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LOG-SINE INTEGRALS INVOLVING SERIES ASSOCIATED WITH THE ZETA FUNCTION AND POLYLOGARITHMS

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Abstract

Motivated essentially by their potential for applications in a wide range of mathematical and physical problems, the Log-Sine integrals have been evaluated, in the existing literature on the subject, in many different ways. The main object of this paper is to show how nicely some general formulas analogous to the generalized Log-Sine integral $\mathrm{Ls}_n^{(m)}\left(\frac{\pi}{3}\right)$ can be obtained by using the theory of Polylogarithms. Relevant connections of the results presented here with those obtained in earlier works are also indicated precisely.

1. Introduction and Preliminaries

Motivated essentially by their potential for applications in a wide range of mathematical and physical problems, the Log-Sine integrals have been evaluated, in the existing literature on the subject, in many different ways. Here, we mainly aim at showing how nicely some general formulas analogous to the generalized Log-Sine integral $\mathrm{Ls}_n^{(m)}\left(\frac{\pi}{3}\right)$ can be obtained by using the theory of Polylogarithms. We also give remarks on some related results on this subject.

For our purpose, we begin by recalling the following definitions and properties, which will be needed in our investigation.

Definition 1. The Log-Sine integrals $\mathrm{Ls}_n(\theta)$ of order n are defined by (1.1)

$$\mathrm{Ls}_n(\theta) := -\int_0^\theta \left(\log\left|2\sin\frac{x}{2}\right|\right)^{n-1} dx \quad (n \in \mathsf{N}\setminus\{1\};\ \mathsf{N} := \{1,2,3,\ldots\});$$

The generalized Log-Sine integrals $\mathrm{Ls}_n^{(m)}(\theta)$ of order n and index m are defined by

$$\mathrm{Ls}_n^{(m)}(\theta) := -\int_0^\theta x^m \left(\log\left|2\sin\frac{x}{2}\right|\right)^{n-m-1} dx \tag{1.2}$$

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DEFINITION 2. The Riemann Zeta function $\zeta(s)$ is defined by

$$\zeta(s) := \begin{cases} \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1-2^{-s}} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} & \big(\Re(s) > 1\big) \\ \displaystyle\frac{1}{1-2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} & \big(\Re(s) > 0;\ s \neq 1\big). \end{cases} \tag{1.3}$$

The Hurwitz (or generalized) Zeta function $\zeta(s, a)$ is defined by

$$\zeta(s, a) := \sum_{k=0}^{\infty} (k+a)^{-s} \qquad \big(\Re(s) > 1;\ a \neq 0, -1, -2, \ldots\big). \tag{1.4}$$

From Definition 2 it is easy to observe that

$$\zeta(s) = \zeta(s, 1) = \big(2^s - 1\big)^{-1} \zeta\left(s, \frac{1}{2}\right) = 1 + \zeta(s, 2). \tag{1.5}$$

DEFINITION 3. The Dilogarithm function $\mathrm{Li}_2(z)$ is defined by

$$\begin{aligned} \mathrm{Li}_2(z) &:= \sum_{n=1}^{\infty} \frac{z^n}{n^2} \qquad (|z| \leqq 1) \\ &= -\int_0^z \frac{\log(1-t)}{t}\, dt. \end{aligned} \tag{1.6}$$

The Polylogarithm function $\mathrm{Li}_n(z)$ is defined by

$$\begin{aligned} \mathrm{Li}_n(z) &:= \sum_{k=1}^{\infty} \frac{z^k}{k^n} \qquad (|z| \leqq 1;\ n \in \mathsf{N} \setminus \{1\}) \\ &= \int_0^z \frac{\mathrm{Li}_{n-1}(t)}{t}\, dt \qquad (n \in \mathsf{N} \setminus \{1,\ 2\}). \end{aligned} \tag{1.7}$$

Clearly, we have

$$\mathrm{Li}_n(1) = \zeta(n) \qquad (n \in \mathsf{N} \setminus \{1\}), \tag{1.8}$$

in terms of the Riemann Zeta function $\zeta(s)$ in (1.3).

DEFINITION 4. The generalized Clausen function $\mathrm{Cl}_n(\theta)$ is defined by

$$\mathrm{Cl}_n(\theta) := \begin{cases} \displaystyle\sum_{k=1}^{\infty} \frac{\sin k\theta}{k^n} & (n \text{ is even}) \\ \displaystyle\sum_{k=1}^{\infty} \frac{\cos k\theta}{k^n} & (n \text{ is odd}). \end{cases} \tag{1.9}$$

The associated Clausen function $\mathrm{Gl}_n(\theta)$ of order n is defined by

$$\mathrm{Gl}_n(\theta) := \begin{cases} \displaystyle\sum_{k=1}^{\infty} \frac{\cos k\theta}{k^n} & (n \text{ is even}) \\ \displaystyle\sum_{k=1}^{\infty} \frac{\sin k\theta}{k^n} & (n \text{ is odd}). \end{cases} \tag{1.10}$$

It is noted that, *only* the special case when $n = 2$, the generalized Clausen function $\mathrm{Cl}_n(\theta)$ satisfies the following relationship:

$$\mathrm{Ls}_2(\theta) = \mathrm{Cl}_2(\theta) \tag{1.11}$$

with the Log-Sine integral $\mathrm{Ls}_n(\theta)$ of order $n = 2$ defined by (1.1).

The following well-known formulas are recorded (see [17, p. 334, Entry (50.5.16)] and [27, p. 98], respectively):

$$x \cdot \cot x = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{2^{2n} B_{2n}}{(2n)!} x^{2n} \qquad (|x| < \pi) \tag{1.12}$$

and

$$\zeta(2n) = (-1)^{n+1} \frac{(2\pi)^{2n}}{2 \cdot (2n)!} B_{2n} \qquad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}), \tag{1.13}$$

where B_n denotes the well-known Bernoulli numbers (see, e.g., [27, p. 59]).

Now, through an eclectic review, we are showing how Log-Sine integrals $\mathrm{Ls}_n(\theta)$ and the generalized Log-Sine integrals $\mathrm{Ls}_n^{(m)}(\theta)$ of order n and index m in Definition 1 have been evaluated in various ways by many mathematicians.

The Log-Sine integrals $\mathrm{Ls}_n(\theta)$ in Definition 1 when the argument $\theta = \pi$ satisfy the following recurrence relation (see, e.g., [19, p. 218, Eq. (7.112)];

see also [27, p. 119, Eq. (111)]):

$$\frac{(-1)^m}{m!}\operatorname{Ls}_{m+2}(\pi) = \pi\left(1-2^{-m}\right)\zeta(m+1) + \sum_{k=2}^{m-1}\frac{(-1)^{k+1}}{k!}\left(1-2^{k-m}\right)\zeta(m-k+1)\operatorname{Ls}_{k+1}(\pi) \qquad (m\in\mathbb{N}), \tag{1.14}$$

where $\zeta(s)$ denotes the Riemann Zeta function given in (1.3).

By mainly analyzing the generalized binomial theorem and the familiar Weierstrass canonical product form of the Gamma function $\Gamma(z)$ (see [27, p. 1, Eq. (2)]), Shen [23, p. 1396, Eq. (19)] evaluated the Log-Sine integral $\operatorname{Ls}_{k+1}(2\pi)$ as follows:

$$\frac{1}{2\pi}\int_0^{2\pi}\left(\log\left|2\sin\frac{x}{2}\right|\right)^k dx = (-1)^k\frac{k!}{2^k}\sum_{n\geqq\frac{k}{2}}^{\infty}\sigma_k^n \qquad (k,n\in\mathbb{N}), \tag{1.15}$$

where σ_k^n are given, in terms of the Stirling numbers $s(n,k)$ of the first kind (see, for details, [27, pp. 56–57]; see also [22]), by

$$\sigma_k^n = \sum_{m=1}^{k-1}\frac{s(n,k-m)}{n!}\frac{s(n,m)}{n!}.$$

By using an idea analogous to that of Shen [23], Beumer [5] presented a recursion formula for

$$D(n) := \frac{(-1)^{n-1}}{2\cdot(n-1)!}\int_0^{\pi}\left[\log\left(\sin\frac{x}{2}\right)\right]^{n-1}dx \qquad (n\in\mathbb{N})$$

in the following form:

$$\sum_{k=1}^{2n-1}(-1)^{k-1}D(k)D(2n-k) = (-1)^{n+1}\frac{2^{2n}-1}{(2n)!}\pi^{2n}B_{2n} \qquad (n\in\mathbb{N}), \tag{1.16}$$

where B_n are the Bernoulli numbers in (1.12) and (1.13), and

$$D(1) = \frac{\pi}{2} \qquad \text{and} \qquad D(2) = \frac{\pi}{2}\log 2.$$

More recently, Batir [4] presented integral representations, involving Log-Sine terms, for some series associated with

$$\binom{2k}{k}^{-1}k^{-n} \qquad \text{and} \qquad \binom{2k}{k}^{-2}k^{-n},$$

and for some closely-related series, by using a number of elementary properties of Polylogarithms.

Lewin [19, pp. 102–103; p. 164] presented the following integral formulas:

$$\int_0^{\frac{\pi}{2}} \log\left(2\sin\frac{x}{2}\right) dx = -G \tag{1.17}$$

and

$$\int_0^{\frac{\pi}{2}} x\log\left(2\sin\frac{x}{2}\right) dx = \frac{35}{32}\zeta(3) - \frac{1}{2}\pi G, \tag{1.18}$$

where G denotes the Catalan constant defined by

$$G := \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^2} \cong 0.91596\,55941\,77219\,015\ldots. \tag{1.19}$$

Several other authors have concentrated upon the problem of evaluation of the Log-Sine integral $\mathrm{Ls}_n(\theta)$ and the generalized Log-Sine integral $\mathrm{Ls}_n^{(m)}(\theta)$ of order n and index m given in Definition 1 with the *argument* θ given by $\theta = \frac{\pi}{3}$. (Throughout this paper, we choose the *principal branch* of the logarithm function $\log z$ in case z is a complex variable.)

In particular, van der Poorten [29] proved that

$$\int_0^{\frac{\pi}{3}} \log^2\left(2\sin\frac{x}{2}\right) dx = \frac{7}{108}\pi^3 \tag{1.20}$$

and

$$\int_0^{\frac{\pi}{3}} x\log^2\left(2\sin\frac{x}{2}\right) dx = \frac{17}{6480}\pi^4. \tag{1.21}$$

Zucker [32] established the following two integral formulas:

$$\int_0^{\frac{\pi}{3}} \left\{\log^4\left(2\sin\frac{x}{2}\right) - \frac{3}{2}x^2\log^2\left(2\sin\frac{x}{2}\right)\right\} dx = \frac{253}{3240}\pi^5 \tag{1.22}$$

and

$$\int_0^{\frac{\pi}{3}} \left\{x\log^4\left(2\sin\frac{x}{2}\right) - \frac{x^3}{2}\log^2\left(2\sin\frac{x}{2}\right)\right\} dx = \frac{313}{408240}\pi^6. \tag{1.23}$$

Zhang and Williams [31] extensively investigated $\mathrm{Ls}_n\left(\frac{\pi}{3}\right)$ and $\mathrm{Ls}_n^{(m)}\left(\frac{\pi}{3}\right)$ along with other integrals in order to present two general formulas (see [31,

p. 272, Eqs. (1.6) and (1.7)]), which include the integral formulas (1.20) to (1.23) as special cases. We choose to recall here one more explicit special case of the Zhang-Williams integral formulas as follows:

$$\begin{aligned}\int_0^{\frac{\pi}{3}} \left\{\log^6\left(2\sin\frac{x}{2}\right) - \frac{15}{4}x^2\log^4\left(2\sin\frac{x}{2}\right)\right. \\ \left. +\frac{15}{16}x^4\log^2\left(2\sin\frac{x}{2}\right)\right\} dx = \frac{77821}{2^6\cdot 3^6\cdot 7}\pi^7.\end{aligned} \tag{1.24}$$

Borwein *et al.* [7] expressed the central binomial sum:

$$S(k) := \sum_{n=1}^{\infty} \frac{1}{n^k\binom{2n}{n}}$$

in an integral form as follows:

$$S(k) = \frac{(-2)^{k-1}}{(k-2)!}\,\mathrm{Ls}_k^{(1)}\left(\frac{\pi}{3}\right) \qquad (k \in \mathsf{N}). \tag{1.25}$$

By doing so, they [7, Theorem 3.3] were able to evaluate the central binomial sums $S(k)$ $(k = 2, \ldots, 8)$ in terms of the multiple Clausen, Glaisher, and Zeta values.

2. Analogous Log-Sine Integrals

In this section, we will show how nicely some general formulas analogous to the generalized Log-Sine integral $\mathrm{Ls}_n^{(m)}\left(\frac{\pi}{3}\right)$ can be obtained by using the theory of Polylogarithms. Indeed, by carrying out repeated integration by parts in (1.7) in conjunction with (1.6), we obtain

LEMMA.

$$\begin{aligned}\mathrm{Li}_n(z) - \mathrm{Li}_n(w) &= \int_w^z \mathrm{Li}_{n-1}(t)\,\frac{dt}{t} \\ &= \left(\sum_{k=1}^{n-2}\frac{(-1)^{k-1}}{k!}(\log t)^k\,\mathrm{Li}_{n-k}(t) + \frac{(-1)^{n-1}}{(n-1)!}(\log t)^{n-1}\log(1-t)\right)\Bigg|_{t=w}^{z} \\ &\quad + \frac{(-1)^{n-1}}{(n-1)!}\int_w^z (\log t)^{n-1}\,\frac{dt}{1-t} \qquad (n \in \mathsf{N}\setminus\{1\}),\end{aligned} \tag{2.1}$$

where (and elsewhere in this paper) an empty sum is understood to be nil; in particular,

$$\begin{aligned}\zeta(n) = \mathrm{Li}_n(1) &= \frac{(-1)^{n-1}}{(n-1)!}\int_0^1 (\log t)^{n-1}\,\frac{dt}{1-t} \\ &= \frac{(-1)^{n-1}}{(n-1)!}\int_0^1 \left[\log(1-t)\right]^{n-1}\,\frac{dt}{t} \qquad (n \in \mathbb{N}\setminus\{1\}).\end{aligned} \tag{2.2}$$

By beginning with (2.2), we can deduce the following analogous Log-Sine integral formulas:

THEOREM 1.

$$\sum_{k=1}^{m} \frac{(-1)^{k+1}}{2^{2k-1}}\binom{2m}{2k-1}\int_0^{\pi/3} (x-\pi)^{2k-1}\log^{2m+1-2k}\left(2\sin\frac{x}{2}\right)dx = (2m)!\sum_{k=0}^{2m-1}\frac{(-1)^{[k/2]}}{k!}\left(\frac{\pi}{3}\right)^k \mathrm{Cl}_{2m+1-k}\left(\frac{\pi}{3}\right) \qquad (m \in \mathbb{N}); \tag{2.3}$$

$$\begin{aligned}\sum_{k=0}^{m} \frac{(-1)^{k+1}}{2^{2k}}\binom{2m}{2k}\int_0^{\frac{\pi}{3}} (x-\pi)^{2k}\log^{2m-2k}\left(2\sin\frac{x}{2}\right)dx &= (-1)^m\left(1+\frac{1}{2m+1}\right)\left(\frac{\pi}{3}\right)^{2m+1} \\ &+ (2m)!\sum_{k=0}^{2m-1}\frac{(-1)^{\left[\frac{k+1}{2}\right]}}{k!}\left(\frac{\pi}{3}\right)^k \mathrm{Gl}_{2m+1-k}\left(\frac{\pi}{3}\right) \qquad (m \in \mathbb{N});\end{aligned} \tag{2.4}$$

$$\begin{aligned}\sum_{k=0}^{m-1} \frac{(-1)^{k}}{2^{2k+1}}\binom{2m-1}{2k+1}\int_0^{\frac{\pi}{3}} (x-\pi)^{2k+1}\log^{2m-2-2k}\left(2\sin\frac{x}{2}\right)dx &= (-1)^{m+1}\left(1+\frac{1}{2m}\right)\left(\frac{\pi}{3}\right)^{2m} - 2(2m-1)!\,\zeta(2m) \\ &+ (2m-1)!\sum_{k=0}^{2m-2}\frac{(-1)^{\left[\frac{k}{2}\right]}}{k!}\left(\frac{\pi}{3}\right)^k \mathrm{Gl}_{2m-k}\left(\frac{\pi}{3}\right) \qquad (m \in \mathbb{N});\end{aligned} \tag{2.5}$$

$$(2.6)\quad \sum_{k=0}^{m-1} \frac{(-1)^k}{2^{2k}} \binom{2m-1}{2k} \int_0^{\frac{\pi}{3}} (x-\pi)^{2k} \log^{2m-1-2k}\left(2\sin\frac{x}{2}\right) dx$$

$$= (2m-1)! \sum_{k=0}^{2m-2} \frac{(-1)^{[\frac{k-1}{2}]}}{k!} \left(\frac{\pi}{3}\right)^k \mathrm{Cl}_{2m-k}\left(\frac{\pi}{3}\right) \qquad (m \in \mathbb{N}),$$

where $\mathrm{Cl}_n(\theta)$ *and* $\mathrm{Gl}_n(\theta)$ *denote the generalized Clausen function and the associated Clausen function given in* (1.9) *and* (1.10)*, respectively.*

PROOF. Substituting $t = u^{-1}$ in (2.2) yields

$$(2.7)\quad (-1)^{n-1} \int_1^{e^{-i\theta}} (\log t)^{n-1} \frac{dt}{1-t}$$

$$= \int_1^{e^{i\theta}} (\log t)^{n-1} \frac{dt}{1-t} + \frac{1}{n}(i\theta)^n \qquad (0 \leqq \theta \leqq \pi).$$

Furthermore, it is easily observed, by setting $t = 1 - e^{ix}$, that

$$(2.8)\quad \int_0^{1-e^{i\theta}} (\log t)^{n-1} \frac{dt}{1-t} = -i \int_0^{\theta} \left(\frac{1}{2}i(x-\pi) + \log\left|2\sin\frac{x}{2}\right|\right)^{n-1} dx$$

or, equivalently, that

$$(2.9)\quad \int_1^{1-e^{i\theta}} (\log t)^{n-1} \frac{dt}{1-t} = -i \int_0^{\theta} \left(\frac{1}{2}i(x-\pi)\right.$$

$$\left. + \log\left|2\sin\frac{x}{2}\right|\right)^{n-1} dx + (-1)^n (n-1)!\zeta(n) \qquad (n \in \mathbb{N} \setminus \{1\})$$

in view of (2.2). In its special case when $n = 2m+1$ ($m \in \mathbb{N}$) and $w = 1$, (2.1) yields

$$(2.10)\quad \int_1^{z} (\log t)^{2m} \frac{dt}{1-t} = (2m)!\,\mathrm{Li}_{2m+1}(z) - (2m)!\zeta(2m+1)$$

$$+ (2m)! \sum_{k=1}^{2m-1} \frac{(-1)^k}{k!} (\log z)^k\, \mathrm{Li}_{2m+1-k}(z) - (\log z)^{2m} \log(1-z) \qquad (m \in \mathbb{N}).$$

Putting $z = e^{i\frac{\pi}{3}}$ in (2.10) and using the following elementary identity:

$$(2.11)\qquad 1 - e^{i\frac{\pi}{3}} = e^{-i\frac{\pi}{3}},$$

we get

$$\text{(2.12)}\qquad \int_1^{e^{i\frac{\pi}{3}}} (\log t)^{2m} \frac{dt}{1-t} = i(-1)^m \left(\frac{\pi}{3}\right)^{2m+1} - (2m)!\,\zeta(2m+1) + (2m)! \sum_{k=0}^{2m-1} \frac{(-1)^k}{k!} \left(i\frac{\pi}{3}\right)^k \mathrm{Li}_{2m+1-k}\left(e^{i\frac{\pi}{3}}\right) \qquad (m \in \mathsf{N}).$$

We now separate the even and odd parts of the sum occurring in (2.12) and make use of (1.8) and (1.11). We thus obtain

$$\text{(2.13)}\qquad \sum_{k=0}^{2m-1} \frac{(-1)^k}{k!} \left(i\frac{\pi}{3}\right)^k \mathrm{Li}_{2m+1-k}\left(e^{i\frac{\pi}{3}}\right) = \sum_{k=0}^{2m-1} \frac{(-1)^{[k/2]}}{k!} \left(\frac{\pi}{3}\right)^k \mathrm{Cl}_{2m+1-k}\left(\frac{\pi}{3}\right) + i \sum_{k=0}^{2m-1} \frac{(-1)^{\left[\frac{k+1}{2}\right]}}{k!} \left(\frac{\pi}{3}\right)^k \mathrm{Gl}_{2m+1-k}\left(\frac{\pi}{3}\right).$$

Upon substituting from (2.13) into (2.12), and equating the real and imaginary parts on each side of the resulting equation, we obtain

$$\text{(2.14)}\qquad \Re\left(\int_1^{e^{i\frac{\pi}{3}}} (\log t)^{2m} \frac{dt}{1-t}\right) = -(2m)!\zeta(2m+1) + (2m)! \sum_{k=0}^{2m-1} \frac{(-1)^{[k/2]}}{k!} \left(\frac{\pi}{3}\right)^k \mathrm{Cl}_{2m+1-k}\left(\frac{\pi}{3}\right) \qquad (m \in \mathsf{N})$$

and

$$\text{(2.15)}\qquad \Im\left(\int_1^{e^{i\frac{\pi}{3}}} (\log\ t)^{2m} \frac{dt}{1-t}\right) = (-1)^m \left(\frac{\pi}{3}\right)^{2m+1} + (2m)! \sum_{k=0}^{2m-1} \frac{(-1)^{\left[\frac{k+1}{2}\right]}}{k!} \left(\frac{\pi}{3}\right)^k \mathrm{Gl}_{2m+1-k}\left(\frac{\pi}{3}\right) \qquad (m \in \mathsf{N}).$$

Setting $n = 2m+1$ ($m \in \mathsf{N}$) and $\theta = \frac{\pi}{3}$ in (2.7), we have

$$\text{(2.16)}\qquad \int_1^{e^{-i\frac{\pi}{3}}} (\log t)^{2m} \frac{dt}{1-t} = \int_1^{e^{i\frac{\pi}{3}}} (\log t)^{2m} \frac{dt}{1-t} + \frac{(-1)^m}{2m+1} \left(\frac{\pi}{3}\right)^{2m+1} i \qquad (m \in \mathsf{N}).$$

Also, by putting $n = 2m + 1$ ($m \in \mathsf{N}$) and $\theta = \frac{\pi}{3}$ in (2.9), and using (2.11), we obtain

$$\text{(2.17)} \quad \int_1^{e^{-i\frac{\pi}{3}}} (\log t)^{2m} \frac{dt}{1-t} = -(2m)!\zeta(2m+1) \\ - i \int_0^{\frac{\pi}{3}} \left[\frac{1}{2} i(x-\pi) + \log\left(2\sin\frac{x}{2}\right)\right]^{2m} dx \qquad (m \in \mathsf{N}).$$

Now, by using the binomial theorem, we find that

$$\text{(2.18)} \quad \left[\frac{1}{2} i(x-\pi) + \log\left(2\sin\frac{x}{2}\right)\right]^{2m} \\ = \sum_{k=0}^{m} (-1)^k \binom{2m}{2k} \left(\frac{x-\pi}{2}\right)^{2k} \log^{2m-2k}\left(2\sin\frac{x}{2}\right) \\ + i \sum_{k=1}^{m} (-1)^{k+1} \binom{2m}{2k-1} \left(\frac{x-\pi}{2}\right)^{2k-1} \log^{2m+1-2k}\left(2\sin\frac{x}{2}\right) \qquad (m \in \mathsf{N}).$$

Substituting (2.18) into the integrand on the right-hand side of (2.17), and equating the real and imaginary parts of each side of the resulting equation, we obtain

$$\text{(2.19)} \quad \Re\left(\int_1^{e^{-i\frac{\pi}{3}}} (\log t)^{2m} \frac{dt}{1-t}\right) = -(2m)!\,\zeta(2m+1) \\ + \sum_{k=1}^{m} \frac{(-1)^{k+1}}{2^{2k-1}} \binom{2m}{2k-1} \int_0^{\frac{\pi}{3}} (x-\pi)^{2k-1} \log^{2m+1-2k}\left(2\sin\frac{x}{2}\right) dx \quad (m \in \mathsf{N})$$

and

$$\text{(2.20)} \quad \Im\left(\int_1^{e^{-i\frac{\pi}{3}}} (\log t)^{2m} \frac{dt}{1-t}\right) \\ = \sum_{k=0}^{m} \frac{(-1)^{k+1}}{2^{2k}} \binom{2m}{2k} \int_0^{\frac{\pi}{3}} (x-\pi)^{2k} \log^{2m-2k}\left(2\sin\frac{x}{2}\right) dx \qquad (m \in \mathsf{N}).$$

Finally, from (2.14), (2.15), (2.16), (2.19), and (2.20), we deduce the analogous Log-Sine integral formulas (2.3) and (2.4).

Just as in our derivations of (2.3) and (2.4), if we set

$$n = 2m \quad (m \in \mathsf{N}), \qquad z = e^{i\frac{\pi}{3}} \qquad \text{and} \qquad w = 1$$

in (2.1), and apply (2.7) and (2.9) with $\theta = \frac{\pi}{3}$, we obtain the other analogous Log-Sine integral formulas (2.5) and (2.6). This completes the proof of Theorem 1.

3. Remarks on $\mathrm{Cl}_n(\theta)$ and $\mathrm{Gl}_n(\theta)$

In view of Equations (2.3) to (2.6), we need to express the generalized Clausen function $\mathrm{Cl}_n(\theta)$ and its associated Clausen function $\mathrm{Gl}_n(\theta)$ as explicitly as possible, at least at the argument $\theta = \pi/3$. To do this, we begin by rewriting the following known formula:

$$\begin{aligned}(3.1)\quad \sum_{k=1}^{\infty} \frac{\cos(2\pi kx)}{k^n} + i\sum_{k=1}^{\infty} \frac{\sin(2\pi kx)}{k^n} &= \frac{(2\pi i)^{n-1}}{(n-1)!}\Bigg(\zeta'(1-n, x) \\ &+ (-1)^{n-1}\zeta'(1-n, 1-x) - \pi i\,\frac{B_n(x)}{n}\Bigg) \qquad (0 < x < 1;\ n \in \mathsf{N}\setminus\{1\})\end{aligned}$$

which was proved by Adamchik [1, Eq. (9)] (see also [21, Eq. (21)]) who used Lerch's transform formula [18] and where $B_n(x)$ are the Bernoulli polynomials of degree n in x (see, for details, [27, pp. 59–61]). Now replacing n by $2n$ and $2n + 1$ in (3.1) and equating the real and imaginary parts in each case, we obtain the following formulas:

$$(3.2)\quad \mathrm{Gl}_n(2\pi x) = (-1)^{1+[\frac{1}{2}n]}2^{n-1}\pi^n\,\frac{B_n(x)}{n!} \qquad (0 \leqq x \leqq 1;\ n \in \mathsf{N}\setminus\{1\}),$$

which is a known result (cf. [19, p. 202, Eq. (7.60)]; see also [27, p. 119, Eq. (109)]); and

$$\begin{aligned}(3.3)\quad \mathrm{Cl}_n(2\pi x) = (-1)^{1+[\frac{1}{2}(n+1)]}\,&\frac{(2\pi)^{n-1}}{(n-1)!}\big[\zeta'(1-n, x) \\ &+ (-1)^{n+1}\zeta'(1-n, 1-x)\big] \qquad (0 < x < 1;\ n \in \mathsf{N}\setminus\{1\}).\end{aligned}$$

Srivastava *et al.* [28, Eq. (3.8) and Eq. (3.17)] presented the following formulas:

$$(3.4)\qquad \mathrm{Cl}_{2n+1}\left(\frac{\pi}{3}\right) = \frac{1}{2}(1 - 2^{-2n})(1 - 3^{-2n})\zeta(2n+1) \qquad (n \in \mathsf{N})$$

and

$$\begin{aligned}(3.5)\quad \mathrm{Cl}_{2n}\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{6^{2n}}\Bigg[&\zeta\left(2n, \frac{1}{3}\right) \\ &+ \zeta\left(2n, \frac{1}{6}\right) - 2^{2n-1}(3^{2n} - 1)\zeta(2n)\Bigg] \qquad (n \in \mathsf{N}).\end{aligned}$$

Cvijović and Klinowski [15, Eq. (10a)] proved formulas which can be specialized in the following form:

$$\begin{aligned}\text{(3.6)}\quad \mathrm{Cl}_n\left(\frac{2\pi p}{q}\right) &= \frac{1}{q^n}\sum_{k=1}^{q}\zeta\left(n,\frac{k}{q}\right)\left[\frac{1+(-1)^n}{2}\sin\left(\frac{2k\pi p}{q}\right)\right.\\ &\left.+\frac{1-(-1)^n}{2}\cos\left(\frac{2k\pi p}{q}\right)\right]\qquad (n,\, q+1\in\mathbb{N}\setminus\{1\};\;\; p\in\mathbb{Z}),\end{aligned}$$

which, upon replacing n by $2n+1$ and $2n$ with $p=1$ and $q=6$, leads to (3.4) and (3.5), respectively.

We are ready now to consider some explicit expressions of (2.3) to (2.6). Upon setting $m = 1, 2, 3$ in (2.4), and $m = 2, 3$ in (2.5), if we apply (3.2), we obtain the following explicit analogous Log-Sine integral formulas.

COROLLARY 1. *The following integral formulas hold true:*

$$\text{(3.7)}\quad \int_0^{\frac{\pi}{3}}\log^2\left(2\sin\frac{x}{2}\right)dx = \frac{7}{108}\pi^3,$$

$$\text{(3.8)}\quad \int_0^{\frac{\pi}{3}}(x-\pi)\log^2\left(2\sin\frac{x}{2}\right)dx = -\frac{403}{2^4\cdot 3^4\cdot 5}\pi^4,$$

$$\text{(3.9)}\quad \int_0^{\frac{\pi}{3}}\left[\frac{3}{2}(x-\pi)^2\log^2\left(2\sin\frac{x}{2}\right)-\log^4\left(2\sin\frac{x}{2}\right)\right]dx = \frac{73}{2^4\cdot 3^4\cdot 5}\pi^5,$$

$$\begin{aligned}\text{(3.10)}\quad \int_0^{\frac{\pi}{3}}&\left[2(x-\pi)\log^4\left(2\sin\frac{x}{2}\right)\right.\\ &\left.-(x-\pi)^3\log^2\left(2\sin\frac{x}{2}\right)\right]dx = -\frac{39883}{2^4\cdot 3^6\cdot 5\cdot 7}\pi^6,\end{aligned}$$

and

$$\begin{aligned}\text{(3.11)}\quad \int_0^{\frac{\pi}{3}}&\left[\log^6\left(2\sin\frac{x}{2}\right)-\frac{15}{4}(x-\pi)^2\log^4\left(2\sin\frac{x}{2}\right)\right.\\ &\left.+\frac{15}{16}(x-\pi)^4\log^2\left(2\sin\frac{x}{2}\right)\right]dx = \frac{697}{2^6\cdot 3^6\cdot 7}\pi^7.\end{aligned}$$

It should be noted that the integral formulas (1.20) to (1.24) can easily be deduced from these last integral formulas (3.7) to (3.11) in Corollary 1.

Upon setting $m = 1, 2$ in (2.3) and (2.6), if we apply (3.4) and (3.5), we obtain Corollary 2 below.

COROLLARY 2. *The following integral formulas hold true:*

$$\int_0^{\frac{\pi}{3}} (x-\pi)\log\left(2\sin\frac{x}{2}\right)dx = \frac{2}{3}\zeta(3) - \frac{4\sqrt{3}}{81}\pi^3 + \frac{\sqrt{3}\pi}{54}\left[\zeta\left(2,\frac{1}{3}\right)+\zeta\left(2,\frac{1}{6}\right)\right], \tag{3.12}$$

$$\int_0^{\frac{\pi}{3}} \left[2(x-\pi)\log^3\left(2\sin\frac{x}{2}\right) - \frac{1}{2}(x-\pi)^3\log\left(2\sin\frac{x}{2}\right)\right]dx = \frac{100}{9}\zeta(5) - \frac{4\pi^2}{9}\zeta(3) - \frac{8\sqrt{3}}{243}\pi^5 + \frac{\sqrt{3}\pi}{162}\left[\zeta\left(4,\frac{1}{3}\right) + \zeta\left(4,\frac{1}{6}\right)\right] - \frac{\sqrt{3}\pi^3}{243}\left[\zeta\left(2,\frac{1}{3}\right)+\zeta\left(2,\frac{1}{6}\right)\right], \tag{3.13}$$

$$\int_0^{\frac{\pi}{3}} \log\left(2\sin\frac{x}{2}\right)dx = \frac{2\sqrt{3}}{27}\pi^2 - \frac{\sqrt{3}}{36}\left[\zeta\left(2,\frac{1}{3}\right)+\zeta\left(2,\frac{1}{6}\right)\right], \tag{3.14}$$

and

$$\int_0^{\frac{\pi}{3}} \left[\log^3\left(2\sin\frac{x}{2}\right) - \frac{3}{4}(x-\pi)^2\log\left(2\sin\frac{x}{2}\right)\right]dx = \frac{2\sqrt{3}}{243}\pi^4 + \frac{2\pi}{3}\zeta(3) + \frac{\sqrt{3}\pi^2}{108}\left[\zeta\left(2,\frac{1}{3}\right) + \zeta\left(2,\frac{1}{6}\right)\right] - \frac{\sqrt{3}}{216}\left[\zeta\left(4,\frac{1}{3}\right)+\zeta\left(4,\frac{1}{6}\right)\right]. \tag{3.15}$$

4. Further Remarks and Observations

We consider the following general integral:

$$\mathrm{Ls}_{m+2}^{(m)}(z) = \int_0^z x^m \log\left(2\sin\frac{x}{2}\right)dx \qquad (m \in \mathsf{N}_0). \tag{4.1}$$

Applying integration by parts in (4.1), and using (1.12) and (1.13), we obtain

$$\int_0^z x^m \log\left(2\sin\frac{x}{2}\right) dx = \frac{z^{m+1}}{m+1}\log\left(2\sin\frac{z}{2}\right) - \frac{z^{m+1}}{(m+1)^2} + \frac{2z^{m+1}}{m+1}\sum_{k=1}^{\infty}\frac{\zeta(2k)}{m+2k+1}\left(\frac{z}{2\pi}\right)^{2k} \qquad (|z|<2\pi;\ m\in\mathsf{N}). \tag{4.2}$$

If the known formulas Eq. (2.16) in [11, p. 515] and Eq. (2.13) in [11, p. 514] are used, the infinite series in (4.2) can be expressed as finite series as follows

THEOREM 2. *The following integral formulas hold true:*

$$\begin{aligned}\int_0^z x^{2m}\log\left(2\sin\frac{x}{2}\right) dx &= \frac{z^{2m+1}}{2m+1}\log\left(2\sin\frac{z}{2}\right) \\ &+ \frac{(2\pi)^{2m+1}}{2m+1}\sum_{k=0}^{2m+1}\binom{2m+1}{k}\cdot\left[\zeta'\left(-k,1-\frac{z}{2\pi}\right)\right. \\ &\left.+(-1)^k\zeta'\left(-k,1+\frac{z}{2\pi}\right)\right]\left(\frac{z}{2\pi}\right)^{2m+1-k} \qquad (|z|<2\pi;\ m\in\mathsf{N}_0)\end{aligned} \tag{4.3}$$

and

$$\begin{aligned}\int_0^z x^{2m-1}\log\left(2\sin\frac{x}{2}\right) dx &= \frac{z^{2m}}{2m}\log\left(2\sin\frac{z}{2}\right) \\ +(-1)^{m+1}(2m-1)!\,\zeta(2m+1) &+ \frac{(2\pi)^{2m}}{2m}\sum_{k=0}^{2m}\binom{2m}{k}\left[\zeta'\left(-k,1-\frac{z}{2\pi}\right)\right. \\ &\left.+(-1)^k\zeta'\left(-k,1+\frac{z}{2\pi}\right)\right]\left(\frac{z}{2\pi}\right)^{2m-k} \qquad (|z|<2\pi;\ m\in\mathsf{N}).\end{aligned} \tag{4.4}$$

The special cases of (4.3) when $m=0$ and (4.4) when $m=1$ would, in light of some of the identities in [27, Chapter 2], readily yield, respectively, the integral formulas (4.5) and (4.6) below.

COROLLARY 3. *The following integral formulas hold true:*

$$\begin{aligned}\int_0^{\frac{\pi}{3}}\log\left(2\sin\frac{x}{2}\right) dx &= 2\pi\left[\zeta'\left(-1,\frac{5}{6}\right)-\zeta'\left(-1,\frac{1}{6}\right)\right] = -\,\mathrm{Cl}_2\left(\frac{\pi}{3}\right) \\ &= -\frac{\pi}{3}\log(2\pi)+2\pi\log\left(\frac{\Gamma_2\left(\frac{5}{6}\right)}{\Gamma_2\left(\frac{7}{6}\right)}\right)\end{aligned} \tag{4.5}$$

and

(4.6)

$$\int_0^{\frac{\pi}{3}} x \log\left(2\sin\frac{x}{2}\right) dx = \frac{2\zeta(3)}{3} + \frac{2\pi^2}{3}\left[\zeta'\left(-1, \frac{5}{6}\right) - \zeta'\left(-1, \frac{1}{6}\right)\right]$$
$$= \frac{2\zeta(3)}{3} - \frac{\pi^2}{9}\log(2\pi) + \frac{2\pi^2}{3}\log\left(\frac{\Gamma_2\left(\frac{5}{6}\right)}{\Gamma_2\left(\frac{7}{6}\right)}\right),$$

where Γ_2 *denotes the double Gamma function* (*cf.* [2], [3]*; see also* [27, pp. 94–96]).

It should be remaked in passing that, in view of Equations (3.3), and (4.3) to (4.6), closed-form expressions for the derivatives of $\zeta(s, a)$ at the negative integer s and rational a are needed, some of which were evaluated by Adamchik [1] and Miller and Adamchik [21].

It is also noted here that Srivastava *et al.* [28] studied extensively some definite integrals in conjunction with series involving the Zeta function such as in (4.2) whose closed-form evaluation has been an attractive and interesting research subject since a letter dated 1729 from Goldbach to Daniel Bernoulli (*cf.* [27, Chapter 3]; see also [9], [11], [12], [13], [16], [20], [24], and [25]).

In case (3.3) is used, (4.3) and (4.4) are readily rewritten as Corollary 4 below.

COROLLARY 4. *The following integral formulas hold true:*

$$\int_0^z x^{2m} \log\left(2\sin\frac{x}{2}\right) dx \tag{4.7}$$
$$= (2m)! \sum_{k=2}^{2m+2} \frac{(-1)^{k+[\frac{1}{2}(k+1)]}}{(2m+2-k)!} \mathrm{Cl}_k(z) z^{2m+2-k} \quad (|z| < 2\pi;\, m \in \mathsf{N}_0)$$

and

$$\int_0^z x^{2m-1} \log\left(2\sin\frac{x}{2}\right) dx = (-1)^{m+1}(2m-1)!\zeta(2m+1) \tag{4.8}$$
$$+ (2m-1)! \sum_{k=2}^{2m+1} \frac{(-1)^{k+[\frac{1}{2}(k+1)]}}{(2m+1-k)!} \mathrm{Cl}_k(z) z^{2m+1-k} \quad (|z| < 2\pi;\, m \in \mathsf{N}).$$

Upon setting $m = 1$ in (4.7) and $m = 2$ in (4.8) with $z = \frac{\pi}{3}$, we can deduce Corollary 5 below.

COROLLARY 5. *The following integral formulas hold true:*

$$\int_0^{\frac{\pi}{3}} x^2 \log\left(2\sin\frac{x}{2}\right) dx = -\frac{2\sqrt{3}}{729}\pi^4 - \frac{2\pi}{9}\zeta(3) - \frac{\sqrt{3}\pi^2}{324}\left[\zeta\left(2,\frac{1}{3}\right)+\zeta\left(2,\frac{1}{6}\right)\right] + \frac{\sqrt{3}}{648}\left[\zeta\left(4,\frac{1}{3}\right)+\zeta\left(4,\frac{1}{6}\right)\right] \tag{4.9}$$

and

$$\int_0^{\frac{\pi}{3}} x^3 \log\left(2\sin\frac{x}{2}\right) dx = -\frac{2\sqrt{3}}{243}\pi^5 - \frac{\pi^2}{9}\zeta(3) - \frac{29}{9}\zeta(5) - \frac{\sqrt{3}\pi^3}{972}\left[\zeta\left(2,\frac{1}{3}\right)+\zeta\left(2,\frac{1}{6}\right)\right] + \frac{\sqrt{3}\pi}{648}\left[\zeta\left(4,\frac{1}{3}\right)+\zeta\left(4,\frac{1}{6}\right)\right]. \tag{4.10}$$

DEFINITION 5. The Polygamma functions $\psi^{(n)}(z)$ $(n \in \mathsf{N})$ are defined by

$$\psi^{(n)}(z) := \frac{d^{n+1}}{dz^{n+1}}\{\log\Gamma(z)\} = \frac{d^n}{dz^n}\{\psi(z)\} \qquad (n \in \mathsf{N}_0;\ z \in \mathsf{C}\setminus\mathsf{Z}_0^-;\ \mathsf{Z}_0^- := \{0,-1,-2,\ldots\}), \tag{4.11}$$

which, in terms of the Hurwitz Zeta function $\zeta(s,a)$ given in (1.4), can be written in the following form:

$$\psi^{(n)}(z) = (-1)^{n+1} n!\,\zeta(n+1,z) \qquad (n \in \mathsf{N};\ z \in \mathsf{C}\setminus\mathsf{Z}_0^-). \tag{4.12}$$

By using (4.12), Shen's procedure (see [23, pp. 1393–1394]) can be shortened considerably and it may also be easier to apply it to other situations of a similar nature (see also [22] and [14]). For example, the integral formula (1.15) can be restated as follows:

$$\frac{1}{2\pi}\int_0^{2\pi}\left[\log\left(2\sin\frac{x}{2}\right)\right]^n dx = \frac{(-1)^n n!}{2^n}a_n \qquad (n \in \mathsf{N}_0), \tag{4.13}$$

where the coefficients a_n are given by

$$\frac{2^{-2z}\Gamma\left(\frac{1}{2}-z\right)}{\sqrt{\pi}\,\Gamma(1-z)} = \sum_{n=0}^{\infty} a_n z^n$$

or, equivalently, by the following simple recursion formula:

$$a_1 = 0 \quad \text{and} \quad (n+1)a_{n+1} = 2\sum_{k=1}^{n} a_{n-k}(2^k-1)\zeta(k+1) \qquad (n \in \mathsf{N}).$$

Similarly, by making use of a known result [19, p. 230, Equation (7.160)], the following formulas can be obtained.

COROLLARY 6. *Each of the following results holds true:*

$$(-1)^m \int_0^{\frac{\pi}{3}} \left(x - \frac{\pi}{3}\right)^{2m+1} \log\left(2\sin\frac{x}{2}\right) dx = -\frac{1}{2}(2m+1)!\left(1-2^{-2m-2}\right)\left(1-3^{-2m-2}\right)\zeta(2m+3) + (2m+1)!\sum_{k=0}^{m}(-1)^k\left(\frac{\pi}{3}\right)^{2k}\frac{\zeta(2m+3-2k)}{(2k)!} \qquad (m \in \mathsf{N}_0), \tag{4.14}$$

$$\int_0^{\frac{\pi}{3}} \left(\log\frac{\sin x}{\sin\left(x+\frac{\pi}{3}\right)}\right)^{2m+1} dx = (-1)^{m+1}\left(\frac{3m+4}{2m+3}\right)\left(\frac{\pi}{3}\right)^{2m+3} + (2m+2)!\sum_{k=0}^{m}(-1)^k\left(\frac{\pi}{3}\right)^{2k+1}\frac{\zeta(2m+2-2k)}{(2k+1)!} \qquad (m \in \mathsf{N}_0), \tag{4.15}$$

$$(-1)^{m+1}\int_0^{\frac{\pi}{3}}\left(x-\frac{\pi}{3}\right)^{2m}\log\left(2\sin\frac{x}{2}\right)dx + \frac{1}{2m+1}\int_0^{\frac{\pi}{3}}\left[\log\left(\frac{\sin x}{\sin\left(x+\frac{\pi}{3}\right)}\right)\right]^{2m}dx = (2m)!\sum_{k=1}^{m}(-1)^k\left(\frac{\pi}{3}\right)^{2k-1}\frac{\zeta(2m+3-2k)}{(2k-1)!} \qquad (m \in \mathsf{N}), \tag{4.16}$$

and

$$\sum_{k=0}^{m}(-1)^k\left(\frac{\pi}{3}\right)^{2k}\frac{\zeta(2m+2-2k)}{(2k)!} = \frac{1}{2}(-1)^m\left(\frac{6m+5}{(2m+2)!}\right)\left(\frac{\pi}{3}\right)^{2m+2} + \frac{1}{2}(1-2^{-2m-1})(1-3^{-2m-1})\zeta(2m+2) \qquad (m \in \mathsf{N}). \tag{4.17}$$

In their special cases when $m = 0$, the integral formulas (4.14) and (4.15) yield, respectively, the following *remarkably simple* results:

$$\int_0^{\frac{\pi}{3}} \left(x - \frac{\pi}{3}\right) \log\left(2\sin\frac{x}{2}\right) dx = \frac{2}{3}\zeta(3) \tag{4.18}$$

and

$$\int_0^{\frac{\pi}{3}} \log\left(\frac{\sin x}{\sin\left(x + \frac{\pi}{3}\right)}\right) dx = \frac{5\pi^3}{81}, \tag{4.19}$$

which is the *corrected* version of the second entry in [19, p. 230, Eq. (7.161)].

We note in conclusion that (4.18) is recorded in [19, p. 230] and can also be obtained from (4.5) and (4.6).

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