tau\, dt \quad \text{for } \sigma \geq 0,$$

$$\langle S^\#_\sigma, f\rangle = \int_{-\infty}^{\infty} \int_0^{\infty} J_0((-\sigma\tau)^{\frac{1}{2}})(Nf(.,t))^{(n-1)}(-\tau)\, d\tau\, dt \quad \text{for } \sigma < 0.$$

For $\sigma \in \mathrm{R}$, $S^\#_\sigma$ is a joint eigendistribution in $\mathcal{H}'$ of $-L$ and U (cf. Theorem 2.8). On the other hand, S_σ is a joint eigendistribution (cf. [5]) of $-L$ and U which, as stated in the introduction, belongs to $\mathcal{H}'$. Thus, for $\sigma \neq 0$, S_σ is a multiple of $S^\#_\sigma$ and so $[S_\sigma] = [S^\#_\sigma]$. Since S_σ converges in $\mathcal{H}'$ to S_0 as σ tends to zero, we get, that also $[S_0] = [S^\#_0]$.

The distributions $S_{\lambda,k}$ can be explicitly written using Laguerre polynomials. For a non negative integer m let L^0_m be the Laguerre polynomial of degree m

and order zero, defined by $L^0_m(\tau) = \sum_{j=0}^m \binom{m}{j}(-1)^j \frac{\tau^j}{j!}$. We have (cf. [14], also [7])

$$S_{\lambda,k} = F_{\lambda,k} \otimes e^{-i\lambda t}, \tag{2.22}$$

where for $k \geq 0$, $\lambda \neq 0$,

$$\begin{aligned}(2.23)\quad &\langle F_{\lambda,k}, g\rangle \\ &= \left\langle (L^0_{k-q+n-1}H)^{(n-1)}, \tau \to \frac{2}{|\lambda|}e^{-\tau/2}Ng\left(\frac{2}{|\lambda|}\tau\right)\right\rangle, \quad g \in \mathscr{S}(\mathsf{C}^n)\end{aligned}$$

and for $k < 0$, $\lambda \neq 0$

$$\begin{aligned}(2.24)\quad &\langle F_{\lambda,k}, g\rangle \\ &= \left\langle (L^0_{-k-p+n-1}H)^{(n-1)}, \tau \to \frac{2}{|\lambda|}e^{-\tau/2}Ng\left(-\frac{2}{|\lambda|}\tau\right)\right\rangle, \quad g \in \mathscr{S}(\mathsf{C}^n).\end{aligned}$$

Using the Leibnitz rule and the change of variable $\tau = \frac{|\lambda|}{2}s$ we get, for $k \geq 0$, $\lambda \neq 0$ and $f \in \mathscr{S}(H_n)$,

$$\begin{aligned}(2.25)\quad &|\lambda|^{n-1}\langle S_{\lambda,k}, f\rangle \\ &= |\lambda|^{n-1}(-1)^{n-1}\frac{2}{|\lambda|} \\ &\qquad \int_{-\infty}^{\infty} e^{-i\lambda t}\int_0^{\infty} L^0_{k-q+n-1}(\tau)\frac{d^{n-1}}{d\tau^{n-1}}\left(e^{-\frac{\tau}{2}}Nf\left(\frac{2}{|\lambda|}\tau, t\right)\right)d\tau\, dt \\ &= \frac{1}{2^{n-1}}\sum_{j=0}^{n-1}\binom{n-1}{j}4^j(-1)^j\,|\lambda|^{n-1-j} \\ &\qquad \times \int_{-\infty}^{\infty} e^{-i\lambda t}\int_0^{\infty} L^0_{k-q+n-1}\left(\frac{|\lambda|}{2}s\right)e^{-\frac{|\lambda|}{4}s}(Nf(.,t))^{(j)}(s)\, ds\, dt\end{aligned}$$

and similarly, for $k < 0$, $\lambda \neq 0$ and $f \in \mathscr{S}(H_n)$,

$$\begin{aligned}(2.26)\quad |\lambda|^{n-1}\langle S_{\lambda,k}, f\rangle &= \frac{1}{2^{n-1}}\sum_{j=0}^{n-1}\binom{n-1}{j}4^j|\lambda|^{n-1-j} \\ &\quad \times \int_{-\infty}^{\infty} e^{-i\lambda t}\int_0^{\infty} L^0_{-k-p+n-1}\left(\frac{|\lambda|}{2}s\right)e^{-\frac{|\lambda|}{4}s}(Nf(.,t))^{(j)}(-s)\, ds\, dt.\end{aligned}$$

3. $\mathscr{E}$ is an homeomorphism

REMARK 3.1. The following result is a Mehler type formula (see for example [6], page 92, or Corollary 4.2 in [3]) :

$$\lim_{m\to 0} L_m^0\left(\frac{x^2}{2(2m+1)}\right)e^{\frac{x^2}{4(2m+1)}} = J_0(x)$$

uniformly on compact subsets of $[0,\infty)$.

PROOF OF THEOREM 1.2. Let E, $\Delta(U(p,q),H_n)$ and $\mathscr{E}$ be as in the introduction and let $\theta : E \to E/\sim$ be the quotient map. The map $\widetilde{\mathscr{E}} : E \to \mathscr{A}(U(p,q),H_n)$ given by $\widetilde{\mathscr{E}}(\Psi) = (-\widehat{L}(\Psi), i\widehat{U}(\Psi))$ is continuous. Indeed, since E is equipped with the pointwise convergence topology, if Ψ_n converges to Ψ (and we set $\Psi_n \rightharpoonup \Psi$) then $L\Psi_n \rightharpoonup L\Psi$. So, denoting by γ_n and γ the eigenvalues associated to Ψ_n and Ψ, respectively, we have that $\gamma_n\Psi_n \rightharpoonup \gamma\Psi$. Choosing some f such that $\langle\Psi, f\rangle \neq 0$, we conclude that $\gamma_n \to \gamma$.

Thus the bijection $\mathscr{E} : \Delta(U(p,q),H_n) - \{[1]\} \to \mathscr{A}(U(p,q),H_n)$ is also continuous.

For $(\sigma,\lambda) \in \mathscr{A}(U(p,q),H_n)$, we say that it is of type I if $\lambda \neq 0$ (and so $\sigma = |\lambda|(2k+p-q)$ with $k \in \mathsf{Z}$). In this case we set $S_{(\sigma,\lambda)} = \frac{|\lambda|^{n-1}}{2^{n-1}}S_{\lambda,k}$. We will say that (σ,λ) is of type II if $\lambda = 0$, and we set $S_{(\sigma,\lambda)} = S_\sigma^{\#}$.

To see that $\mathscr{E}^{-1}$ is continuous it enough to show that if $\{(\sigma_m,\lambda_m)\}_{m\in\mathsf{N}}$ is a sequence in $\mathscr{A}(U(p,q),H_n)$, either of type I or of type II, and if $\lim_{m\to\infty}(\sigma_m,\lambda_m) = (\sigma,\lambda)$, then

$$\lim_{m\to\infty} S_{(\sigma_m,\lambda_m)} = S_{(\sigma,\lambda)} \tag{3.1}$$

with convergence in $\mathscr{S}'(H_n)$.

Consider the case when $\sigma > 0$, $\lambda = 0$. If $\{(\sigma_m,\lambda_m)\}_{m\in\mathsf{N}}$ is of type I then $\sigma_m = |\lambda_m|(2k_m+p-q)$ with $k_m \in Z$. Since $\lambda_m \to 0$ and $2|\lambda_m|k_m \to \sigma$ we have $k_m > 0$ for m large enough.

Fix $s \geq 0$ and let $x_m = ((2k_m+1)|\lambda_m|s)^{\frac{1}{2}}$. Then $\lim_{m\to\infty} x_m = (\sigma s)^{\frac{1}{2}}$. Since $\frac{|\lambda_m|}{2}s = \frac{x_m^2}{2(2k_m+1)}$ the uniform convergence in Remark 3.1 and dominated convergence gives that for $j = 0,\ldots,n-1$,

$$\lim_{m\to\infty}\int_0^\infty L^0_{k_m-q+n-1}\left(\frac{|\lambda_m|}{2}s\right)e^{-\frac{|\lambda_m|}{4}s}(Nf(.,t))^{(j)}(s)\,ds = \int_0^\infty J_0((\sigma s)^{\frac{1}{2}})(Nf(.,t))^{(j)}(s)\,ds$$

Thus, taking into account of (2.25), we obtain (3.1). If $\{(\sigma_m,\lambda_m)\}_{m\in\mathsf{N}}$ is of type II, since J_0 is continuous, dominated convergence gives $\lim_{m\to\infty} S_{(\sigma_m,\lambda_m)} = S_\sigma^{\#}$.

The case $\sigma < 0$, $\lambda = 0$ follows the sames lines: in this case $k_m < 0$ for m large enough, and so (2.26) and the definition of $S^{\#}_{\sigma}$ for $\sigma < 0$ imply (3.1).

The origin $\sigma = 0$, $\lambda = 0$ has not additional work. As above, by (2.25) and (2.26), we see that $\lim_{m\to\infty} S_{(\sigma_m,\lambda_m)} = S^{\#}_0$. In particular this shows that the equivalence class of 1 is an isolated point of $\Delta(U(p,q), H_n)$.

The proof for the cases where $\lambda \neq 0$ are obvious.

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