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ON THE HEART OF HYPERMODULES

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Abstract

This paper presents some types of hypermodules, associated to an arbitrary hypermodule, studying properties and characterizing their hearts. Also, we establish a few results concerning the sequence of heart, which can be associated to a hypermodule, in connection with subhypermodules generated by a non-empty set, by a union of subhypermodules or by the intersection of subhypermodules. Finally, we study several properties of 1-hypermodules.

1. Introduction

A *hypergroupoid* $(H, \circ)$ is a non-empty set H with a *hyperoperation* $\circ$ defined on H, that is, a mapping of $H \times H$ into the family of non-empty subsets of H. If A, B are non-empty subsets of H then $A \circ B$ is given by $A \circ B = \bigcup\{x \circ y \mid x \in A, y \in B\}$. $x \circ A$ is used for $\{x\} \circ A$ and $A \circ x$ for $A \circ \{x\}$. A hypergroupoid $(H, \circ)$ is called a *hypergroup* in the sense of Marty if for all $x, y, z \in H$ the following two conditions hold: (i) $x \circ (y \circ z) = (x \circ y) \circ z$, (ii) $x \circ H = H \circ x = H$. If $(H, \circ)$ satisfies only the first axiom, then it is called a *semi-hypergroup*. An exhaustive review updated to 1992 of hypergroup theory appears in [2]. A recent book [3] contains a wealth of applications.

Let H be a hypergroup and ρ an equivalence relation on H. Let $\rho(a)$ be the equivalence class of a with respect to ρ and let $H/\rho = \{\rho(a) \mid a \in H\}$. A hyperoperation $\otimes$ is defined on H/ρ by $\rho(a) \otimes \rho(b) = \{\rho(x) \mid x \in \rho(a) \circ \rho(b)\}$. If ρ is strongly regular, then it readily follows that $\rho(a) \otimes \rho(b) = \{\rho(x) \mid x \in a \circ b\}$. It is well known for ρ strongly regular that $(H/\rho, \otimes)$ is a group (see Theorem 31 in [2]), that is $\rho(a) \otimes \rho(b) = \rho(c)$ for all $c \in a \circ b$.

A *hyperring* is a multi-valued system $(R, +, \circ)$ which satisfies the ring-like axioms in the following way: $(R, +)$ is a hypergroup in the sense of Marty, $(R, \circ)$ is a semi-hypergroup and the multiplication is distributive with respect to the hyperoperation $+$. Let R be a hyperring. We recall the relation Γ defined as follows: $x \Gamma y \iff \exists n \in \mathsf{N}, \exists (k_1, \dots, k_n) \in \mathsf{N}^n$, and $[\exists (x_{i1}, \dots, x_{ik_i}) \in$

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R^{k_i}, $(i = 1, \ldots, n)$] such that

$$x, y \in \sum_{i=1}^{n} \Big(\prod_{j=1}^{k_i} x_{ij} \Big).$$

The relation Γ is reflexive and symmetric. Let Γ^* be the transitive closure of Γ, then Γ^* is a strongly regular relation both on $(R, +)$ and $(R, \cdot)$, and the quotient R/Γ^* is a ring [5], [6].

Let $(M, +)$ be a hypergroup and $(R, +, \cdot)$ be a hyperring. According to [7] M is said to be a hypermodule over R if there exists

$$\cdot : R \times M \to \wp^*(M); \qquad (a, m) \mapsto a \cdot m$$

such that for all $a, b \in R$ and $m_1, m_2, m \in M$, we have

(1) $a \cdot (m_1 + m_2) = a \cdot m_1 + a \cdot m_2$,
(2) $(a + b) \cdot m = (a \cdot m) + (b \cdot m)$,
(3) $(a \cdot b) \cdot m = a \cdot (b \cdot m)$.

Let M be an R-hypermodule, N_1 and N_2 two subhypermodules of M; we say that N_2 is N_1-conjugable if N_2 as a subhypergroup is conjugable, and an R-hypermodule M is regular if M as a hypergroup is regular.

Let R be a hyperring and M be a hypermodule over R. Let $x, y \in M$, the relation ϵ on M defined as follows [4]:

$$x \epsilon y \Leftrightarrow x, y \in \sum_{i=1}^{n} m_i'; \quad m_i' = m_i \qquad \text{or} \qquad m_i' = \sum_{j=1}^{n_i} \Big(\prod_{k=1}^{k_{ij}} x_{ijk} \Big) z_i,$$
$$m_i \in M, \quad x_{ijk} \in R, \quad z_i \in M.$$

If M is an R-hypermodule, then we set

$$\epsilon_0 = \{(m, m) \mid m \in M\}$$

and for every integer $n \geq 1$, ϵ_n is the relation defined as follows:

$$x \epsilon_n y \Leftrightarrow x, y \in \sum_{i=1}^{n} m_i'.$$

Obviously, for every $n \geq 0$, the relation ϵ_n is symmetric, and the relation $\epsilon = \bigcup_{n \geq 0} \epsilon_n$ is reflexive and symmetric.

The fundamental relation ϵ^* on M can be defined as the smallest equivalence relation such that the quotient M/ϵ^* is a module over the corresponding fundamental ring R/Γ^* such that M/ϵ^* as a group is not abelian.

Let M and N be R-hypermodules. A function $f : M \to N$ is called an R-homomorphism, if for every $(x, y) \in M^2$ and $r \in R$

$$f(x + y) = f(x) + f(y) \qquad \text{and} \qquad f(r \cdot x) = r \cdot f(x).$$

If H is an R-module and $f : M \to H$ is an R-homomorphism, we let $\operatorname{Ker} f = \{m \in M \mid f(m) = 0_H\}$. Moreover, the canonical projection $\phi_M : M \to M/\epsilon^*$ by $\phi_M(m) = \epsilon^*(m)$, is an R-homomorphism and $\omega(M) := \operatorname{Ker} \phi_M = \{m \in M \mid \phi_M(m) = 0_{M/\epsilon^*}\}$.

If M is a hypermodule and $\rho \subseteq M \times M$ is an equivalence relation then for all pairs (A, B) of non-empty subsets of M, we set $A\overline{\overline{\rho}}B$ if and only if $a\rho b$ for all $a \in A$ and $b \in B$. The relation ρ is said to be *strongly regular to the right* if $x\rho y$ implies $x \circ a\overline{\overline{\rho}}y \circ a$ and $x.r\overline{\overline{\rho}}y.r$ for all $x, y, a \in M$ and $r \in R$. Analogously, we can define *strongly regular to the left*. Moreover ρ is called *strongly regular* if it is strongly regular to the right and to the left.

THEOREM 1.1. *Let M be an R-hypermodule and ρ be a strongly regular relation on M. Then $(M/\rho, \oplus)$ is an R-hypermodule if and only if for every $x \in M$ and $r \in R$, $r \cdot \rho(x) = \rho(r \cdot x)$.*

PROOF. Since ρ is strongly regular, the scalar hyperoperation $r \cdot \rho(x) := \rho(r \cdot x)$ is well defined. Since M is an R-hypermodule, the properties of M as an R-hypermodule, guarantee that the hypergroup M/ρ is an R-hypermodule.

2. Heart of a hypermodule

In the following m'_i, z'_i and y'_i are the notations in the definition of ϵ.

Let M be an R-hypermodule and A be a non-empty subset of M. Then the intersection of the subsets of M which are complete and contain A is called the complete closure of A in M; it will be denoted $C(A)$. If $K_1(A) = A$, and

$$K_{n+1}(A) = \left\{x \in M \;\middle|\; \exists p \in \mathsf{N}, \exists (m'_1, m'_2, \ldots, m'_p) : x \in \sum_{i=1}^{p} m'_i \cap K_n \neq \emptyset\right\},$$

then $K(A) = C(A)$, and the relation $xKy \Leftrightarrow x \in C(\{y\})$ is an equivalence. Also, for every $x, y \in M$, we have $xKy \Leftrightarrow x\epsilon^*y$. Furthermore, if B is a non-empty subset of M, we have $C(B) = \bigcup_{b\in B} C(b)$, where $C(b) = C(\{b\})$ [1].

THEOREM 2.1. *Let M be an R-hypermodule, $\phi_M : M \to M/K$ the canonical projection. If N is a hypermodule with ordinary group and $f : M \to N$ is an R-homomorphism, then there exists $g : M/K \to N$ such that $g\phi_M = f$.*

PROOF. It is enough to observe for every $x \in M$ that $g\phi_M(x) = f(x)$. First, g is well defined: in fact $\phi_M(x) = \phi_M(y) \Rightarrow xKy$. Since N is a hypermodule (with ordinary group), it follows that $f(x) = f(y)$. Furthermore, g is an R-homomorphism because for every $x, y \in M$, and $u \in x + y$, we have $g(\phi_M(x) + \phi_M(y)) = g\phi_M(x + y) = g\phi_M(u) = f(u) = f(x + y) = f(x) + f(y) = g\phi_M(x) + g\phi_M(y)$. Also, for every $r \in R$, and $v \in r.x$ we have $g(\phi_M(r \cdot x)) = g(\phi_M(v)) = f(v) = f(r \cdot x) = r \cdot f(x) = r \cdot (g\phi_M(x))$.

THEOREM 2.2. *Let $f : M \to M'$ be an R-homomorphism, then:*

(1) $\forall x \in M,\ f(C(x)) \subseteq C(f(x))$.

(2) *f determines an R-homomorphism $f^* : M/K \to M'/K'$ defined by $f^*(\phi_M(x)) = \phi_{M'}(f(x))$.*

PROOF. (1) It is sufficient to observe that for every $n \in \mathsf{N}$, the implication $x\epsilon_n y \Rightarrow f(x)\epsilon_n f(x)$ is valid.

(2) f^* is well defined, in fact $\phi_M(x) = \phi_M(y)$, i.e., xKy implies by (1) $f(x)Kf(y)$, hence $f^*(\phi_M(x)) = f^*(\phi_M(y))$. Clearly, f^* is an R-homomorphism, because for every $u \in x + y$, $f^*(\phi_M(x) + \phi_M(y)) = f^*(\phi_M(f(x + y))) = \phi_{M'}(f(u)) = \phi_{M'}(f(u)) = \phi_{M'}(f(x) + f(y)) = \phi_{M'}(f(x)) + \phi_{M'}(f(y)) = f^*(\phi_M(x)) + f^*(\phi_M(y))$, and for every $r \in R$ and $v \in r.x$, we have $f^*(\phi_M(r \cdot x)) = \phi_{M'}(f(v)) = \phi_{M'}(f(r \cdot x)) = r \cdot \phi_M(f(x)) = r \cdot f^*(\phi_M(x))$.

LEMMA 2.3. *For every non empty subset H of an R-hypermodule M, we have*

(1) $\phi_M^{-1}(\phi_M(H)) = \omega(M) + H = H + \omega(M)$.

(2) *If H is a complete part of M, then $\phi_M^{-1}(\phi_M(H)) = H$.*

PROOF. (1) For every $x \in \omega(M)+H$, there exists a pair $(a, b) \in \omega(M) \times H$ such that $x \in a+b$, so $\phi_M(x) \subseteq \phi_M(a)+\phi_M(b) = 0_{M/\epsilon^*} + \phi_M(b) = \phi_M(b)$. Therefore $x \in \phi_M^{-1}(\phi_M(b)) \subseteq \phi_M^{-1}(\phi_M(H))$.

Conversely, for every $x \in \phi_M^{-1}(\phi_M(H))$, there exists an element $b \in H$ such that $\phi_M(x) = \phi_M(b)$. By the reproducibility there exists $a \in M$ such that $x \in a + b$, so $\phi_M(b) = \phi_M(x) = \phi_M(a) + \phi_M(b)$, hence $\phi_M(a) = 0_{M/\epsilon^*}$ and $a \in \phi_M^{-1}(0_{M/\epsilon^*}) = \omega(M)$. Therefore, $x \in a + b \subseteq \omega(M) + H$. This proves that $\phi_M^{-1}(\phi_M(M)) = \omega(M) + H$. In the same way, it is possible to prove that $\phi_M^{-1}(\phi_M(H)) = H + \omega(M)$.

(2) It is obvious that $H \subseteq \phi_M^{-1}(\phi_M(H))$. Moreover, if $x \in \phi_M^{-1}(\phi_M(H))$, then there exists an element $b \in H$ such that $\phi_M(x) = \phi_M(b)$. Hence $x \in \epsilon^*(x) = \epsilon^*(b) \subseteq H$ and $\phi_M^{-1}(\phi_M(H)) \subseteq H$.

LEMMA 2.4. *Let M be an R-hypermodule. Then $\omega(M)$ is the intersection of all R-subhypermodules of M that are complete parts.*

PROOF. By Lemma 2.3, we have $\omega(M)+\omega(M) = \omega(M)$ as a hypermodule. Let $A \in \bigcap M_i$, where every M_i is a complete part subhypergroup of M. Then $A + \omega(M) = A$. Also, A is an invertible subhypermodule of M, hence

$$\forall (a, x) \in A \times \omega(M), \exists b \in A : a \in b + x \Rightarrow a \in A + x \Rightarrow x \in A + a = A.$$

Therefore $\omega(M) \subseteq A$.

THEOREM 2.5. *Let M be an R-hypermodule and B the union of summations of finite numbers of $\sum_{i=1}^{n} m_i'$, containing at least one right and at least one left identity and be scalar multiplicatively closed. Then $B = \omega(M)$.*

PROOF. We set E_l (E_r) equal to the set of left (right) identities and $T = \{P \in B \mid P \cap E_l \neq \emptyset, P \cap E_r \neq \emptyset\}$. Furthermore, for every $x \in M$, we denote by $i_l(x)$ ($i_r(x)$) the set of left (right) inverses of x. First, we prove that for every $a \in B, i_l(a) \subseteq B \supseteq i_r(a)$. Let $a \in B$, then there exists a $\sum_{i=1}^{n} m_i' = P \in T$ such that $a \in P$. If $a' \in i_l(a)$, then $e' \in E_l$ exists such that $e' \in a' + a$; if $a'' \in i_l(a)$, then $e'' \in E_r$ exists such that $e'' \in a + a''$. We now consider the $P_1 = a' + \sum_{i=1}^{n} m_i' + a + a''$, we have $P_1 \subseteq T$, in fact $\{e', e''\} \subseteq e' + e'' \subseteq a' + a + a + a'' \subseteq P_1$. Furthermore, $\{a', a''\} \subseteq P_1$; in fact $a' + a + a'' \subseteq P_1$ and $a' \in a' + e'' \subseteq a' + a + a''$, also $a'' \in e' + a'' \subseteq a' + a + a''$.

Now, we prove that B is a complete part of M. Let $a \in \sum_{i=1}^{n} m_i' \cap B \neq \emptyset$, hence a $\sum_{i=1}^{t} z_i' = P \in T$ exists such that $a \in P$. Now let e', e'' be respectively the left and right identities, $a', a'' \in M$, such that $e' \in a' + a, e'' \in a + a''$. Then $\sum_{i=1}^{n} m_i' \subseteq e' + \sum_{i=1}^{n} m_i' + e'' \subseteq a' + a + \sum_{i=1}^{n} m_i' + a + a'' \subseteq a' + P + \sum_{i=1}^{n} m_i' + P + a'' \supseteq a' + a + a + a'' \supseteq \{e', e''\}$, thus $a' + P + \sum_{i=1}^{n} m_i' + P + a'' = P_1$. Therefore $\sum_{i=1}^{n} m_i' \subseteq P_1 \in T$ and for this reason $\sum_{i=1}^{n} m_i' \subseteq B$.

Let $a, b \in M$, such that $a \in P, b \in Q$ where $P, Q \in T$. Then $a + b \in B$. Also for every $r \in R, r \cdot a \subseteq B$.

Also, B satisfies the conditions of reproducibility. Since M is an R-hypermodule, the properties of M as an R-hypermodule, guarantee that the hypergroup B is an R-hypermodule. It is clear that $B \subseteq \omega(M)$. As seen from the above, it turns out that B is a complete part subhypermodule, thus by Lemma 2.4, $\omega(M) \subseteq B$.

We denote by $\sum_C(A)$ the set of hypersums A of elements of M such that $C(A) = A$.

THEOREM 2.6. *Let M be an R-hypermodule and $(x_1', \ldots, x_n')$ such that $\sum_{i=1}^{n} x_i' \in \sum_C(M)$, then there exists $(y_1', \ldots, y_n')$ such that $\sum_{i=1}^{n} x_i' + \sum_{i=1}^{n} y_i' = \omega(M)$.*

PROOF. We set $x_i' = \sum_{j=1}^{n_i}\big(\prod_{k=1}^{k_{ij}} r_{ijk}\big)x_i$. For $1 \leq t \leq n$, let a_t be an element of $\omega(M)$, then there exists $y_t \in M$ such that $a_t \in x_t + y_t$, hence

$$\sum_{j=1}^{n_t}\Big(\prod_{k=1}^{k_{tj}} r_{tjk}\Big)a_t \subseteq \sum_{j=1}^{n_t}\Big(\prod_{k=1}^{k_{tj}} r_{tjk}\Big)x_t + \sum_{j=1}^{n_t}\Big(\prod_{k=1}^{k_{tj}} r_{tjk}\Big)y_t = x_t' + y_t'.$$

Since $\omega(M)$ is a complete part, we have $x_t' + y_t' \subseteq \omega(M)$. Therefore

$$\begin{aligned}\sum_{i=1}^{n} x_i' + y_n' = \omega(M) + \sum_{i=1}^{n} x_i' + y_n' &= \sum_{i=1}^{n-1} x_i' + \omega(M) + x_n' + y_n' \\ &= \sum_{i=1}^{n-1} x_i' + \omega(M) = \omega(M) + \sum_{i=1}^{n-1} x_i'\end{aligned}$$

and so

$$\sum_{i=1}^{n} x_i' + y_n' + y_{n-1}' = \omega(M) + \sum_{i=1}^{n-2} x_i' + x_{n-1}' + y_{n-1}' = \omega(M) + \sum_{i=1}^{n-2} x_i'.$$

Going on the same way one arrives to

$$\sum_{i=1}^{n} x_i' + \sum_{i=1}^{n} y_i' = \omega(M) + x_1' + y_1' = \omega(M).$$

LEMMA 2.7. *Let $(M, +)$ be an R-hypermodule, then*

(1) *$M - \omega(M)$ is a complete part of M.*

(2) *If $M - \omega(M)$ is a hypersum, then $\omega(M)$ is also a hypersum.*

PROOF. (1) Obvious.

(2) By (1), $M - \omega(M)$ is a complete part. Now, by using Theorem 2.6, the proof is completed.

REMARK 2.8. Let M be an R-hypermodule endowed with a complete hypersum. The following implication is satisfied for every $A \in \wp^*(M)$: $A \cap \sum_{i=1}^{n} m_i' = \emptyset \Rightarrow C(A) \cap \sum_{i=1}^{n} m_i' = \emptyset$.

Suppose that $z \in C(A) \cap \sum_{i=1}^{n} m_i'$. Then $a \in A$ exists such that $z \in C(a)$, hence $C(a) = C(z)$. The hypothesis $\sum_{i=1}^{n} m_i' = C\big(\sum_{i=1}^{n} m_i'\big)$ implies

$$C(z) \subseteq \bigcup_{y \in \sum_{i=1}^{n} m_i'} C(y) = C\Big(\sum_{i=1}^{n} m_i'\Big) = \sum_{i=1}^{n} m_i'.$$

Therefore $a \in A$, $a \in C(z) \subseteq \sum_{i=1}^{n} m'_i$, where $\sum_{i=1}^{n} m'_i \cap A \neq \emptyset$ which is absurd.

Let $(M, +)$ be an R-hypermodule. Let's consider the sequence

$$(*) \quad M \supseteq \omega(M) = \omega_1 \supseteq \omega(\omega(M)) = \omega_2 \supseteq \cdots \supseteq \omega_k \supseteq \omega_{k+1} \supseteq \cdots \supseteq \omega_n \supseteq \cdots$$

PROPOSITION 2.9. *Let M be an R-hypermodule. Then the following conditions are equivalent:*

(1) *The sequence $(*)$ is finite;*

(2) *there is $(n, k) \in \mathsf{N}^2$, where $n > k + 1$, such that ω_n is a complete part of ω_k;*

(3) *there is $(n, k) \in \mathsf{N}^2$ where $n > k + 1$, such that for any $(x, y) \in (\omega_k - \omega_n) \times (\omega_k - \omega_n)$; $(x + y) \cap (\omega_k - \omega_n) \neq \emptyset$ implies $x + y \subseteq \omega_k - \omega_n$;*

(4) *there is $(n, k) \in \mathsf{N}^2$, where $n > k + 1$, such that for any ω_n is a ω_k-conjugable.*

PROOF. $(1) \Rightarrow (2)$. If the sequence $(*)$ is finite, then there is $n \in \mathsf{N}$ such that $\omega_n = \omega_{n-1}$, hence ω_n is a complete part of ω_{n-2}.

$(2) \Rightarrow (3)$. If ω_n is a complete part of ω_k, then $\omega_k - \omega_n$ is a complete part of ω_k.

$(3) \Rightarrow (4)$. One proves easily that for any $s \in \mathsf{N}$, ω_s is a closed subhypermodule of M. Moreover, for all $a, b \in \omega_k$, if $\{a, b\} \subseteq \omega_k - \omega_n$, we have $a + b \subseteq \omega_k$, if $a \neq b$ and $|\{a, b\} \cap \omega_n| = 1$, we have $a + b \subseteq \omega_k - \omega_n$, since ω_n is a closed subhypermodule of ω_k. Then, we obtain that ω_n is ω_k-conjugable.

$(4) \Rightarrow (1)$. We know ω_n is a complete part subhypermodule of ω_k. Hence $\omega_{k+1} = \omega(\omega_k) \subseteq \omega_n \subseteq \omega_{k+1}$ from which $\omega_n = \omega_{k+1}$. So, we have: $\omega_{n+1} = \omega(\omega_n) = \omega(\omega_{k+1}) = \omega_{k+2} \supseteq \omega_n = \omega_{k+1} \supseteq \omega_{k+2}$. Therefore, $\omega_n = \omega_{k+2} = \omega_{n+1}$. Let $\omega_{n+s} = \omega_{k+1}$. It follows $\omega_{n+s+1} = \omega(\omega_{n+s}) = \omega(\omega_{k+1}) = \omega_{k+2} = \omega_{k+1}$. Then, for any m such that $m \geq n$, we have $\omega_m = \omega_n$.

THEOREM 2.10. *Let $(M, +)$ be an R-hypermodule such that the sequence $(*)$ is finite, and let N be a complete part subhypermodule of M. Then there is $p \in \mathsf{N}$ such that $\omega_{p+1}(N) = \omega_{p+1}(M)$.*

PROOF. Notice that $\omega(N)$ is a subhypermodule of $\omega(M)$. Indeed, for any $a \in \omega(N)$, there is $e \in N$ such that $a \in a + e$, it's clear that $a \in \epsilon_N(e) \subseteq \epsilon_M(e) = \omega(M)$. Moreover, since N is a complete part subhypermodule of M, we have $\omega(M) \subseteq N$. Then $\omega_1(N) \subseteq \omega_1(M) \subseteq N$. For any $s \geq 1$, from $\omega_s(N) \subseteq \omega_s(M) \subseteq \omega_{s-1}(N)$, one obtains $\omega_{s+1}(N) \subseteq \omega_{s+1}(M) \subseteq \omega_s(N)$, hence a sequence $N \supseteq \omega_1(M) \supseteq \omega_1(N) \supseteq \omega_2(M) \supseteq \omega_2(N) \supseteq \cdots$.

By Proposition 2.9, there is $(n, p) \in \mathsf{N} \times \mathsf{N}$, where $n > p + 1$, such that $\omega_n(M) = \omega_{p+1}(M)$, therefore $\omega_{p+1}(M) = \omega_{p+1}(N)$.

REMARK 2.11. If N_1, N_2 be subhypermodules of M, then

$$\omega(N_1 \cap N_2) \leq \omega(N_1) \cap \omega(N_2).$$

Generally, we do not have equality.

Let M be an R-hypermodule, for which $\omega(M) \neq M$ and let $x, y \in M$ be arbitrary in M. Let's define on $M' = M \cup \{b, c, d\}$ $(\{b, c, d\} \cap M = \emptyset)$ the following hyperoperation:

$+'$	x	b	c	d
y	$y+x$	b	c	d
b	b	M	d	c
c	c	d	M	b
d	d	c	b	M

and for every $r \in R, m \in M$ the scalar multiplication $r \cdot' m = r \cdot m$ and $r \cdot' a = a$, $r \cdot' b = b$ and $r \cdot' c = c$. We can easily verify $(M', +)$ with scalar multiplication $\cdot'$ is an R-hypermodule. We consider subhypermodules $M'_1 = M \cup \{b\}, M'_2 = M \cup \{c\}, M'_3 = M \cup \{d\}$ of M', then $\omega(M'_1) = \omega(M'_2) = \omega(M'_3) = M$, $\omega(M'_1 \cap M'_2 \cap M'_3) \neq M$.

But for an R-hypermodule M, and $N_1, N_2 \leq M$ whose sequence $(*)$ is finite, we can find the following relation between $\omega(N_1 \cap N_2)$ and $\omega(N_1), \omega(N_2)$:

PROPOSITION 2.12. *If $N_1, N_2 \leq M$, where M has a finite sequence $(*)$, then there exist $p \in \mathsf{N}$, such that $\omega_{p+1}(N_1 \cap N_2) = \omega_{p+1}(\omega(N_1) \cap \omega(N_2))$.*

PROOF. Let's consider $\overline{M} := N_1 \cap N_2$ and $\overline{N} := \omega(N_1) \cap \omega(N_2)$. Then $\overline{N}$ is a subhypermodule, complete part of $\overline{M}$. (We can verify this using the definition of a complete part of a hypermodule.) Therefore we use the proof of Theorem 2.10.

Also, we can give a relation for R-subhypermodule of M :

$$\exists p \in \mathsf{N}, \quad \omega_{p+1}(N_1 \cap N_2 \cap \cdots \cap N_m) = \omega_{p+1}(\omega(N_1) \cap \omega(N_2) \cap \cdots \cap \omega(N_m)).$$

REMARK 2.13. If $N_1, N_2 \leq M$, then $\omega(N_1) \subseteq N_1 \cap \omega(\langle N_1 \cup N_2 \rangle)$.

Generally, we have not equality. Let M_1 and M_2 be two R-hypermodules with the scalar hyperoperation $\cdot_1$ and $._2$ respectively. Let m_1, n_1 arbitrary in M_1

and m_2, n_2 arbitrary in M_2. Let's define on $M = M_1 \cup M_2 \cup \{a\}$ $(a \notin M_1 \cup M_2)$ with the following hyperoperations:

$+'$	m_1	a	m_2
n_1	$n_1 + m_1$	a	M
a	a	M_1	M
n_2	M	M	$n_2 + m_2$

and for every $r \in R, x \in M_1$ and $y \in M_2$ the scalar multiplication $r \cdot' x = r \cdot_1 x$, $r \cdot' y = r \cdot_2 y$ and $r \cdot' a = a$. We can easily verify $(M, +')$ with scalar multiplication $\cdot'$ is an R-hypermodule. We consider subhypermodules $N_1 = M_1 \cup \{a\}$, $N_2 = M_2$, $N_1 \cup N_2 = M$, $\langle N_1 \cup N_2 \rangle = M$, then $\omega(\langle N_1 \cup N_2 \rangle) = M$. So

$$\omega(N_1) = M_1 \subset N_1 \cap \omega(\langle N_1 \cup N_2 \rangle) = N_1 = M_1 \cup \{a\}.$$

THEOREM 2.14. *Let M be an R-hypermodule with commutative hypergroup and N_1, N_2 be subhypermodules of M. If for any $a \in \langle N_1 \cup N_2 \rangle - (N_1 \cup N_2)$, there exists $(n_1, n_2) \in N_1 \times N_2$, such that $a \in n_1 + n_2$ and if $\langle \omega(N_1) \cup \omega(N_2) \rangle$ is a closed subhypermodule of $\omega(\langle N_1 \cup N_2 \rangle)$ then*

$$\langle \omega(N_1) \cup \omega(N_2) \rangle = \omega(\langle N_1 \cup N_2 \rangle).$$

PROOF. We shall prove that $\langle \omega(N_1) \cup \omega(N_2) \rangle$ is conjugable in $\langle N_1 \cup N_2 \rangle$ as hypermodule. $\langle \omega(N_1) \cup \omega(N_2) \rangle$ is closed in $\langle N_1 \cup N_2 \rangle$ because, from $x \in a + b$, where $(a, b) \in \langle \omega(N_1) \cup \omega(N_2) \rangle^2$ and $x \in \langle N_1 \cup N_2 \rangle$, it results $(a, b) \in \omega(\langle N_1 \cup N_2 \rangle)^2$ and so $x \in \omega(\langle N_1 \cup N_2 \rangle)$. Using now the condition given in the proposition, $x \in \langle \omega(N_1) \cup \omega(N_2) \rangle$.

As regards an arbitrary element $a \in \langle N_1 \cup N_2 \rangle$, we have three cases:

$$a \in N_1 \Rightarrow \exists a' \in N_1, a + a' \subseteq \omega(N_1) \subseteq \langle \omega(N_1) \cup \omega(N_2) \rangle;$$
$$a \in N_2 \Rightarrow \exists a' \in N_2, a + a' \subseteq \omega(N_2) \subseteq \langle \omega(N_1) \cup \omega(N_2) \rangle;$$
$$a \in \langle N_1 \cup N_2 \rangle - (N_1 \cup N_2) \Rightarrow \exists n_1 \in N_1, \exists n_2 \in N_2, a \in n_1 + n_2.$$

For n_i there exists $n_i' \in N_i$, such that $n_i + n_i' \in \omega_{n_i}$, $i = 1, 2$.

So, $a + n_1' + n_2' \subseteq (n_1 + n_2') + (n_2 + n_2') \subseteq \omega(N_1) \oplus \omega(N_2) \subseteq < \omega(N_1) \cup \omega(N_2) >$, whence for every $t \in n_1' + n_2'$, $a + t \subseteq < \omega(N_1) \cup \omega(N_2) >$.

A hypermodule M is said to be ϵ_n^*-complete hypermodule if there exists $n \in \mathsf{N} \cup \{0\}$, and n is the smallest integer such that $\epsilon_n^* = \epsilon^*$ and $\epsilon_n^* \neq \epsilon_{n-1}^*$.

LEMMA 2.15. *A hypermodule M is ϵ_0^*-complete if and only if M is a hypermodule* (*with ordinary group*).

PROOF. Suppose that M is a ϵ_0^*-complete hypermodule, so $\epsilon_0^* = \epsilon^*$, hence $\epsilon_2 \subseteq \epsilon_0$ and $\epsilon_1 \subseteq \epsilon_0$. Now, for every $x, y \in m_1 + m_2$, we have $x\epsilon_2 y$, so $x = y$. Also for every $x, y \in r.m$, we have $x\epsilon_1 y$, so $x = y$. Therefore $m_1 + m_2$ and $r \cdot m$ both are singletons, and so M is a hypermodule over the hyperring R.

Conversely, if M is a module, then for every $\sum_{i=1}^n m_i'$, we have $\left|\sum_{i=1}^n m_i'\right| = 1$. By definition, $x\epsilon_n y$ if and only if $x = \sum_{i=1}^n m_i' = y$, thus $x = y$ and $x\epsilon_0 y$.

COROLLARY 2.16. *If M is a ϵ_n^*-complete R-hypermodule, then M/ϵ_n^* is an R/Γ^*-module.*

PROPOSITION 2.17. *Every finite hypermodule is ϵ_n^*-complete.*

PROOF. Since M is finite, the chain $\epsilon_1^* \subseteq \epsilon_2^* \subseteq \cdots$ is stationary. Thus there exists $n \in \mathsf{N}$ such that $\epsilon_n^* = \epsilon^*$ and $\epsilon_n^* \neq \epsilon_{n-1}^*$.

THEOREM 2.18. *We have*

(1) *If $\forall(v, w) \in (\omega(M))^2$, $v\epsilon_n w$, then $\epsilon = \epsilon_{n+1}$.*

(2) *If $\forall(v, w) \in (\omega(M))^2$, $v\epsilon_n^* w$, then $\epsilon^* = \epsilon_{n+1}^*$.*

PROOF. (1) If $x\epsilon y$, since $\omega(M) + M = M + \omega(M) = M$ then there exists $(v, w) \in (\omega(M))^2$ such that $y \in x + v$ and $y \in x + w$, by the hypothesis $v\epsilon_n w$. Now, we have $(x + v)\overline{\overline{\epsilon}}_{n+1}(x + w)$, whence $x\epsilon_{n+1} y$, so $\epsilon \subseteq \epsilon_{n+1}$.

(2) It follows from (1).

Let M be an R-hypermodule. M is called 1-hypermodule if $\omega(M)$ is a singleton.

THEOREM 2.19. *Let M be an 1-hypermodule and $\omega(M) = \{e\}$. Then*

(1) *The ϵ^*-classes are the summations $e + a$, where $a \in M$.*

(2) *Every R-subhypermodule of M is complete part.*

(3) *If $\{M_i\}_{i\in I}$ is a family of R-subhypermodules of M, then $\bigcap_{i\in I} M_i$ is an R-subhypermodule of M.*

(4) *The direct product of 1-hypermodules is a 1-hypermodule.*

PROOF. (1) It is clear.

(2) If N is a subhypermodule of M, we have $N \cap \omega(M) \neq \emptyset$, which implies $\omega(M) \subseteq N$, hence $N = N + \omega(M)$ and therefore N is a complete part.

(3) For (2), $\forall i \in I$, $e \in M_i$, we set $N = \bigcap_{i\in I} M_i$, hence $N \neq \emptyset$. Then for every $x, y \in N$ there exists $b \in M$ such that $y \in b + x$, but M_i is ($\forall i \in I$) a closed submodule by (2), thus $b \in M_i$. Also, for every $r \in R, m \in N$, we have $r \cdot m \subseteq N$.

(4) Set $N = \prod_{i\in I} N_i$, $x' = (x_i')_{i\in I} \in N$, $e = (e_i)_{i\in I}$. We have $x\epsilon_n e$ if and only if $z'^1 = (z_i'^1)_{i\in I}, z'^2 = (z_i'^2)_{i\in I}, \ldots, z'^n = (z_i'^n)_{i\in I}$ exist such

that $x, e \in \sum_{i=1}^{n} z'^k$, that is if and only if $\forall i \in I, x'_i, e_i \in \sum_{k=1}^{n} z_i'^k$. Then $z'_i = \sum_{k=1}^{n} z_i'^k = e_i$, whence $x = e$, for this reason $\omega(N) = \{e\}$.

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