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A RESULT ON FRACTIONAL k-DELETED GRAPHS

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Abstract

Let $k \geq 2$ be an integer, and let G be a graph of order n with $n \geq 4k - 5$. A graph G is a fractional k-deleted graph if there exists a fractional k-factor after deleting any edge of G. The binding number of G is defined as

$$\mathrm{bind}(G) = \min\left\{\frac{|N_G(X)|}{|X|} : \emptyset \neq X \subseteq V(G), N_G(X) \neq V(G)\right\}.$$

In this paper, it is proved that if $\mathrm{bind}(G) > \frac{(2k-1)(n-1)}{k(n-2)}$, then G is a fractional k-deleted graph. Furthermore, it is shown that the result in this paper is best possible in some sense.

1. Introduction

We consider only finite undirected simple graph G with vertex set $V(G)$ and edge set $E(G)$. For $x \in V(G)$, we use $N_G(x)$ for the set of vertices of $V(G)$ adjacent to x, and $d_G(x)$ for the degree of x in G. The minimum vertex degree of G is denoted by $\delta(G)$. For any $S \subseteq V(G)$, we define $N_G(S) = \bigcup_{x\in S} N_G(x)$. We denote by $G[S]$ the subgraph of G induced by S, by $G - S$ the subgraph obtained from G by deleting vertices in S together with the edges incident to vertices in S. A vertex set $S \subseteq V(G)$ is called independent if $G[S]$ has no edges. The binding number of G is defined as

$$\mathrm{bind}(G) = \min\left\{\frac{|N_G(X)|}{|X|} : \emptyset \neq X \subseteq V(G), N_G(X) \neq V(G)\right\}.$$

Let k be an integer such that $k \geq 1$. Then a spanning subgraph F of G is called a k-factor if $d_F(x) = k$ for all $x \in V(G)$. A fractional k-factor is a function h that assigns to each edge of a graph G a number in [0, 1], so that for each vertex x we have $d_G^h(x) = k$, where $d_G^h(x) = \sum_{e\ni x} h(e)$ (the sum is taken over all edges incident to x) is a fractional degree of x in G. A graph G is a fractional k-deleted graph if there exists a fractional k-factor after deleting any edge of G. The other terminologies and notations not given in this paper can be found in [1] and [10].

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Many authors have investigated factors [2], [3], [4], [7], [9], [12], and fractional factors [8]. Li, Yan and Zhang gave a necessary and sufficient condition for a graph to be a fractional k-deleted graph [5]. Li, Zhang and Yan showed a sufficient condition for a graph to be a fractional k-deleted graph [6]. Recently, Zhou and Duan obtained a sufficient condition for a graph to be a fractional k-deleted graph [13]. In this paper, we give a new sufficient condition for a graph to be a fractional k-deleted graph.

The following results on fractional k-deleted graphs are known.

THEOREM 1.1 ([6]). *Let G be a graph, and let $k \geq 2$ be an integer. If $\delta(G) \geq k+1$ and $I(G) > k$, then G is a fractional k-deleted graph.*

THEOREM 1.2 ([13]). *Let G be a graph. Then G is a fractional 2-deleted graph if $\delta(G) \geq 3$ and* $\mathrm{bind}(G) \geq 2$.

We prove the following theorem for a graph to be a fractional k-deleted graph, which is an extension of Theorem 1.2.

THEOREM 1.3. *Let $k \geq 2$ be an integer, and let G be a graph of order n with $n \geq 4k-5$. If*

$$\mathrm{bind}(G) > \frac{(2k-1)(n-1)}{k(n-2)},$$

then G is a fractional k-deleted graph.

The following two results are essential to the proof of Theorem 1.3.

THEOREM 1.4 ([5]). *A graph G is a fractional k-deleted graph if and only if for any $S \subseteq V(G)$ and $T = \{x : x \in V(G) \setminus S, d_{G-S}(x) \leq k\}$*

$$\delta_G(S,T) = k|S| + d_{G-S}(T) - k|T| \geq \varepsilon(S,T),$$

where $\varepsilon(S,T)$ is defined as follows,

$$\varepsilon(S,T) = \begin{cases} 2, & \text{if } T \text{ is not independent,} \\ 1, & \text{if } T \text{ is independent, and } e_G(T, V(G)\setminus(S\cup T)) \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

THEOREM 1.5 ([11]). *Let G be a graph of order n with* $\mathrm{bind}(G) > c$. *Then $\delta(G) > n - \frac{n-1}{c}$.*

2. Proof of Theorem 1.3

PROOF. Suppose that G satisfies the assumption of the theorem, but it is not a fractional k-deleted graph. Then by Theorem 1.4, there exist some $S \subseteq V(G)$ and $T = \{x : x \in V(G) \setminus S, d_{G-S}(x) \leq k\}$ such that

$$\delta_G(S,T) = k|S| + d_{G-S}(T) - k|T| \leq \varepsilon(S,T) - 1. \tag{1}$$

CLAIM 1. $|T| \geq k+1$.

PROOF. In view of Theorem 1.5, we have

$$\begin{aligned}|S| + d_{G-S}(x) \geq d_G(x) \geq \delta(G) &> n - \frac{n-1}{\frac{(2k-1)(n-1)}{k(n-2)}} \\ &= n - \frac{k(n-2)}{2k-1} = \frac{(k-1)n+2k}{2k-1} \\ &\geq \frac{(k-1)(4k-5)+2k}{2k-1} = 2(k-1) - \frac{k-3}{2k-1}.\end{aligned}$$

If $k \geq 3$, then according to the integrity of $\delta(G)$ we obtain

$$|S| + d_{G-S}(x) \geq \delta(G) \geq 2k-2. \tag{2}$$

If $k = 2$, then by the integrity of $\delta(G)$ we get

$$|S| + d_{G-S}(x) \geq \delta(G) \geq 2k-1. \tag{3}$$

Let $|T| \leq k$ and $k \geq 3$, then by (1) and (2), we have

$$\begin{aligned}\varepsilon(S,T) - 1 &\geq \delta_G(S,T) = k|S| + d_{G-S}(T) - k|T| \\ &\geq |T||S| + d_{G-S}(T) - k|T| \\ &= \sum_{x \in T}(|S| + d_{G-S}(x) - k) \geq \sum_{x \in T}(2k-2-k) \\ &= \sum_{x \in T}(k-2) = (k-2)|T| \geq |T| \geq \varepsilon(S,T),\end{aligned}$$

which is a contradiction.

Let $|T| \leq k$ and $k = 2$, then by (1) and (3), we have

$$\begin{aligned}\varepsilon(S,T) - 1 &\geq \delta_G(S,T) = k|S| + d_{G-S}(T) - k|T| \\ &\geq |T||S| + d_{G-S}(T) - k|T| \\ &= \sum_{x \in T}(|S| + d_{G-S}(x) - k) \geq \sum_{x \in T}(2k-1-k) \\ &= \sum_{x \in T}(k-1) = (k-1)|T| = |T| \geq \varepsilon(S,T),\end{aligned}$$

a contradiction.

CLAIM 2. $S \neq \emptyset$.

PROOF. Let $S = \emptyset$. If $k \geq 3$, then by (1) and (2) we get that

$$\begin{aligned}\varepsilon(S,T) - 1 \geq \delta_G(S,T) &= k|S| + d_{G-S}(T) - k|T| \\ &= d_G(T) - k|T| \geq (\delta(G) - k)|T| \\ &\geq (2k-2-k)|T| = (k-2)|T| \geq |T| \geq \varepsilon(S,T),\end{aligned}$$

this is a contradiction.

If $k = 2$, then by (1) and (3) we have

$$\begin{aligned}\varepsilon(S,T) - 1 \geq \delta_G(S,T) &= k|S| + d_{G-S}(T) - k|T| \\ &= d_G(T) - k|T| \geq (\delta(G) - k)|T| \\ &\geq (2k-1-k)|T| = (k-1)|T| = |T| \geq \varepsilon(S,T),\end{aligned}$$

which is a contradiction.

CLAIM 3. *There exists $x \in T$ such that $d_{G-S}(x) \leq k-1$.*

PROOF. If $d_{G-S}(x) \geq k$ for all $x \in T$, then we get from Claim 2

$$\delta_G(S,T) = k|S| + d_{G-S}(T) - k|T| \geq k|S| \geq k \geq 2 \geq \varepsilon(S,T),$$

which contradicts (1).

Define

$$h = \min\{d_{G-S}(x) \mid x \in T\}.$$

Then by Claim 3, we obtain

$$0 \leq h \leq k-1.$$

By Theorem 1.5 and $\delta(G) \leq |S| + h$, we get

$$|S| \geq \delta(G) - h > n - \frac{k(n-2)}{2k-1} - h = \frac{(k-1)n+2k}{2k-1} - h. \tag{4}$$

The proof splits into two cases.

Case 1. $h = 0$.

First, we prove the following claim.

CLAIM 4. $\frac{k(n-2)}{n-1} \geq 1$.

PROOF. In view of $k \geq 2$ and $n \geq 4k-5$, we get

$$k(n-2) - (n-1) = (k-1)(n-2) - 1 \geq 0.$$

Thus, we obtain

$$\frac{k(n-2)}{n-1} \geq 1.$$

Let m be the number of vertices x in T such that $d_{G-S}(x) = 0$, and let $Y = V(G) \setminus S$. Then $N_G(Y) \neq V(G)$ since $h = 0$, and $Y \neq \emptyset$ by Claim 1, and so $|N_G(Y)| \geq \text{bind}(G)|Y|$. Thus

$$n - m \geq |N_G(Y)| \geq \text{bind}(G)|Y| = \text{bind}(G)(n - |S|),$$

that is,

$$|S| \geq n - \frac{n-m}{\text{bind}(G)} > n - \frac{k(n-2)(n-m)}{(2k-1)(n-1)}. \tag{5}$$

According to (1), (5), Claim 4 and $|T| \leq n - |S|$, we have

$$\begin{aligned}
\varepsilon(S,T) - 1 &\geq \delta_G(S,T) = k|S| + d_{G-S}(T) - k|T| \\
&\geq k|S| - k|T| + |T| - m \\
&\geq k|S| - (k-1)(n - |S|) - m \\
&= (2k-1)|S| - kn + n - m \\
&> (2k-1)\left(n - \frac{k(n-2)(n-m)}{(2k-1)(n-1)}\right) - kn + n - m \\
&= kn - \frac{k(n-2)(n-m)}{n-1} - m \\
&\geq kn - \frac{k(n-2)(n-1)}{n-1} - 1 \\
&= kn - k(n-2) - 1 = 2k - 1 > 2 \geq \varepsilon(S,T).
\end{aligned}$$

This is a contradiction.

Case 2. $1 \leq h \leq k-1$.
In view of Claim 1, we obtain

$$|T| \geq k + 1 > h + 1.$$

Let v be a vertex in T such that $d_{G-S}(v) = h$, and put $Y = T - N_{G-S}(v)$. Then $|Y| \geq |T| - h > 1$ and $N_G(Y) \neq V(G)$. Thus, we get

$$\frac{n-1}{|T|-h} \geq \frac{|N_G(Y)|}{|Y|} \geq \text{bind}(G) > \frac{(2k-1)(n-1)}{k(n-2)},$$

that is,

$$|T| < \frac{k(n-2)}{2k-1} + h. \tag{6}$$

By (4) and (6), we have

$$\begin{aligned}\delta_G(S,T) &= k|S| + d_{G-S}(T) - k|T| \\ &\geq k|S| - k|T| + h|T| = k|S| - (k-h)|T| \\ &> k\left(\frac{(k-1)n+2k}{2k-1} - h\right) - (k-h)\left(\frac{k(n-2)}{2k-1} + h\right).\end{aligned}$$

Subcase 2.1. $h = 1$.
Obviously, we obtain

$$\begin{aligned}\delta_G(S,T) &> k\left(\frac{(k-1)n+2k}{2k-1} - 1\right) - (k-1)\left(\frac{k(n-2)}{2k-1} + 1\right) \\ &= k \cdot \frac{(k-1)n+1}{2k-1} - (k-1)\cdot\frac{kn-1}{2k-1} = \frac{2k-1}{2k-1} = 1.\end{aligned}$$

According to the integrity of $\delta_G(S,T)$, we get that

$$\delta_G(S,T) \geq 2 \geq \varepsilon(S,T),$$

this contradicts (1).

Subcase 2.2. $2 \leq h \leq k-1$.
Clearly, $k \geq 3$. Let $f(h) = k\big(\frac{(k-1)n+2k}{2k-1} - h\big) - (k-h)\big(\frac{k(n-2)}{2k-1} + h\big)$. Then

$$\delta_G(S,T) > f(h), \tag{7}$$

and

$$f^{'}(h) = -2k + 2h + \frac{k(n-2)}{2k-1}.$$

Since $2 \leq h \leq k-1$ and $n \geq 4k-5$, we have

$$\begin{aligned}f^{'}(h) &\geq -2k + 4 + \frac{k(n-2)}{2k-1} = \frac{-4k^2+2k+8k-4+kn-2k}{2k-1} \\ &= \frac{kn-4k^2+8k-4}{2k-1} \geq \frac{k(4k-5)-4k^2+8k-4}{2k-1} = \frac{3k-4}{2k-1} > 0.\end{aligned}$$

Thus, we get

$$f(h) \geq f(2). \tag{8}$$

From (7), (8) and $k \geq 3$, we obtain

$$\begin{aligned}\delta_G(S,T) &> f(h) \geq f(2)\\ &= k\left(\frac{(k-1)n+2k}{2k-1}-2\right)-(k-2)\left(\frac{k(n-2)}{2k-1}+2\right)\\ &= \frac{k(k-1)n+2k^2-4k^2+2k-k(k-2)n-2k^2+6k-4}{2k-1}\\ &= \frac{kn-4k^2+8k-4}{2k-1} \geq \frac{k(4k-5)-4k^2+8k-4}{2k-1}\\ &= \frac{3k-4}{2k-1} = 1+\frac{k-3}{2k-1} \geq 1.\end{aligned}$$

By the integrity of $\delta_G(S,T)$, we have

$$\delta_G(S,T) \geq 2 \geq \varepsilon(S,T),$$

which contradicts (1).

From all the cases above, we deduced the contradiction. Hence, G is a fractional k-deleted graph. This completes the proof of Theorem 1.3.

REMARK 1. Let us show that the condition $\mathrm{bind}(G) > \frac{(2k-1)(n-1)}{k(n-2)}$ in Theorem 1.3 can not be replaced by $\mathrm{bind}(G) \geq \frac{(2k-1)(n-1)}{k(n-2)}$. Let $r \geq 1, k \geq 3$ be two odd positive integer and let $l = \frac{5kr+1}{2}$ and $m = 5kr - 5r + 1$, so that $n = m + 2l = 10kr - 5r + 2$. Let $H = K_m \bigvee lK_2$ and $X = V(lK_2)$. Then for any $x \in X$, $|N_H(X \setminus x)| = n - 1$. By the definition of $bind(H)$, $bind(H) = \frac{|N_H(X\setminus x)|}{|X\setminus x|} = \frac{n-1}{2l-1} = \frac{n-1}{5kr} = \frac{(2k-1)(n-1)}{k(n-2)}$. Let $S = V(K_m) \subseteq V(H)$, $T = V(lK_2) \subseteq V(H)$. Then $|S| = m$, $|T| = 2l$. Obviously, T is not independent, then $\varepsilon(S,T) = 2$. Thus, we obtain

$$\begin{aligned}\delta_H(S,T) &= k|S| - k|T| + d_{H-S}(T)\\ &= k|S| - k|T| + |T| = k|S| - (k-1)|T|\\ &= km - 2(k-1)l = k(5kr-5r+1) - (k-1)(5kr+1)\\ &= 1 < 2 = \varepsilon(S,T).\end{aligned}$$

By Theorem 1.4, H is not a fractional k-deleted graph. In the above sense, the result in Theorem 1.3 is best possible.

REMARK 2. We don't know whether the result can be strengthened to the form that if $\mathrm{bind}(G) > \frac{(2k-1)(n-1)}{k(n-2)}$ then G is k-deleted. We guess that the above result can hold for kn even.

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