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WEAK COMPACTNESS IN THE DUAL SPACE OF A JB*-TRIPLE IS COMMUTATIVELY DETERMINED

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Abstract

We prove the following criterium of weak compactness in the dual of a JB*-triple: a bounded set K in the dual of a JB*-triple E is not relatively weakly compact if and only if there exist a sequence of pairwise orthogonal elements (a_n) in the closed unit ball of E, a sequence (φ_n) in K, and $\vartheta > 0$ satisfying that $|\varphi_n(a_n)| > \vartheta$ for all $n \in \mathsf{N}$. This solves a question stimulated by the main result in [11] and posed in [9].

1. Introduction and Preliminaries

Relatively weakly compact subsets in the dual of a C*-algebra have been intensively studied during the last fifty years. The first precedent appears in a paper by A. Grothendieck in 1953 (see [15]). This forerunner establishes the following characterization of weak compactness in the dual of a $C(\Omega)$-space: a bounded subset $K \subseteq C(\Omega)^*$ is not relatively weakly compact if and only if there exists a sequence (O_n) of pairwise disjoint open subsets of Ω such that $\lim_{n\to\infty} \sup\{|\mu(O_n)| : \mu \in K\} \neq 0$. Urysohn's lemma allows us to replace the O_n's by norm-one positive continuous functions on Ω with mutually disjoint supports.

When K is a bounded set in the predual of a von Neumann algebra M, M. Takesaki [26] and C. Akemann [1] (see also [27, Theorem III.5.4]) proved that K is not relatively weakly compact if and only if there exists a sequence (p_n) of pairwise orthogonal projections in M such that $\lim_{n\to\infty} \sup\{|\phi(p_n)| : \phi \in K\} \neq 0$. That is, weak compactness in M_* is determined by the abelian subalgebras of M. Consequently, relatively weakly compact subsets in the dual of a C*-algebra A are commutatively determined by the abelian subalgebras of A^{**}.

In [24] H. Pfitzner showed that weak compactness in the dual of a C*-algebra A is in fact determined by the abelian subalgebras of A. Concretely, a bounded set $K \subseteq A^*$ fails to be relatively weakly compact if and only if there

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exist a positive θ, a sequence (a_n) of pairwise orthogonal positive elements in the closed unit ball of A and a sequence (φ_n) in K satisfying $|\varphi_n(a_n)| > \theta$, for every $n \in \mathbb{N}$ (compare [12] for a new and shorter proof).

C*-algebras belong to a more general class of complex Banach spaces in which the geometric, holomorphic, and algebraic structure mutually interplay. We are referring to the class of JB*-triples. We recall (see [21]) that a *JB**-*triple* is a complex Banach space E equipped with a continuous triple product $\{\cdot,\cdot,\cdot\} : E \times E \times E \to E$, which is symmetric and linear in the first and third variables, conjugate linear in the second variable and satisfies:

(i) (Jordan Identity) $L(a, b)L(x, y) = L(x, y)L(a, b) + L(L(a, b)x, y) - L(x, L(b, a)y)$, where $L(a, b)$ is the operator on E given by $L(a, b)x = \{a, b, x\}$;

(ii) $L(a, a)$ is a hermitian operator with non-negative spectrum;

(iii) $\|L(a, a)\| = \|a\|^2$.

Every C*-algebra is a JB*-triple with respect to the product $\{x, y, z\} = \frac{1}{2}(xy^*z + zy^*x)$, and every JB*-algebra is a JB*-triple under the triple product $\{x, y, z\} = (x \circ y^*) \circ z + (z \circ y^*) \circ x - (x \circ z) \circ y^*$.

A JBW*-triple is a JB*-triple which is also a dual Banach space (with a unique isometric predual [3]). It is known that the second dual of a JB*-triple is a JBW*-triple (compare [8]). Further, the triple product of every JBW*-triple is separately weak*-continuous [3].

The above quoted results of Takesaki and Akemann were extended in [23] to characterize relatively weakly compact subsets in the predual of a JBW*-triple.

A *JC*-triple* is a norm-closed subspace of a C*-algebra which is closed under the ternary product $\{x, y, z\} = \frac{1}{2}(xy^*z + zy^*x)$. JC*-triples form an intermediate class of complex Banach spaces between C*-algebras and JB*-triples. A criterium for weak compactness in the dual of a JC*-triple, which is also a generalization of Pfitzner's result, was established in [11]. This criterium assures that a bounded subset in the dual space of a JC*-triple E is relatively weakly compact if and only if its restriction to any abelian maximal subtriple C of E is relatively weakly compact in the dual of C. However, as pointed out by C. M. Edwards in [9], "whether the results hold for general JB*-triples remains an open question". The main result of this paper gives a positive answer to this question for general JB*-triples (see Theorem 2.3). The solution presented in this paper is itself a novelty which simplifies the results in [11] with a new and shorter orthogonalization process based on Bergmann operators.

Reference [6] is a basic forerunner of the problem studied in this paper. Briefly speaking, we could say [6] contains a partial answer for our problem in terms of Pelczynski's Property (V). We recall that a series $\sum_{n\geq 1} z_n$ in a Banach space X is called *weakly unconditionally convergent* (w.u.c. for short) if for each $\varphi \in X^*$ we have $\sum_{n=1}^{\infty} |\varphi(z_n)| < \infty$, equivalently, there exists $C > 0$ such that for any finite subset $\mathscr{F} \subset \mathsf{N}$ and $|\varepsilon_k| = 1$ in C we have $\| \sum_{k\in\mathscr{F}} \varepsilon_k z_k \| \leq C$, (see, for example, [7, Theorem 6 in Chapter 5]). It is clear that every bounded linear operator between Banach spaces preserves w.u.c. series. A Banach space X has property (V) if for any (bounded) non relatively weakly compact set $K \subseteq X^*$ there exists a w.u.c. series $\sum_n x_n$ in X such that $\sup_{\varphi\in X^*} |\varphi(x_n)|$ does not converge to zero. It is established in [6] that every JB*-triple satisfies property (V). We shall see later that every bounded sequence of mutually orthogonal elements in a JB*-triple defines a w.u.c. series, however the reciprocal statement need not hold in general. We shall establish a new orthogonalization method to construct sequences of mutually orthogonal elements from w.u.c. series.

1.1. Preliminaries

Let X and Y be two Banach spaces, throughout the paper, the symbol $L(X, Y)$ will stand for the space of all bounded linear operators from X to Y. We shall write $L(X)$ for the space $L(X, X)$.

A JB*-triple E is said to be *abelian* if $\{\{x, y, z\}, u, v\} = \{x, y, \{z, u, v\}\} = \{x, \{y, z, u\}, v\}$, for all $x, y, z, u, v \in E$. The JB*-subtriple generated by a single element is always abelian.

Let x be an element in a JB*-triple E. Throughout the paper the symbol E_x will denote the norm-closed subtriple of E generated by x. It is known that E_x is JB*-triple isomorphic to the C*-algebra $C_0(L)$ of all complex-valued continuous functions on L vanishing at 0, where L is a locally compact subset of $(0, \|x\|]$ satisfying that $L \cup \{0\}$ is compact. Further, there exists a JB*-triple isomorphism $\Psi : E_x \to C_0(L)$ which satisfies $\Psi(x)(t) = t$, for all t in L (compare [20, 4.8] and [21, 1.15]). In particular, given a natural n, the symbol $x^{\frac{1}{2n-1}}$ makes sense as an element of $E_x \cong C_0(L)$.

An element u in a JB*-triple E is said to be a *tripotent* if $u = \{u, u, u\}$. Given a tripotent $u \in E$, the mappings $P_i(u) : E \to E_i$, $(i = 0, 1, 2)$, defined by

$$\begin{aligned} P_2(u) &= L(u, u)(2L(u, u) - \mathrm{id}_E),\\ P_1(u) &= 4L(u, u)(\mathrm{id}_E - L(u, u)), \qquad \text{and}\\ P_0(u) &= (\mathrm{id}_E - L(u, u))(\mathrm{id}_E - 2L(u, u)),\end{aligned}$$

are contractive linear operators. For each $j = 0, 1, 2$, $P_j(u)$ is the projection onto the eigenspace $E_j(u)$ of $L(u, u)$ corresponding to the eigenvalue $\frac{j}{2}$ and

$$E = E_2(u) \oplus E_1(u) \oplus E_0(u)$$

is the *Peirce decomposition* of E relative to u. Furthermore, the following

Peirce rules are satisfied,

$$\{E_2(u), E_0(u), E\} = \{E_0(u), E_2(u), E\} = 0, \tag{1}$$

$$\left\{E_i(u), E_j(u), E_k(u)\right\} \subseteq E_{i-j+k}(u), \tag{2}$$

where $E_{i-j+k}(u) = 0$ whenever $i - j + k \notin \{0, 1, 2\}$ (compare [13]).

When W is a JBW*-triple, the JBW*-subtriple generated by a norm-one element $x \in W$ coincides with the weak*-closure, $\overline{W_x}^{w^*}$, of W_x. By [18, Lemma 3.11] there exists a JBW*-triple isomorphism, Ψ, between $\overline{W_x}^{w^*}$ and a commutative W*-algebra C. We shall write $r(x) = \Psi^{-1}(1)$, where 1 denotes the unit element in C. It is clear that $r(x)$, commonly termed the range tripotent of x, is a tripotent in W. Moreover, $r(x)$ coincides with the weak*-limit of the sequence $x^{\frac{1}{2n-1}}$, $(n \in \mathbb{N})$. It is also known that the JBW*-algebra $E_2^{**}(r(x))$ contains x as a positive element (compare [10]).

Given a JBW*-triple W, a norm-one element φ in W_* and a norm-one element z in W with $\varphi(z) = 1$, it follows from [2, Proposition 1.2] that the assignment

$$(x, y) \mapsto \varphi\{x, y, z\}$$

defines a positive sesquilinear form on W. Further, for every norm-one element w in W satisfying $\varphi(w) = 1$, we have $\varphi\{x, y, z\} = \varphi\{x, y, w\}$, for all $x, y \in W$. The mapping $x \mapsto \|x\|_\varphi := (\varphi\{x, x, z\})^{\frac{1}{2}}$, defines a prehilbertian seminorm on W. The Strong*-topology (noted by $S^*(W, W_*)$) is the topology on W generated by the family $\{\|\cdot\|_\varphi : \varphi \in W_*, \|\varphi\| = 1\}$. This topology was introduced by T. J. Barton and Y. Friedman in [2].

When φ is an element in the dual of a JB*-triple E, the prehilbertian seminorm $\|.\|_\varphi$ is defined on E^{**} (and hence on E) by the assignment

$$x \mapsto \|x\|_\varphi := (\varphi\{x, x, z\})^{\frac{1}{2}},$$

where z is a norm-one element in E^{**} with $\varphi(z) = \|\varphi\|$. The inequality

$$\|\{x, y, z\}\| \le \|x\|\,\|y\|\,\|z\|$$

holds for every x, y and z in a JB*-triple E (compare [14, Corollary 3]). Consequently,

$$\|x\|_\varphi \le \|\varphi\|^{\frac{1}{2}}\|x\|,$$

for all $\varphi \in E^*$ and $x \in E$.

For each element a in a JB*-triple E, the conjugate linear mapping $Q(a)$ from E to itself is defined, for each element b in E, by $Q(a)(b) := \{a, b, a\}$. Let x, y be two elements in E. The *Bergmann operator* $B(x, y) : E \to E$

is defined by $B(x, y)(z) = z - 2L(x, y)(z) + Q(x)Q(y)(z)$, for all z in E (compare [22] or [28, page 305]). In the particular case of u being a tripotent, it is known that $P_0(u) = B(u, u)$.

Let x be a symmetric element in a unital JB*-algebra A. The operator $U_x : A \to A$ is defined by $U_x(y) := 2(y \circ x) \circ x - x^2 \circ y$, for all y in A. When A is regarded as a JB*-triple, we have $U_x(y) = Q(x)(y^*)$, $\forall y \in A$. Since by [16, Lemma 2.4.21] $U_x^2 = U_{x^2}$, we deduce that

$$Q(x)^2(y) = U_x^2(y) = U_{x^2}(y) = Q(x^2)(y^*), \qquad \forall y \in A.$$

We also have $2L(x, x)(y) = 2(x^2 \circ y + (y \circ x) \circ x - (y \circ x) \circ x) = 2x^2 \circ y$, for all $y \in A$. Therefore, for each $y \in A$ we have

$$B(x, x)(y) = y - 2L(x, x)(y) + Q(x)^2(y) = Q(1 - x^2)(y^*),$$

which implies that $\|B(x, x)\| \leq 1$, whenever x belongs to the closed unit ball of A.

A tripotent u, in a JB*-triple E, is said to be *bounded* if there exists a norm-one element $x \in E$ such that $L(u, u)x = u$. The element x is a bound of u and in this case we write $u \leq x$. We shall write $y \leq u$ whenever y is a positive element in the JB*-algebra $E_2(u)$ (compare [11, pages 79–80]).

LEMMA 1.1. *Let x be a symmetric element in the closed unit ball of a JB*-algebra A. Then $B(x, x)$ is a contractive operator. Moreover, if p is a projection in A with $p \leq x$, then $B(x, x)(y)$ belongs to $A_0(p)$, for every y in A.*

PROOF. We may assume that A is unital. The comments preceding this lemma guarantee that $\|B(x, x)\| \leq 1$ and $B(x, x)y = Q(1-x^2)(y^*)$, $(y \in A)$. Since $p \leq x^2 \leq 1$, we have $0 \leq 1 - x^2 \leq 1 - p$, and hence $1 - x^2$ belongs to $A_0(p)$. Finally, it follows, by Peirce rules, that $B(x, x)y \in A_0(p)$.

Lemma 1.1 above can be now extended to JB*-triples.

LEMMA 1.2. *Let E be a JB*-triple, e a tripotent in E, and x a norm-one element in E with $e \leq x$. Then $B(x, x)$ is a contractive operator and $B(x, x)(y)$ belongs to $E_0(e)$, for every y in E.*

PROOF. By [14, Corollary 1] we may suppose that E embeds as a subtriple into a JBW*-algebra, A, of the form $L(H) \bigoplus^\infty N$, where H is a complex Hilbert space and N is an ℓ_∞-sum of finite-dimensional simple JB*-algebras.

We may then assume that

$$e \leq x \ (\leq r(x))$$

in the JBW*-algebra A, where $r(x)$ is the range tripotent of x in A. From [4, Lemma 2.3] and [22, Corollary 5.12] there exists a weak*-continuous isometric triple embedding T from A into A, such that $T(r(x))$ (and hence $T(e)$) is a projection in A. It is easy to check that $0 \leq T(e) \leq T(x) \leq T(r(x))$. By Lemma 1.1, we have $T(B(x,x)(y)) = B(T(x),T(x))(T(y)) \in A_0(T(e))$, for every $T(y) \in T(E) \subseteq A$. Therefore, $B(x,x)(y) \in A_0(e) \cap E = E_0(e)$, for all $y \in E$.

Another central notion in the paper is the concept of orthogonality. Two elements a, b in a JB*-triple, E, are said to be *orthogonal* (written $a \perp b$) if $L(a,b) = 0$. Lemma 1 in [5] shows that $a \perp b$ if and only if one of the following statements holds:

$$\{a,a,b\} = 0; \quad a \perp r(b); \quad r(a) \perp r(b); \quad E_2^{**}(r(a)) \perp E_2^{**}(r(b));$$
$$r(a) \in E_0^{**}(r(b)); \quad a \in E_0^{**}(r(b)); \quad b \in E_0^{**}(r(a)); \quad E_a \perp E_b.$$

The Peirce rule (1) shows that for each tripotent u in a JB*-triple E, $E_0(u) \perp E_2(u)$. The Jordan identity and the above reformulations assure that

$$a \perp \{x,y,z\}, \qquad \text{whenever} \quad a \perp x, y, z. \tag{3}$$

Let A be a C*-algebra. Two elements $a, b \in A$ are said to be orthogonal for the C*-algebra product if $ab^* = b^*a = 0$. However, A also enjoys a structure of JB*-triple. We have, a priory, two notions of orthogonality in A. It can be checked, from the above reformulations, that two elements a, b in A are orthogonal for the C*-algebra product if and only if they are orthogonal when A is considered as a JB*-triple.

For every tripotent e in a JB*-triple E, the formula

$$\|P_2(e)(x) + P_0(e)(x)\| = \max\{\|P_2(e)(x)\|, \|P_0(e)(x)\|\},$$

holds for every x in E (compare [13, Lemma 1.3]). In particular, if $\{x_1, \ldots, x_m\}$ is a set of mutually orthogonal elements in a JB*-triple E, it follows from the above equivalent reformulations of orthogonality and the previous formula, that the JB*-subtriple generated by the set $\{x_1, \ldots, x_m\}$ coincides with the ℓ_∞-sum $\bigoplus_{k=1,\ldots,m}^{\infty} E_{x_k}$ and hence it is JB*-triple isomorphic to an abelian C*-algebra.

We deduce from the above paragraph that every bounded sequence of pairwise orthogonal elements in a JB*-triple defines a w.u.c. series.

2. Main result

The aim of this section is to prove that weak compactness in the dual of a JB*-triple is commutatively determined. Bergmann operators, wisely used, turn to be a powerful tool in orthogonalization processes. More concretely, we shall make use of appropriated Bergmann operators to orthogonalize weakly unconditional convergent series in JB*-triples.

LEMMA 2.1. *Let E be a JB*-triple, v a tripotent in E, and φ an element in the closed unit ball of E^*. Then for each $y \in E_2(v)$ with $\|y\| \leq 1$ we have*

$$|\varphi(x - B(y, y)(x))| < 21\|x\|\|v\|_{\varphi}, \tag{4}$$

for every $x \in E$.

PROOF. By Peirce rules we have $L(y, y)(x) \in E_2(v) \oplus E_1(v)$ and $Q(y)^2(x) \in E_2(v)$. Since $x - B(y, y)(x) = 2L(y, y)(x) - Q(y)^2(x)$, the desired statement follows from [11, Lemma 3.2].

We shall also need the following strengthening version of [11, Lemma 3.4].

LEMMA 2.2. *Let E be a JB*-triple, $\theta > 0$, $\delta_n > 0$ ($n \in \mathbb{N}$), and let $\{\varphi_1\} \cup \{\varphi_n\}_{n \geq 2}$ be a sequence of functionals in the closed unit ball of E^*. Given an element x in the closed unit ball of E, satisfying $|\varphi_1(x)| > \theta$ and $\|x\|_{\varphi_n} < \delta_n$, $n \geq 2$, there exist two elements a, y in the unit ball of E_x, and two tripotents u, v in $(E_x)^{**}$ such that $a \leq u \leq y \leq v$, $|\varphi_1(a)| > \frac{3}{4}\theta$, and $\|v\|_{\varphi_n} < \frac{8}{\theta}\delta_n$, $n \geq 2$.*

PROOF. We have already commented that E_x is JB*-triple isomorphic to the C*-algebra $C_0(L)$, where L is a locally compact subset of $(0, \|x\|]$ satisfying that $L \cup \{0\}$ is compact. Moreover, there exists a JB*-triple isomorphism $\Psi : E_x \to C_0(L)$ satisfying $\Psi(x)(t) = t$, for all t in L. By slight abuse of notation, E_x and $C_0(L)$ will be identified.

Let $a, y \in C_0(L)$ be the functions defined by

$$a(t) := \begin{cases} 0, & \text{if } 0 \leq t \leq \frac{\theta}{4} \\ 2t - \frac{\theta}{2}, & \text{if } \frac{\theta}{4} \leq t \leq \frac{\theta}{2} \\ t, & \text{if } \frac{\theta}{2} \leq t \leq \|x\| \end{cases} ;$$

$$y(t) := \begin{cases} 0, & \text{if } 0 \leq t \leq \frac{\theta}{8} \\ \frac{8}{\theta}\left(t - \frac{\theta}{8}\right), & \text{if } \frac{\theta}{8} \leq t \leq \frac{\theta}{4} \\ 1, & \text{if } \frac{\theta}{4} \leq t \leq \|x\|. \end{cases}$$

Since $\|x-a\| < \frac{\theta}{4}$ and $|\varphi_1(x)| > \theta$ it follows that $|\varphi_1(a)| > \frac{3}{4}\theta$.

The element x decomposes as the sum of two orthogonal elements $x = x\chi_{[\frac{\theta}{8},\|x\|]} + x\chi_{[0,\frac{\theta}{8})}$ (in $(E_x)^{**}$). Since $\|\cdot\|_{\varphi_n}^2$ is additive when applied to the sum of orthogonal elements, we get $\left\|x\chi_{[\frac{\theta}{8},\|x\|]}\right\|_{\varphi_n} < \delta_n$. We define $u = \chi_{[\frac{\theta}{4},\|x\|]}$, $v = \chi_{[\frac{\theta}{8},\|x\|]}$ (in $(E_x)^{**}$), which clearly satisfy that $a \le u \le y \le v$.

Since $\|\cdot\|_\varphi$ is an order-preserving map on the set of positive elements in $(E_x)^{**}$ ([11, Lemma 3.3]), we have that $\|v\|_{\varphi_n} \le \|\frac{8}{\theta}x\chi_{[\frac{\theta}{8},\|x\|]}\|_{\varphi_n} < \frac{8}{\theta}\delta_n$ $(n \ge 2)$, which finishes the proof.

Our main result can be stated now.

THEOREM 2.3. *Let E be a JB*-triple and K be a bounded subset in E^*. The following are equivalent:*

a) *K is not relatively weakly compact.*

b) *There exist a sequence of pairwise orthogonal elements (a_n) in the closed unit ball of E, a sequence (φ_n) in K, and $\vartheta > 0$ satisfying that $|\varphi_n(a_n)| > \vartheta$ for all $n \in \mathsf{N}$.*

b′) *There exists a subtriple $\mathscr{C}$ of E isometric to an abelian C*-algebra such that the restriction of K to it is not relatively weakly compact.*

PROOF. a) $\Rightarrow$ b). Since JB*-triples have Pelczynski's Property (V) (compare [6]) there exist $\theta > 0$, $(\varphi_n) \subset K$ and a w.u.c. series $\sum_{n\ge1} z_n$ in E with $\|z_n\| \le 1$, such that $|\varphi_n(z_n)| > \theta$, $\forall n \in \mathsf{N}$. We may assume that K is contained in the closed unit ball of E^*.

Let us fix a decreasing sequence (δ_n) of positive numbers satisfying $\frac{336}{\theta}\sum_{n=1}^{\infty}\delta_n < \frac{\theta}{2}$. We shall construct, inductively, a sequence (a_n) of mutually orthogonal elements in the closed unit ball of E, infinite subsets $\mathsf{N} \supsetneqq N_1 \supsetneqq N_2 \supsetneqq \cdots \supsetneqq N_{n-1} \supsetneqq N_n \supsetneqq \cdots$ and a strictly increasing mapping $\sigma : \mathsf{N} \to \mathsf{N}$ such that for each natural n there exists a w.u.c. series $\sum_{k\in N_n} z_{n,k}$ in E with $\|z_{n,k}\| \le 1$,

$$z_{n,k} \perp a_j, \qquad \text{for all} \quad j \in \{1,\dots,n\}, \quad k \in N_n,$$

$$|\varphi_{\sigma(i)}(a_i)| > \frac{3}{8}\theta, \qquad i = 1,\dots,n,$$

and
$$|\varphi_k(z_{n,k})| > \theta - \frac{336}{\theta}\sum_{j=1}^{n}\delta_j > \frac{\theta}{2}, \qquad k \in N_n.$$

To define a_1, choose $j_1 \in \mathsf{N}$ with $\frac{1}{j_1} < \frac{1}{C^2}\delta_1^2$, where C is the positive constant associated to the w.u.c. series $\sum_{n\ge1} z_n$ (see comments in the Introduction).

Since every Hilbert space is of cotype 2 (compare [25, page 32]) we have

$$\frac{1}{j_1}\sum_{k=1}^{j_1}\|z_k\|_{\varphi_m}^2 \le \frac{1}{j_1}\int_D \left\|\sum_{k=1}^{j_1}\varepsilon_k z_k\right\|_{\varphi_m}^2 d\mu$$
$$\le \frac{1}{j_1}\int_D \|\varphi_m\| \left\|\sum_{k=1}^{j_1}\varepsilon_k z_k\right\|^2 d\mu \le \frac{C^2}{j_1} < \delta_1^2,$$

where $m \in \mathsf{N}$, $D = \{-1, 1\}^{\mathsf{N}}$, $\varepsilon_k \in \{\pm 1\}$ and μ is the uniform probability measure on D. Since the above inequality is satisfied for every $m \in \mathsf{N}$, there exist $\sigma(1) \in \{1, \ldots, j_1\}$ and an infinite subset $N_1 \subset \mathsf{N}$ such that $\sigma(1) < \min N_1$ and $\|z_{\sigma(1)}\|_{\varphi_m} < \delta_1$, for every $m \in N_1$.

Applying Lemma 2.2 to $z_{\sigma(1)}$ and $\{\varphi_{\sigma(1)}\} \cup \{\varphi_m\}_{m\in N_1}$ we obtain two elements a_1, y_1 in the closed unit ball of $E_{z_{\sigma(1)}}$ and two tripotents $u_1, v_1 \in E^{**}$ such that $a_1 \le u_1 \le y_1 \le v_1$,

$$|\varphi_{\sigma(1)}(a_1)| > \frac{3}{4}\theta > \frac{3}{8}\theta, \qquad \text{and} \qquad \|v_1\|_{\varphi_m} < \frac{8}{\theta}\delta_1 < \frac{16}{\theta}\delta_1, \quad m \in N_1.$$

We define $z_{1,k} := B(y_1, y_1)z_k$, $(k \in N_1)$, which are elements in the closed unit ball of E by Lemma 1.2. Clearly $\sum_{k\in N_1} z_{1,k}$ also is a w.u.c. series. Lemma 1.2 assures that $z_{1,k}$ is contained in $E \cap E_0^{**}(u_1)$. Since $a_1 \in E_2^{**}(u_1)$, we deduce that $a_1 \perp z_{1,k}, \forall k \in \mathsf{N}$ (compare with the reformulations of orthogonality given in page 312). Moreover $\left\|\sum_{k\in\mathscr{F}} \varepsilon_k z_{1,k}\right\| = \left\|B(y_1, y_1)\left(\sum_{k\in\mathscr{F}} \varepsilon_k z_k\right)\right\| \le C$, for every finite $\mathscr{F} \in N_1$ and $|\varepsilon_k|$ in C. Now, noticing that $y_1 \in E_2^{**}(v_1)$, Lemma 2.1 applies to assure that

$$|\varphi_k(z_{1,k})| \ge |\varphi_k(z_k)| - |\varphi_k(z_k - z_{1,k})| > \theta - 21\frac{16}{\theta}\delta_1 \left(> \frac{\theta}{2}\right),$$

for all $k \in N_1$.

Suppose now, in our inductive step, that $a_1, \ldots, a_n$, $N_n \subsetneq N_{n-1} \subsetneq \cdots \subsetneq N_1 \subsetneq \mathsf{N}$, $\sigma(1) < \sigma(2) < \cdots < \sigma(n)$, and the w.u.c. series $\sum_{k\in N_n} z_{n,k}$ in E have been constructed satisfying the corresponding induction hypothesis.

Take $j_{n+1} \in \mathsf{N}$ with $\frac{1}{j_{n+1}} < \frac{1}{C^2}\delta_{n+1}^2$ and a subset $D_{n+1} \subset N_n$ with exactly j_{n+1} elements. As before, for $m \in N_n$ we have

$$\frac{1}{j_{n+1}}\sum_{k\in D_{n+1}}\|z_{n,k}\|_{\varphi_m}^2 \le \frac{1}{j_{n+1}}\int_D \left\|\sum_{k\in D_{n+1}}\varepsilon_k z_{n,k}\right\|_{\varphi_m}^2 d\mu$$
$$\le \frac{1}{j_{n+1}}\int_D \|\varphi_m\| \left\|\sum_{k\in D_{n+1}}\varepsilon_k z_{n,k}\right\|^2 d\mu \le \frac{C^2}{j_{n+1}} < \delta_{n+1}^2,$$

hence there exist $\sigma(n+1) \in D_{n+1}$ and an infinite subset $N_{n+1} \subseteq N_n$ such that $\sigma(n+1) < \min N_{n+1}$ and $\|z_{n,\sigma(n+1)}\|_{\varphi_m} < \delta_{n+1}$, for every $m \in N_{n+1}$.

Applying Lemma 2.2 to $z_{n,\sigma(n+1)}$ and $\{\varphi_{\sigma(n+1)}\} \cup \{\varphi_m\}_{m\in N_{n+1}}$ we obtain two elements a_{n+1}, y_{n+1} in the closed unit ball of $E_{z_{n,\sigma(n+1)}}$ and two tripotents $u_{n+1}, v_{n+1} \in (E_{z_{n,\sigma(n+1)}})^{**}$ such that $a_{n+1} \leq u_{n+1} \leq y_{n+1} \leq v_{n+1}$,

$$|\varphi_{\sigma(n+1)}(a_{n+1})| > \frac{3}{8}\theta, \qquad \text{and} \qquad \|v_{n+1}\|_{\varphi_m} < \frac{16}{\theta}\delta_{n+1}, \quad m \in N_{n+1}.$$

By the induction hypothesis, $z_{n,k} \perp a_j$, for all $j \in \{1, \dots, n\}$, $k \in N_n$. Since $a_{n+1}, y_{n+1}, u_{n+1}$, and v_{n+1} belong to $(E_{z_{n,\sigma(n+1)}})^{**}$, the equivalent reformulations of orthogonality given in page 312, guarantee that they are all orthogonal to a_j, for all $j \in \{1, \dots, n\}$.

We define $z_{n+1,k} := B(y_{n+1}, y_{n+1})(z_{n,k})$, $k \in N_{n+1}$. Again, Lemma 1.2 assures that $z_{n+1,k}$ is contained in $E \cap E_0^{**}(u_{n+1})$. Since $a_{n+1} \in E_2^{**}(u_{n+1})$, we deduce that a_{n+1} is orthogonal to each $z_{n+1,k}$, $\forall k \in N_{n+1}$. Since y_{n+1} and $z_{n,k}$ are orthogonal to a_j for all $j \in \{1, \dots, n\}$, $k \in N_{n+1}$, using (3), it can be seen that

$$z_{n+1,k} = B(y_{n+1}, y_{n+1})(z_{n,k}) = z_{n,k} - 2L(y_{n+1}, y_{n+1})(z_{n,k}) + Q(y_{n+1})^2(z_{n,k})$$

is orthogonal to a_j, for all $j \in \{1, \dots, n\}$, $k \in N_{n+1}$. Moreover,

$$\Big\|\sum_{k\in\mathscr{F}} \varepsilon_k z_{n+1,k}\Big\| = \Big\|B(y_{n+1}, y_{n+1})\Big(\sum_{k\in\mathscr{F}} \varepsilon_k z_{n,k}\Big)\Big\| \leq C,$$

for any finite subset $\mathscr{F} \subset N_{n+1}$, and $|\varepsilon_k| = 1$ in $\mathbb{C}$.

Finally, since $y_{n+1} \in E_2^{**}(v_{n+1})$, Lemma 2.1 assures that

$$\begin{aligned}|\varphi_k(z_{n+1,k})| &\geq |\varphi_k(z_{n,k})| - |\varphi_k(z_{n,k} - z_{n+1,k})| \\ &> \theta - \frac{336}{\theta}\sum_{j=1}^{n}\delta_j - 21\frac{16}{\theta}\delta_{n+1} \\ &= \theta - \frac{336}{\theta}\sum_{j=1}^{n+1}\delta_j \quad \left(> \frac{\theta}{2}\right) \qquad \text{for all} \quad k \in N_{n+1}.\end{aligned}$$

b) $\Rightarrow$ b′) Since the elements (a_n) are mutually orthogonal, the subtriple $\mathscr{C}$ generated by the family $\{a_n : n \in \mathbb{N}\}$ coincides with the ℓ_∞-sum $\bigoplus_n^\infty E_{a_n}$. We recall that each E_{a_n} is isomorphic to $C_0(L)$, for a suitable locally compact Hausdorff space. Therefore $\mathscr{C}$ is triple-isomorphic to an abelian C*-algebra and the restriction of K to $\mathscr{C}$ cannot be relatively weakly compact.

b′) $\Rightarrow$ a) is obvious.

A Dieudonné-type theorem for JC*-triples was established in [11, Theorem 4.2]. When in the proof of the just quoted result, Theorem 2.3 replaces [11, Theorem 3.5], we obtain the following generalization of Dieudonné's theorem in the more general setting of JB*-triples.

THEOREM 2.4. *Let* (ϕ_n) *be a sequence in the dual of a JB*-triple* E *such that the sequence* $(\phi_n(r(x)))$ *converges whenever* $r(x)$ *is the range tripotent of a norm-one element* x *in* E. *Then there exists* ϕ *in* E^* *satisfying that* (ϕ_n) *converges weakly to* ϕ. *In particular, if* $(\phi_n(r(x))) \to 0$, *for every range tripotent,* $r(x)$, *of a norm-one element* x *in* E, *then* (ϕ_n) *is a weakly null sequence in* E^*.

The vector-valued version of the above theorem follows now as a consequence. The following corollary also generalizes the main result in [19] with a shorter and simpler proof.

COROLLARY 2.5. *Let* E *be a JB*-triple,* X *a Banach space and* (T_n) *a sequence of weakly compact operators from* E *to* X. *Suppose that* $\lim T_n^{**}(r(x))$ *exists whenever* $r(x)$ *is the range tripotent of a norm-one element* x *in* E. *Then there exists a unique weakly compact operator* $T : E \to X$, *such that* $T^{**}(z) = \lim T_n^{**}(z)$, *for every* $z \in E^{**}$.

PROOF. We claim that for each $z \in E^{**}$, $(T_n^{**}(z))$ is a norm convergent sequence. Otherwise, there exist $z \in E^{**}$, $\varepsilon > 0$, and $(\sigma(n)) \subset \mathbb{N}$ such that $\|T_{\sigma(n+1)}^{**}(z) - T_{\sigma(n)}^{**}(z)\| > \varepsilon$, $\forall n \in \mathbb{N}$. Defining $S_k = T_{\sigma(k+1)}^{**} - T_{\sigma(k)}^{**}$, we can find norm-one functionals $\psi_k \in X^*$ satisfying $|\psi_k(S_k(z))| > \varepsilon$ $(\forall k \in \mathbb{N})$. Since $T_k^{**} : E^{**} \to X^{**}$ is weak*-to-weak* continuous, the sequence $(\psi_k T_k^{**})_{k\in\mathbb{N}}$ lies, in fact, in E^*. In particular, the sequence $(\psi_k S_k) \subseteq E^*$ satisfies, by hypothesis, that $\lim \psi_k S_k(r) = 0$, for every range tripotent, $r = r(x)$, of a norm-one element x in E. Theorem 2.4 assures that $(\psi_k S_k)$ is weakly null in E^*, which contradicts $|\psi_k S_k(z)| = |\psi_k S_k(z))| > \varepsilon$, $(k \in \mathbb{N})$.

The assignment $z \mapsto Lz := \lim T_n^{**}(z)$ defines a linear mapping $L : E^{**} \to X^{**}$, which is continuous by the Uniform Boundedness Principle. Since each T_n is weakly compact we have $T_n^{**}(E^{**}) \subseteq X$, $\forall n \in \mathbb{N}$. In particular $L(E^{**}) \subseteq X$. Therefore $T := L_{|E} : E \to X$ is a well-defined bounded linear operator.

Theorem 2.4 implies that, for each $\psi \in X^*$ the $\psi T_n^{**} = T_n^*(\psi) \in E^*$ converge weakly to some $\varphi \in E^*$. Thus $\psi L = \varphi \in E^*$, which proves that L is weak*-to-weak* continuous. It is now clear that $T^{**} = L$. Finally, the expression $T^{**}(E^{**}) = L(E^{**}) \subseteq X$ shows that T is weakly compact.

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