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ON ESSENTIAL AND CONTINUOUS SPECTRA OF THE LINEARIZED WATER-WAVE PROBLEM IN A FINITE POND

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Abstract

We show that the spectrum of the Laplace equation with the Steklov spectral boundary condition, in the connection of the linearized theory of water-waves, can have a nontrivial essential component even in case of a bounded basin with a horizontal water surface. The appearance of the essential spectrum is caused by the boundary irregularities of the type of a rotational cusp or a cuspidal edge. In a previous paper the authors have proven a similar result for the Steklov spectral problem in a bounded domain with a sharp peak.

1. Introduction

1.1. Preamble

Let us consider water-waves in a canal, open sea or other unbounded domain. Within the linear theory, water-waves are described by a mixed boundary value problem for the Laplace equation with the Steklov spectral boundary condition on the horizontal water surface (see [17], [6], [8] for the physical background). It is well-known that the wave propagation phenomenon occurs provided the Steklov spectral parameter λ belongs to the continuous spectrum σ_c of the problem. In an infinite basin Ω this spectrum is not empty and usually includes the positive real axis $\overline{\mathsf{R}}_+$ of the complex plane C. The inclusion $\lambda \in \sigma_c$ frustrates the Fredholm property of the problem operator in the Sobolev space $H^2(\Omega)$, and in order to provide solvability of the problem, one needs to reduce the data space and to impose radiation conditions which distinguish between the waves incoming from and outgoing to the infinity (see the books cited above for the mathematical background).

In this paper we show that the continuous spectrum of the Steklov problem may be nonempty even in a bounded three-dimensional pond. This phenomenon may be due to either a submerged body touching the water surface,

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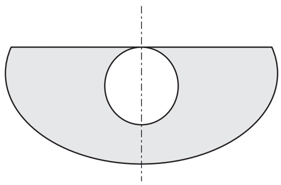

FIGURE 1

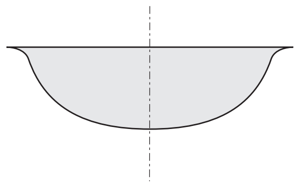

FIGURE 2

or, a sharp edge of the pond (see, respectively, the Figures 1 and 2 with cross-sections in a rotationally symmetric situation).

Our investigation is based on certain tools developed in [15] for the study of the continuous and essential spectra of the Steklov problem in a finite peak shaped domain (see Fig. 3). We emphasize that the result in [15] serves for the two-dimensional variants of the above mentioned problem (erase the dotted line in Fig. 1 and 2 indicating the rotation axis).

Although the results in the present paper are of the same type as in [15], the transition to the new geometrical singularities, namely, the rotational peak in Fig. 4 and the peak edge in Fig. 5, require serious modifications of the tools.

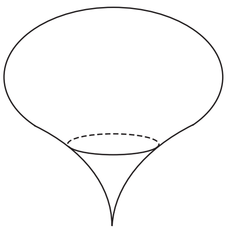

FIGURE 3

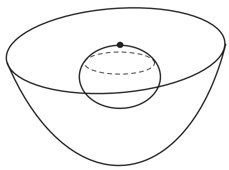

FIGURE 4

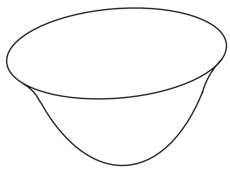

FIGURE 5

Statement of the radiation conditions at the tangency point on Fig. 1 and the edge on Fig. 2 will become the subject of forthcoming papers.

1.2. Statement of the problems

Let Λ be a two-dimensional domain in the horizontal plane $\Pi = \{x : x_3 = 0\}$ of the Euclidean space R^3, and let the smooth, closed simple contour γ be the edge of Λ. Let also Σ be a smooth surface in the lower half-space $\mathsf{R}^3_- = \{x : x_3 < 0\}$ with the same edge γ, and let $\Xi \subset \mathsf{R}^3$ be the domain bounded by $\Lambda \cup \Sigma \cup \gamma$. By Θ we understand a subdomain of Ξ with a smooth two-dimensional boundary Γ; we assume that the origin $\mathscr{O}$ of the Cartesian x-coordinates is the only common point of Γ and $\partial\Xi$ and, moreover, that Γ touches Λ at this point (see Fig. 4 and its two-dimensional version, Fig. 1).

We denote $\mathsf{B}_R := \{y \in \mathsf{R}^2 : |y| < R\}$ and assume that in the cylinder

$\mathsf{B}_R \times (-d, d) \ni \mathscr{O}$, the domain $\Omega = \Xi \setminus \overline{\Theta}$ is determined by the inequalities

$$-h(y) < z < 0 \tag{1}$$

where $y = (y_1, y_2) = (x_1, x_2)$ and $z = x_3$. Let us also define the degenerate ligament $\Upsilon_R := \{x : y \in \mathsf{B}_R, -h(y) < z < 0\}$. In (1), h is a smooth function such that $h(0) = 0$ and $\nabla_y h(y) = 0$, where $\nabla_y = (\partial/\partial y_1, \partial/\partial y_2)$ is the two-dimensional gradient. We assume that

$$|\nabla_y^k h(y) - \nabla_y^k h_0(y)| \leq c_k |y|^{m+1-k}, \qquad k \in \mathsf{N}_0 := \{0, 1, 2, \ldots\}, \tag{2}$$

where $\nabla_y^k h$ is the family of all derivatives of h of order k, and h_0 is a positive homogeneous polynomial of even degree $m \geq 2$:

$$h_0(ty) = t^m h_0(y) \quad \text{for } t \in \mathsf{R}^3, \qquad h_0(y) > 0 \quad \text{for } y \in \mathsf{R}^2 \setminus \{0\}. \tag{3}$$

In the case $m = 2$ the submerged body $\overline{\Theta}$, touching the water surface Λ, may be a ball or an ellipsoid.

In the domain Ω we consider the problem of the linearized theory of water-waves (see [17], [6], [8] and others)

$$-\Delta_x \Phi(x) = 0, \qquad x \in \Omega, \tag{4}$$

$$\partial_n \Phi(x) = 0, \qquad x \in \Sigma \cup (\Gamma \setminus \mathscr{O}), \tag{5}$$

$$\partial_z \Phi(x) = \lambda \Phi(x), \qquad x \in \Lambda \setminus \mathscr{O}, \tag{6}$$

where $\Delta_x = \nabla_x \cdot \nabla_x$ is the Laplacian, ∂_n is the derivative along the outward normal n (so $\partial_n = \partial_z$ on Λ), and Φ denotes the velocity potential and λ the spectral parameter proportional to the square of the frequency of harmonic oscillations. The conditions (6) are called the Steklov spectral boundary conditions.

We also consider the same problem in the entire domain Ξ:

$$-\Delta_x \Phi(x) = 0, \qquad x \in \Xi, \tag{7}$$

$$\partial_n \Phi(x) = 0, \qquad x \in \Sigma, \tag{8}$$

$$\partial_z \Phi(x) = \lambda \Phi(x), \qquad x \in \Lambda. \tag{9}$$

In the d-neighborhood $\mathscr{V}_d \subset \Pi$ of the contour $\gamma \subset \Pi$ we introduce the natural system of the curvilinear coordinates (ν, τ), where ν is the oriented distance to γ, $\nu > 0$ inside Λ, and τ is the curve length on γ. We assume that in the vicinity of $\gamma \subset \mathsf{R}^3$ the domain Ξ is given by the inequalities

$$-\nu^{\mathbf{m}} \mathbf{h}(\nu, \tau) < z < 0, \qquad \nu > 0, \tag{10}$$

where $\mathbf{m} \geq 1$ is a real number and $\mathbf{h}$ a smooth positive function on $[0, d] \times \gamma$ (see Fig. 5 and its two dimensional version, Fig. 2).

The domain Ξ can be interpreted as a lake or pond with water, like, for example, Loch Ness; the set Θ would then describe the body of the monster (see [3]).

In the following we denote by c, c', C etc. positive constants which may vary from place to place and which are independent of the functions or variables in the given expressions.

2. Spectra of the problem: preliminary description of the results

2.1. The discrete spectrum

If $\mathbf{m} = 1$ in (10), the boundary becomes Lipschitz and, owing to the compactness of the embedding $H^1(\Xi) \subset L^2(\Lambda)$ (see, e.g., [10]) and the theory of self-adjoint operators in Hilbert space (see, e.g. [2]), the problem (7)–(9) has the discrete spectrum formed by the eigenvalue sequence

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots \to \infty, \tag{11}$$

where the eigenvalues λ_k are repeated according to their multiplicities. The corresponding eigenfunctions $\Phi_0 = \text{const.}, \Phi_1, \Phi_2, \dots, \Phi_k, \dots$ in the Sobolev space $H^1(\Xi)$ can be subject to the orthogonality and normalization conditions

$$\langle \Phi_k, \Phi_j \rangle_\Xi = \delta_{k,j}, \qquad k, j \in \mathsf{N}_0, \tag{12}$$

where $\delta_{k,j}$ is the Kronecker symbol,

$$\langle \Phi, \Psi \rangle_\Xi = (\nabla_x \Phi, \nabla_x \Psi)_\Xi + (\Phi, \Psi)_\Lambda \tag{13}$$

is a specific inner product in $H^1(\Xi)$, while $(\cdot, \cdot)_\Xi$ and $(\cdot, \cdot)_\Lambda$ are the intrinsic inner produtcs in the Lebesgue spaces $L^2(\Xi)$ and $L^2(\Lambda)$, respectively. Note that $\frac{1}{2}\langle \Phi, \Phi \rangle_\Xi$ is the energy functional for the problem (7)–(9).

We emphasize that in the case when the embedding $H^1(\Xi) \subset L^2(\Lambda)$ fails, the space $\mathscr{H}(\Xi; \Lambda)$ is smaller than $H^1(\Xi)$; here $\mathscr{H}(\Xi; \Lambda)$ is obtained by completing the space $C_c^\infty(\overline{\Xi} \setminus \gamma)$ of compactly supported smooth functions with respect to the norm $\langle \Phi, \Phi \rangle_\Xi^{1/2}$ (see [4]).

2.2. The continuous and essential spectra

If $\mathbf{m} > 1$ (see Fig. 5), the surface Σ touches the plane Π along the contour γ, and the boundary $\partial \Xi$ is not Lipschitz. The same is thue for the domain Ω, where the surfaces Γ and Λ touch each other at $\mathcal{O}$. The goal of this paper is to prove that the spectrum of the problem (4)–(6) in Ω is not discrete, and that the same is true for the problem (7)–(9) in the case

$$\mathbf{m} \geq 2. \tag{14}$$

Similar properties of the continuous and essential spectra were discovered for the Steklov spectral problem in [15] in peak-shaped domains (see Fig. 3). In [14] an algebraic criterion was found for the existence of a nonempty essential spectrum for the Neumann problem of a second order selfadjoint elliptic system on the peak-shaped domains in Fig. 3.

The geometic irregularity γ on the surface $\partial\Xi$ given by the formula (10) with $\mathbf{m} > 1$ can be regarded as a cuspidal edge. The point $\mathcal{O}$ at which the two connected components of the boundary $\partial\Omega$ touch each other, is also of the cuspidal type. Indeed, if the peak on Fig. 4 is to be called a sharpened rod (or an owl), the region between Γ and Λ, desribed by the inequalities (1), may be regarded as a plate with degenerating thickness. We emphasize that elements of the Weyl singular sequences, used in [15] and Section 2 to detect a point of the essential spectrum, are obtained by employing asymptotic ansätze for thin rods and plates (see, e.g., [11], [13]).

2.3. *Formulation of the results on spectra*

In Section 2 we will check the following assertion about the problem (7)–(9).

THEOREM 2.1. *If* $\mathbf{m} \in [1, 2)$*, the spectrum of the problem* (7)–(9) *is discrete. The spectrum cannot be discrete in the case* (14)*, while for* $\mathbf{m} > 2$*, the point* $\lambda = 0$ *belongs to the continuous spectrum.*

It is worth to generalize the geometry of the domain Ω and assume that h_0 in (2) is a positive homogeneous function of degree $m > 1$, i.e., the hypotheses (3) hold true even with not necessarily integer exponents m.

THEOREM 2.2. *Let* $\mathbf{m} = \mathbf{1}$ *in the relation* (10)*. The spectrum of the problem* (4)–(6) *is discrete, if and only if* $m \in (1, 2)$*. In the case* $m > 2$ *the point* $\lambda = 0$ *belongs to the continuous spectrum. In the case* $m = 2$ *there exists a threshold* $\lambda_0 > 0$ *such that the ray* $[\lambda_0, +\infty)$ *belongs to the essential spectrum.*

An equivalent reformulation of the problem (7)–(9) or (4)–(6) as the spectral problem with a positive self-adjoint operator in a Hilbert space is given in Section 3.1 and, hence, the complement of the closed positive real axis $\overline{\mathsf{R}}_+$ in the plane C is clearly free of the spectrum.

The first two assertions of Theorem 2.2 are verified in Section 3 in parallel to Theorem 2.1. The third assertion is much more delicate so that its proof is completed in Section 4 only.

The structure of the whole spectrum of the problem in Ω with $m > 2$ remains as an open question, and the same is true for the problem in Ξ with any $\mathbf{m} \geq 2$.

3. Studying the spectra

3.1. The operator formulation of the problems

In view of the boundary irregularities, the problem (7)–(9) is to be considered as the integral identity (see [9])

$$(\nabla_x \Phi, \nabla_x \Psi)_\Xi = \lambda(\Phi, \Psi)_\Lambda, \qquad \Psi \in \mathscr{H}(\Xi, \Lambda). \tag{15}$$

In the Hilbert space $\mathscr{H}(\Xi, \Lambda) = H^1(\Xi) \cap L^2(\Lambda)$, endowed with the inner product (13), we introduce the symmetric, continuous, and therefore self-adjoint operator K_Ξ by the formula

$$\langle K_\Xi \Phi, \Psi\rangle_\Xi = (\Phi, \Psi)_\Lambda, \qquad \Phi, \Psi \in \mathscr{H}(\Xi, \Lambda). \tag{16}$$

The change of the spectral parameter

$$\lambda \mapsto \mu = (1+\lambda)^{-1} \tag{17}$$

reduces the variational problem (15) to the abstract equation

$$K_\Xi \Phi = \mu \Phi \in \mathscr{H}(\Xi, \Lambda). \tag{18}$$

The operator K_Ξ is positive and has the unit norm while $\mu = 1$ is an eigenvalue with a constant eigenfunction. Hence, its spectrum lies on the closed segment

$$[0, 1] = \{\mu \in \mathbb{C} : \operatorname{Re} \mu \in [0, 1], \operatorname{Im} \mu = 0\} \tag{19}$$

of the complex plane. Furthermore, $\mu = 0$ is an eigenvalue with the infinite-dimensional eigenspace $H^1_0(\Xi; \Lambda) = \{\Phi \in \mathscr{H}(\Xi; \Lambda) : \Phi = 0 \text{ on } \Lambda\} \subset \mathscr{H}(\Xi; \lambda)$.

Changing the entire domain Ξ for the domain Ω, not containing $\overline{\Theta}$, in all above formulae, we obtain the operator K_Ω in the Hilbert space $\mathscr{H}(\Xi; \Lambda)$ which possesses all the properties listed above; we refer to (18) as the corresponding abstact reformulation of the problem (4)–(6).

3.2. Weighted Poincaré inequalities and the discrete spectra

The first assertion in the following lemma contains a weighted inequality in the domain Ω. It has been proved in [16] and the proof is based on a trick proposed in [15]. We present below a detailed proof of the second assertion which gives a similar weighted inequality in the domain Ξ; the proof is a modification of those in [15], [16].

LEMMA 3.1. 1) *Let* $\mathbf{m} = 1$ *in the formula* (10) *for the domains* Ξ *near the irregularity contour* $\gamma \subset \partial\Omega$. *Under the conditions* (1) *and* (2), *the inequality*

$$\| \, |x|^{-1}\Phi; L^2(\Omega)\| + \| \, |x|^{-1+m/2}\Phi; L^2(\partial\Omega)\| \le c\|\Phi; H^1(\Omega)\| \tag{20}$$

is valid and $c > 0$ is independent of the function $\Phi \in H^1(\Omega)$.

2) *Under the condition (10) with the sharpness exponent $\mathbf{m} > 1$, the inequality*

$$\| |\rho|^{-1}\Phi; L^2(\Xi)\| + \| |\rho|^{-1+\mathbf{m}/2}\Phi; L^2(\partial\Xi)\| \le c\|\Phi; H^1(\Xi)\| \tag{21}$$

with $\rho(x) = \operatorname{dist}(x, \gamma)$ is valid, and c is independent of the function $\Phi \in H^1(\Xi)$.

PROOF. Without loss of generality, we assume that the support of the function Φ is included in the cylinder $\Xi_d := \{x = (y, z) : y \in \mathscr{V}_d\}$, where $\mathscr{V}_d$ is as after (9). As regards to (21), we can thus replace $\rho = \rho(x)$ by the equivalent function $\nu = \nu(y)$ in the following arguments. Let us denote by $J(\nu, \tau)$ the Jacobian of the coordinate transform $y \mapsto (\nu, \tau) \in \mathscr{V}_d$. By the smoothness of γ, we have $0 < c \le J(\nu, \tau) \le C$ for some positive constants c, C. With slight abuse of notation we write $\Phi = \Phi(y, z) = \Phi(\nu, \tau, z)$, and we represent Φ in the form

$$\Phi(y, z) = \overline{\Phi}(y) + \Phi_\perp(y, z), \tag{22}$$

where

$$\overline{\Phi}(y) = \int_{-\nu^{\mathbf{m}}\mathbf{h}(\nu,\tau)}^{0} \Phi(y, z)\, dz \qquad \text{and} \qquad \int_{-\nu^{\mathbf{m}}\mathbf{h}(\nu,\tau)}^{0} \Phi_\perp(y, z)\, dz = 0 \tag{23}$$

for almost all $y \in \mathscr{V}_d$.

By this orthogonality condition the Poincaré inequality holds for $\Phi_\perp$ (considered as a function of $z \in (-\nu^{\mathbf{m}}\mathbf{h}(\nu, \tau), 0)$ with fixed (ν, τ)). We denote $\partial_z := \partial/\partial z$ and write

$$\begin{aligned}
&\int_{\Xi_d} |\partial_z \Phi(x)|^2\, dx \\
&\quad = \int_\gamma \int_0^d \int_{-\nu^{\mathbf{m}}\mathbf{h}(\nu,\tau)}^{0} |\partial_z \Phi_\perp(\nu, \tau, z)|^2 J(\nu, \tau)\, dz\, d\nu\, d\tau \\
&\quad \ge \pi^2 \int_\gamma \int_0^d \int_{-\nu^{\mathbf{m}}\mathbf{h}(\nu,\tau)}^{0} (\nu^{\mathbf{m}}\mathbf{h}(\nu, \tau))^{-2} |\Phi_\perp(\nu, \tau, z)|^2 J(\nu, \tau)\, dz\, d\nu\, d\tau \\
&\quad \ge C \int_{\Xi_d} \nu^{-2\mathbf{m}} |\Phi_\perp(x)|^2\, dx, \qquad C > 0. \qquad (24)
\end{aligned}$$

Denoting $\Lambda_d = \mathscr{V}_d \subset \Lambda$ and $\Sigma_d = \{x = (y, z) \in \Sigma : y \in \mathscr{V}_d\}$, we obtain from standard trace inequalities (see, e.g., [9])

$$I_0 := \int_{\Sigma_d \cup \Lambda_d} \nu(y)^{-\mathbf{m}} |\Phi_\perp(x)|^2\, ds_x$$

$$\le c \int_{\Lambda_d} \nu(y)^{-\mathbf{m}} |\Phi_\perp(y, -\nu^{\mathbf{m}}\mathbf{h}(\nu,\tau))|^2\,dy + c\int_{\Lambda_d} \nu(y)^{-\mathbf{m}} |\Phi_\perp(y,0)|^2\,dy$$

$$\le c\int_{\Xi_d} \big(|\partial_z \Phi_\perp(y,z)|^2 + \nu(y)^{-2\mathbf{m}} |\Phi_\perp(y,z)|^2\big)\,dy\,dz; \tag{25}$$

where we used the fact that by our assumptions, the boundary Jacobians on the surfaces Σ_d and Λ_d are bounded from above and below.

By the smoothness of γ, we have $|\nabla_y \Phi(\nu,\tau,z)| \ge C|\partial_\nu \Phi(\nu,\tau,z)|$ for all ν, τ, z. Hence,

$$\begin{aligned}
c\int_{\Xi_d} &|\nabla_x \Phi(x)|^2\,dx \\
&\ge \int_{\Xi_d} |\partial_\nu \Phi(x)|^2\,dx \\
&= \int_\gamma \int_0^d \nu^{\mathbf{m}}\mathbf{h}(\nu,\tau) |\partial_\nu \overline{\Phi}(\nu,\tau)|^2 J(\nu,\tau)\,d\nu\,d\tau \\
&\qquad + \int_\gamma\int_0^d \int_{-\nu^{\mathbf{m}}\mathbf{h}(\nu,\tau)}^0 |\partial_\nu \Phi_\perp(\nu,\tau,z)|^2 J(\nu,\tau)\,dz\,d\nu\,d\tau \\
&\qquad + 2\int_\gamma\int_0^d \partial_\nu\overline{\Phi}(\nu,\tau) \int_{-\nu^{\mathbf{m}}\mathbf{h}(\nu,\tau)}^0 \partial_\nu \Phi_\perp(\nu,\tau,z) J(\nu,\tau)\,dz\,d\nu\,d\tau \\
&=: I_1 + I_2 + I_3.
\end{aligned} \tag{26}$$

To evaluate I_1 we recall the one dimensional Hardy inequality (see [5]),

$$\int_0^R r^{\alpha-1} |U(r)|^2\,dr \le \frac{4}{\alpha^2}\int_0^R r^{\alpha+1} |\partial_r U(r)|^2\,dr, \tag{27}$$

which is valid, if $U(R) = 0$. We use it for the ν-integration with $\alpha = \mathbf{m} - 1 > 0$: recalling that J is bounded from above and below, we get

$$\begin{aligned}
I_1 &\ge c\int_\gamma\int_0^d \nu^{-2+\mathbf{m}} |\overline{\Phi}(\nu,\tau)|^2 J(\nu,\tau)\,d\nu\,d\tau \\
&\ge c\int_{\Xi_d} \nu^{-2} |\overline{\Phi}(y)|^2\,dx, \qquad c > 0.
\end{aligned} \tag{28}$$

For the analysis of the integral I_3 we make use of the rule of differentation of integrals with variable limits and obtain

$$\begin{aligned}
&\left|\partial_\nu \int_{-\nu^{\mathbf{m}}\mathbf{h}(\nu,\tau)}^{0} \Phi_\perp(\nu,\tau,z)\,dz - \int_{-\nu^{\mathbf{m}}\mathbf{h}(\nu,\tau)}^{0} \partial_\nu \Phi_\perp(\nu,\tau,z)\,dz\right| \\
&\qquad \le \partial_\nu(\nu^{\mathbf{m}}\mathbf{h}(\nu,\tau))|\Phi_\perp(\nu,\tau,-\nu^{\mathbf{m}}\mathbf{h}(\nu,\tau))| \\
&\qquad \le C|\Phi_\perp(\nu,\tau,-\nu^{\mathbf{m}}\mathbf{h}(\nu,\tau))| \\
&\qquad \le C|\Phi_\perp(\nu,\tau,-\nu^{\mathbf{m}}\mathbf{h}(\nu,\tau))| + C|\Phi_\perp(\nu,\tau,0)|
\end{aligned}$$

for almost all (ν, τ). Since the first integral on the left-hand side is null by (23), we have

$$\begin{aligned}
|I_3| &\le C \int_\gamma \int_0^d |\partial_\nu \overline{\Phi}(\nu,\tau)|\big(|\Phi_\perp(\nu,\tau,-\nu^{\mathbf{m}}\mathbf{h}(\nu,\tau))| \\
&\qquad + |\Phi_\perp(\nu,\tau,0)|\big) J(\nu,\tau)\,d\nu\,d\tau \\
&\le C\Big(\int_\gamma \int_0^d \nu^{\mathbf{m}} |\partial_\nu \overline{\Phi}(\nu,\tau)|^2 J(\nu,\tau)\,d\nu\,d\tau\Big)^{1/2} \\
&\qquad \cdot \Big(\int_{\Sigma_d \cup \Lambda_d} \nu(y)^{-\mathbf{m}} |\Phi_\perp(x)|^2\,ds_x\Big)^{1/2} \\
&\le \varepsilon I_1 + c\varepsilon^{-1} I_0, \qquad (29)
\end{aligned}$$

with an arbitrary $\varepsilon > 0$. Assume that $\varepsilon = 1/4$; then from (26), (25) and (29) it follows that

$$I_j \le c\|\nabla_x \Phi; L^2(\Xi_d)\| \qquad \text{for} \quad j = 1, 2. \qquad (30)$$

As a conclusion, the inequality

$$\|\,|\rho|^{-1+\mathbf{m}/2}\Phi; L^2(\partial\Xi)\| \le c\|\,|\nu|^{-1+\mathbf{m}/2}\Phi; L^2(\partial\Xi)\| \le c'\|\Phi; H^1(\Xi)\|$$

follows for the $\overline{\Phi}$-component from the first inequality of (28) and (30), and for the $\Phi_\perp$-component by combining (25) and (24).

The inequality

$$\|\,|\rho|^{-1}\Phi; L^2(\Xi)\| \le c\|\,|\nu|^{-1}\Phi; L^2(\Xi)\| \le c'\|\Phi; H^1(\Xi)\|$$

follows for the $\overline{\Phi}$-component from the latter inequality (28), and for the $\Phi_\perp$-component by (24).

Finally, for general functions Φ, not satisfying the support condition in the beginning of the proof, one applies standard trace inequalities outside the sets Ξ_d and Λ_d.

REMARK 3.2. From Lemma 3.1 it follows that $\mathcal{H}(\Omega; \Lambda) = H^1(\Omega)$ only in the case $m \in (1, 2]$, while for $m \geq 2$ the relation $\mathcal{H}(\Omega; \Lambda) \subsetneqq H^1(\Omega)$ will be proven in Section 3.4. In the same way, $\mathcal{H}(\Xi; \Lambda) = H^1(\Xi)$ for $\mathbf{m} \in [1, 2]$, but, as shown in Section 3.4, $\mathcal{H}(\Xi; \Lambda) \subsetneqq H^1(\Xi)$ in the case $\mathbf{m} > 2$.

Let us assume that $\mathbf{m} \in (1, 2)$. The embedding $H^1(\Xi) \subset L^2(\Lambda)$ becomes compact because for any $\varepsilon > 0$ the embedding $H^1(\Xi \setminus \mathcal{U}_\varepsilon) \subset L^2(\Lambda \setminus \mathcal{U}_\varepsilon)$ is compact and, thanks to the weight multiplier $\rho^{\mathbf{m}-2}$ on the left hand side of (21), the norm of the trace operator $H^1(\Xi \cap \mathcal{U}_\varepsilon) \subset L^2(\Lambda \cap \mathcal{U}_\varepsilon)$ does not exceed $c\varepsilon^{2-\mathbf{m}}$. Here $\mathcal{U}_\varepsilon := \{x : \rho(x) < \varepsilon\}$ is the ε-neighborhood of the contour $\gamma \subset \mathsf{R}^3$ so that the domain $\Xi \setminus \overline{\mathcal{U}_\varepsilon}$ has a Lipschitz boundary. Since the norm of the above mentioned trace operator can be made arbitrarily small using a small enough ε, the above observation shows that the operator K_Ξ is compact. Hence, for $\mathbf{m} \in (0, 1)$ its spectrum is discrete on the half-open interval $(0, 1]$ and has the accumulation point $\mu = 0$:

$$1 = \mu_0 > \mu_1 \geq \mu_2 \geq \cdots \geq \mu_k \geq \cdots \to +0. \tag{31}$$

The inverse transformation (17) turns the eigenvalue sequence (31) into the sequence (11) and, thus, the spectrum of the problem (7)–(9) is discrete. Note that $\mu = 0$ transforms into $\lambda = \infty$ which does not influence the spectrum of the problem.

If one assumes that h_0 in (2) is a positive homogeneous function of degree $m \in (1, 2)$, i.e. the hypotheses (3) hold true, then the spectrum of the operator K_Ω consists of the point $\mu = 0$ in the essential spectrum and of the eigenvalues (31). Moreover, the problem (4)–(6) has the discrete spectrum (11) with the only accumulation point at the infinity.

3.3. The essential spectrum of the problem in Ξ

Let us consider the functions

$$\Psi_j(x) = a_j \psi(2^j d^{-1} \nu), \qquad j \in \mathsf{N}_0, \tag{32}$$

where a_j is a normalization factor, $\psi \in C^\infty(\mathsf{R})$, $\psi(t) > 0$ for $t \in (1, 2)$ and $\psi(t) = 0$ for $t \notin (1, 2)$. By definition, the support of Ψ_j is contained in $\overline{\Xi}_d$. We have

$$\|\Psi_j; L^2(\Lambda)\|^2 = a_j^2 \int_\gamma \int_{2^{-j}d}^{2^{1-j}d} |\psi(2^j d^{-1}\gamma)|^2 J(\nu, \tau)\, d\nu\, d\tau \sim a_j^2 2^{-j}, \tag{33}$$

where $J(\nu, \tau)$ is as in the proof of Lemma 3.1 and the notation $A \sim B$ means that there exist positive constants c and C such that $cA \leq B \leq CA$. A similar calculation, using (10) and (32), shows that

$$\begin{aligned}(34)\quad &\|\nabla_x \Psi_j; L^2(\Xi)\|^2 \\ &= \int_\gamma \int_{2^{-j}d}^{2^{1-j}d} \int_{-\nu^{\mathbf{m}}\mathbf{h}(\nu,\tau)}^{0} |\nabla_x \Psi_j(x)|^2 J(\nu,\tau)\, dz\, d\nu\, d\tau \sim a_j^2 2^{-j} 2^{-(\mathbf{m}-2)j}.\end{aligned}$$

Let $a_j = 2^{-j/2}$. Then, in the case $\mathbf{m} \geq 2$ the norms $\|\Psi_j; \mathcal{H}(\Xi; \Lambda)\|^2$ are uniformly bounded in $j \in \mathsf{N}_0$, by virtue of (33) and (34). Since the supports of the functions Ψ_j and Ψ_k with $j \neq k$ are disjoint, the sequence (Ψ_j) converges weakly in $\mathcal{H}(\Xi; \Lambda)$ to the null function as $j \to +\infty$. Moreover, $\|\Psi_j; \mathcal{H}(\Xi; \Lambda)\| \geq c > 0$ and owing to (34) and (32), (13), we have

$$\begin{aligned}\|K_\Xi \Psi_j - \Psi_j; \mathcal{H}(\Xi; \Lambda)\| &= \sup_{\cdots} |\langle K_\Xi \Psi_j - \Psi_j, \Psi\rangle_\Xi| \\ &= \sup_{\cdots} |(\nabla_x \Psi_j, \nabla_x \Psi)_\Xi| \qquad (35) \\ &\leq C 2^{-(\mathbf{m}-2)j/2},\end{aligned}$$

where the dots stand for "$\Psi \in \mathcal{H}(\Xi; \Lambda) : \|\Psi; \mathcal{H}(\Xi; \Lambda)\| = 1$".

In the case $\mathbf{m} \geq 2$ the norms (35) form an infinitesimal sequence so that (Ψ_j) is a Weyl singular sequence for the operator K_Ξ at the point $\mu = 1$. Hence, this point belongs to the essential spectrum (cf. Theorem 9.1.2. of [2]). Since $\mu = 1$ is an eigenvalue of finite multiplicity (actually a simple eigenvalue), this is but a point of the continuous spectrum.

In the case $\mathbf{m} = 2$ the relations (33) and (34) indicate only that the embedding $\mathcal{H}(\Xi; \Lambda) \subset L^2(\Lambda)$ is not compact. Hence, Theorem 9.2.1 of [2] warrants a non-empty essential spectrum on the half-open interval (0, 1].

The above observations and the relation (17) of the spectral parameters λ and μ furnish the proof of Theorem 2.1. The formulas (33) and (34) prove the statement in Remark 3.2 as well.

3.4. The essential spectrum of the problem in Ω

Repeating the calculations (33)–(35) for the operator K_Ω of the problem (4)–(6) and the functions

$$(36)\qquad \Psi_j(x) = a_j \psi(2^j R^{-1} r), \qquad j \in \mathsf{N}_0,$$

where (r, θ) are the polar coordinates on the y-plane and $R > 0$ is a small radius, we obtain the relations

(37)
$$
\begin{aligned}
\|\Psi_j; L^2(\Lambda)\|^2 &= a_j^2 \int_0^{2\pi} \int_{2^{-j}R}^{2^{1-j}R} |\psi(2^j R^{-1} r)|^2 r\, dr\, d\theta \\
&\sim a_j^2 2^{-2j} \\
\|\nabla_x \Psi_j; L^2(\Omega)\|^2 &= a_j^2 \int_0^{2\pi} \int_{2^{-j}R}^{2^{1-j}R} \int_{-h(y)}^{0} |\nabla_x \psi(2^j R^{-1} r)|^2 r\, dz\, dr\, d\theta \\
&\sim a_j^2 2^{-2j} 2^{-(m-2)j},
\end{aligned}
$$
$$
\|K_\Omega \Psi_j - \Psi_j; H^1(\Omega)\| \le C a_j 2^{-m_j/2}.
$$

Putting $a_j = 2^j$ provides the same inferences on the spectrum of the operator K_Ω as those for the operator K_Ξ in the previous subsection. The relations (37) also imply that the Hilbert space $\mathscr{H}(\Omega; \Lambda)$ is smaller than the Sobolev space $H^1(\Omega)$ (cf. Remark 3.2).

4. The case $m = 2$

4.1. The model equation

To make further conclusions on the structure of the spectrum in the case $m = 2$, we need a much more delicate construction of the Weyl sequence. In [15], [13], [16] the authors used the asymptotic theory of elliptic problems in thin domains to derive the following model differential equation on the punctured plane. It describes the behaviour of the solutions of the Steklov problem near cuspidal singularities of the boundary:

$$
-\nabla_y \cdot h_0(y) \nabla_y U(y) = \lambda U(y), \qquad y \in \mathsf{R}^2 \setminus \{0\}. \tag{38}
$$

We are going to use special solutions of this equation to create entries of the Weyl sequence.

By a power-law solution of the differential equation (38) we understand a solution of the form

$$
U(y) = r^{\sigma - 1} u(\theta), \tag{39}
$$

where σ is a complex number and u a smooth function on the unit circle S^1.

4.2. The auxiliary spectral problem

We set $h_0(y) = r^2 H_0(\theta)$, where $H_0 \in C^\infty(\mathsf{S}^1)$ and $H_0 > 0$. Taking into account the formula in the polar coordinates (r, θ)

$$
\nabla_y \cdot h_0(y) \nabla_y = r^{-1} \partial_r r^3 H_0(\theta) \partial_r + \partial_\theta H_0(\theta) \partial_\theta, \tag{40}
$$

we insert (39) into (38) and after separation of variables we arrive at the spectral problem for an elliptic ordinary differential equation on the unit circle

$$-\partial_\theta H_0(\theta)\partial_\theta u(\theta) - (\sigma^2 - 1)H_0(\theta)u(\theta) = \lambda u(\theta), \qquad \theta \in \mathsf{S}^1. \tag{41}$$

Since the left hand side contains a square of the spectral parameter, the equation (41) ought to be treated as a polynomial (quadratic) pencil. It is known (cf. [1]) that the spectrum of (41) consists of normal eigenvalues in the union of a double angle and a strip

$$\{\sigma \in \mathsf{C} : |\operatorname{Im}\lambda| \le k_\lambda |\operatorname{Re}\lambda|\} \cup \{\sigma \in \mathsf{C} : |\operatorname{Im}\lambda| \le l_\lambda\} \tag{42}$$

without finite accumulation points; here $k_\lambda > 0$ and $l_\lambda > 0$ are certain numbers depending on H_0 and λ. Since the equation with $\sigma, \lambda \in \mathsf{R}$ is formally self-adjoint and the coefficients are real, the spectrum has the central symmetry with the origin $\sigma = 0$ and the mirror symmetry with respect to the real axis.

The general information given above, which is valid even for systems of differential equations, can be applied to the scalar case.

In the complex Lebesgue space $L^2(\mathsf{S}^1)$ we introduce the unbounded operator $A(\tau)$ with the differential expression $-\partial_\theta H_0(\theta)\partial_\theta - \tau H_0(\theta)$ and the domain $H^2(\mathsf{S}^1)$; here τ is a real parameter. Clearly, the operator $A(\tau)$ is self-adjoint and semi-bounded from below. Moreover, the compactness of the embedding $H^2(\mathsf{S}^1) \subset L^2(\mathsf{S}^1)$ and Theorem 10.1.5, [2], prove the spectrum of $A(\tau)$ to be discrete and bounded from below. Let $\lambda_0(\tau)$ be the first (smallest) eigenvalue which is simple due to the maximum principle. The evident inequality

$$(A(\tau_1)v, v)_{\mathsf{S}^1} \le (A(\tau_2)v, v)_{\mathsf{S}^1} \qquad \text{for} \quad \tau_1 \ge \tau_2$$

implies the relation $A(\tau_1) < A(\tau_2)$ while, by Theorem 10.2.4. of [2], the function $\tau \mapsto \lambda_0(\tau)$ is strictly monotone decreasing and continuous. The last fact follows e.g. from the max-min principle for eigenvalues. We emphasize that by the same references, all ordered eigenvalues in the operator family $\{A(\tau)\}$ have the monotonicity and continuity properties.

In view of the formula for the roots of the quadratic equation $\sigma^2 - 1 = \tau(\lambda)$, the following assertion has become evident ("i" stands for the imaginary unit).

LEMMA 4.1. 1) *If* $\lambda < \lambda_0 := \lambda_0(-1)$, *then the eigenvales of the problem* (41) *take the form* $\sigma_k^\pm(\lambda) = \pm\sigma_k(\lambda)$, *where* $k \in \mathsf{N}$ *and* $(\sigma_k(\lambda))_{k\in\mathsf{N}}$ *is a positive, increasing, unbounded sequence. The corresponding eigenfunctions* $u_k^\pm(\cdot;\lambda) = u_k(\cdot;\lambda)$ *can be subject to the orthogonality and normalization conditions*

$$(H_0 u_j, u_k)_{\mathsf{S}^1} = \delta_{j,k} \tag{43}$$

where $j, k \in \mathsf{N}$.

2) *If* $\lambda \geq \lambda_0$, *then in addition to the infinite positive and negative sequences* $(\sigma_k^{\pm}(\lambda))_{k\in\mathbb{N}} = (\pm\sigma_k(\lambda))_{k\in\mathbb{N}}$ *of eigenvalues, the problem* (41) *has a finite number of eigenvalues* $\sigma_{-j}^{\pm}(\lambda) = \pm i\sigma_{-j}(\lambda)$, $j = 0, \ldots, J(\lambda) - 1$, *on the imaginary axis. The corresponding eigenfucntions* $u_p^{\pm}(\cdot;\lambda) = u_p(\cdot;\lambda)$ *can be subject to the orthogonality and normalization conditions* (43) *where* j, $k \in \{-J(\lambda)+1, -J(\lambda)+2, \ldots\}$.

3) *The eigenvalues* $\lambda_0^{(p)} = \lambda_p(-1)$ *of the operator* $A(-1)$ *form the sequence of thresholds*

$$\lambda_0^{(0)} := \lambda_0 < \lambda_0^{(1)} \leq \cdots \leq \lambda_0^{(p)} \leq \cdots \to +\infty. \tag{44}$$

If $\lambda = \lambda_0^{(p)}$, *then the point* $\sigma = 0$ *is an eigenvalue of the problem (41), the algebraic multiplicity of which is equal to* 2 *and the geometric multiplicity is either* 1 *or* 2. *The corresponding eigenfunction* $u(\cdot;\lambda_0^{(p)})$ *has the associated eigenfunction* $u'(\cdot;\lambda_0^{(p)}) = 0$, *and in addition to the power-law solution* $r^{-1}u(\theta;\lambda_0^{(p)})$ *the equation* (38) *gets the power-logarithmic solution* $r^{-1}\ln r u(\theta;\lambda_0^{(p)})$.

All other eigenvalues in the case 2) *and all the eigenvalues* $\sigma_k^{\pm}(\lambda)$ *in the case* 1) *are algebraically simple.*

EXAMPLE 4.2. If the body Θ is rotationally symmetric, then $H_0(\theta) = H_0 = \text{const} > 0$, and the equation (41) gets the explicit solution

$$u(\theta) = \exp(\pm iq\theta), \qquad \sigma = \pm(1 + q^2 - H_0^{-1}\lambda)^{1/2},$$

where $q \in \mathbb{N}_0$. We see that $\lambda_0^{(p)} = H_0(1 + p^2)$ in (44). Moreover, all eigenvalues, except the one corresponding to $q = 0$, are of the geometrical multiplicity 2.

4.3. Construction of the Weyl sequence

According to Lemma 4.1, the equation (38) admits in the case $\lambda \geq \lambda_0$ a power law solution (39) with the exponent $\sigma = \pm i|\sigma|$. We consider the functions

$$\Psi_j(x) = a_j X_j(-\ln r)U(y), \tag{45}$$

where a_j is a normalization factor,

$$X_j(t) = \chi(t - 2^j)\chi(2^{j+1} - t), \tag{46}$$

$\chi \in C^{\infty}(\mathbb{R})$ is a cut-off function, $\chi(t) = 1$ for $t \geq 1$ and $\chi(t) = 0$ for $t \leq 0$ while the index j is a large positive integer such that the support of the function (45) lies inside the region given by the inequalities (1). The graph of the function (46) is given in Fig. 6. More precisely, $\operatorname{supp}(\Psi_j)$ is located in the curved closed annulus $\mathscr{A}^0_{2^j}$ with the base $\mathscr{B}^0_{2^j}$ while $X_j(-\ln r) = 1$ for $x \in \mathscr{A}^1_{2^j}$; here

$$\mathscr{A}^\delta_T = \{x = (y, z) : T + \delta \le |\ln r| \le 2T - \delta, 0 \ge z \ge -h(y)\},$$
$$\mathscr{B}^\delta_T = \{y \in \mathsf{R}^2 : T + \delta \le |\ln r| \le 2T - \delta\}.$$

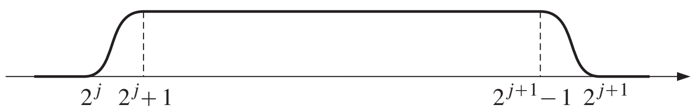

FIGURE 6

Repeating the calculations (33) and (34) for functions (45), we obtain

$$\begin{aligned}\|\Psi_j; L^2(\Lambda)\|^2 &\le a_j^2 \int_{\mathscr{B}^0_{2^j}} r^2 |u(\theta)|^2\, dy \le C a_j^2 \int_{\exp(-2^{1+j})}^{\exp(-2^j)} r^{-2} r\, dr \\ &= C a_j^2 (2^{j+1} - 2^j) = C a_j^2 2^j,\end{aligned}$$

$$\|\Psi_j; L^2(\Lambda)\|^2 \ge a_j^2 \int_{\mathscr{B}^1_{2^j}} r^2 |u(\theta)|^2\, dy \ge c a_j^2 2^j, \qquad c > 0,$$

$$\begin{aligned}\|\nabla_x \Psi_j; L^2(\Omega)\|^2 &\le a_j^2 \int_{\mathscr{A}^0_{2^j}} r^{-4} (|u(\theta)|^2 + |\partial_\theta u(\theta)|^2)\, dy\, dz \\ &\le C a_j^2 \int_{\exp(-2^{1+j})}^{\exp(-2^j)} r^{-4} r^2 r\, dr = C a_j^2 2^j. \end{aligned} \tag{47}$$

Hence, putting $a_j = 2^{-j/2}$ leads to the relation

$$0 < c \le \|\Psi_j; \mathscr{H}(\Omega; \Lambda)\| \le C, \quad \Psi_j \to 0 \text{ weakly in } \mathscr{H}(\Omega; \Lambda) \text{ as } j \to +\infty. \tag{48}$$

Let us process the norm

$$\begin{aligned}&\|K_\Omega \Psi_j + (1+\lambda)^{-1} \Psi_j; \mathscr{H}(\Omega; \Lambda)\| \\ &\qquad = \sup_{\cdots} |\langle K_\Omega \Psi_j - (1+\lambda)^{-1} \Psi_j, \Psi\rangle_\Omega| \\ &\qquad = (1+\lambda)^{-1} \sup_{\cdots} |(\nabla_x \Psi_j, \nabla_x \Psi)_\Omega - \lambda(\Psi_j, \Psi)_\Lambda|\end{aligned} \tag{49}$$

Here the inner product in $H^1(\Omega)$ is as in (13) and the operator K_Ω is as in (16), and the dots again stand for "$\Psi \in \mathscr{H}(\Omega; \Lambda) : \|\Psi; \mathscr{H}(\Omega; \Lambda)\| = 1$".

We set

$$\Psi(y,z)=\overline{\Psi}(y)+\Psi_\perp(y,z),\qquad \overline{\Psi}(y)=h(y)^{-1}\int_{-h(y)}^{0}\Psi(y,z)\,dz, \tag{50}$$

and recall the notation B_R from Section 1.2.

LEMMA 4.3. *The following inequalities hold true:*

$$\|rh^{-1/2}(\nabla_y\overline{\Psi}-\overline{\nabla_y\Psi});L^2(\mathsf{B}_R)\|\le c\|\partial_z\Psi;L^2(\Xi_R)\|, \tag{51}$$

$$\|h^{-1/2}(\overline{\Psi}-\Psi(\cdot,0));L^2(\mathsf{B}_R)\|\le c\|\partial_z\Psi;L^2(\Xi_R)\|, \tag{52}$$

where $\Psi(\cdot,0)$ is the trace of Ψ on the surface Λ and $\Xi_R=\{x:y\in\mathsf{B}_R,-h(y)<z<0\}$ (cf. (1)).

PROOF. The Newton-Leibniz formula yields

$$\Psi(y,\zeta)-\Psi(y,-h(y))=\int_{-h(y)}^{\zeta}\partial_z\Psi(y,z)\,dz$$

so that, integrating over $\zeta\in(-h(y),0)$, we get

$$\begin{aligned}\overline{\Psi}(y)-\Psi(y,-h(y))&=\frac{1}{h(y)}\int_{-h(y)}^{0}\int_{-h(y)}^{\zeta}\partial_z\Psi(y,z)\,dz\,d\zeta\\&=\frac{1}{h(y)}\int_{-h(y)}^{0}z\partial_z\Psi(y,z)\,dz.\end{aligned}$$

Thus,

$$\begin{aligned}&\int_{\mathsf{B}_R}h(y)^{-1}|\overline{\Psi}(y)-\Psi(y,-h(y))|^2\,dy\\&\qquad\le\int_{\mathsf{B}_R}h(y)^{-1}\left|\int_{-h(y)}^{0}\frac{z}{h(y)}\partial_z\Psi(y,z)\,dz\right|^2dy\\&\qquad\le\int_{\mathsf{B}_R}\int_{-h(y)}^{0}|\partial_z\Psi(y,z)|^2\,dy\,dz.\end{aligned}\tag{53}$$

An evident modification of the calculation yields the inequality (52).

Using (50), we continue by writing

$$\nabla_y\overline{\Psi}(y)-\overline{\nabla_y\Psi(y)}=-h(y)^{-1}\nabla_yh(y)(\overline{\Psi}(y)-\Psi(y,h(y))).$$

Notice that $|h(y)^{-1}\nabla_y h(y)| \le cr^{-1}$ by virtue of (2) and (3). Applying (53), we now obtain

$$\begin{aligned}\int_{\mathsf{B}_R} & r^2 h(y)^{-1} |\nabla_y \overline{\Psi}(y) - \overline{\nabla_y \Psi(y)}|^2\,dy \\ &\le C \int_{\mathsf{B}_R} h(y)^{-1} |\overline{\Psi}(y) - \Psi(y, -h(y))|^2\,dy \\ &\le C\|\partial_z \Psi; L^2(\Upsilon_R)\|^2.\end{aligned}$$

Since the function (45) does not depend on z, we have

$$\begin{aligned}(\nabla_x & \Psi_j, \nabla_x \Psi)_\Omega - \lambda(\Psi_j, \Psi(\cdot, 0))_\Lambda \\ &= (h\nabla_y \Psi_j, \overline{\nabla_y \Psi})_{\mathsf{B}_R} - \lambda(\Psi_j, \Psi(\cdot, 0))_{\mathsf{B}_R} \\ &= ((h_0 \nabla_y \Psi_j, \nabla_y \overline{\Psi})_{\mathsf{B}_R} - \lambda(\Psi_j, \overline{\Psi})_{\mathsf{B}_R}) + ((h - h_0)\nabla_y \Psi_j, \nabla_y \overline{\Psi})_{\mathsf{B}_R} \\ &\qquad + (h\nabla_y \Psi_j, \overline{\nabla_y \Psi} - \nabla_y \overline{\Psi})_{\mathsf{B}_R} - \lambda(\Psi_j, \Psi(\cdot, 0) - \overline{\Psi})_{\mathsf{B}_R} \\ &=: I_1 + I_2 + I_3 - I_4. \qquad (54)\end{aligned}$$

Estimating the addenda in (54), we start with I_4. Using the relation $h(y) = \mathcal{O}(r^2)$ (see (2) with $m = 2$) and the first inequality

$$|\Psi_j(x)| \le c2^{-j/2} r^{-1}, \qquad |\nabla_x \Psi_j(x)| \le c2^{-j/2} r^{-2}, \qquad (55)$$

(which is a consequence of the definitions of Ψ_j, X_j and a_j, by virtue of the estimate (52)), we obtain

$$\begin{aligned}|I_4| &\le ca_j \int_{\mathscr{B}^0_{2j}} r^{-1} |\Psi(y, 0) - \overline{\Psi}(y)|\,dy \\ &\le c2^{-j/2} (\operatorname{mes}_2 \mathscr{B}^0_{2j})^{1/2} \|h^{-1/2}(\overline{\Psi} - \Psi(\cdot, 0)); L^2(\mathsf{B}_R)\| \\ &\le c2^{-j/2} \exp(-2^j) \|\nabla_x \Psi; L^2(\mathsf{B}_R)\| \le c2^{-j/2} \exp(-2^j). \qquad (56)\end{aligned}$$

In view fo the second inequality (55) we get

$$\begin{aligned}|I_3| &\le ca_j \int_{\mathscr{B}^0_{2j}} h(y) r^{-2} |\nabla_y \overline{\Psi}(y) - \overline{\nabla \Psi}(y)|\,dy \\ &\le ca_j \int_{\mathscr{B}^0_{2j}} r h(y)^{-1/2} |\nabla_y \overline{\Psi}(y) - \overline{\nabla \Psi}(y)|\,dy \\ &\le c2^{-j/2} \exp(-2^j).\end{aligned}$$

To deal with I_2, we recall that, first, $|h(y) - h_0(y)| \le cr^3$ due to (2) with $m = 2$, and, second, the definition (50) and the relation (51) yield

$$\begin{aligned}
&\|h^{1/2}\nabla_y \overline{\Psi}; L^2(\mathsf{B}_R)\|^2 \\
&\quad\le c\|h^{1/2}\overline{\nabla_y \Psi}; L^2(\mathsf{B}_R)\|^2 + c\|\partial_z \Psi; L^2(\Upsilon_R)\|^2 \\
&\quad\le c\int_{\mathsf{B}_R} h(y)^{-1} \left| \int_{-h(y)}^0 \nabla_y \Psi(y,z)\,dz \right|^2 dy + c\|\partial_z \Psi; L^2(\Upsilon_R)\|^2 \\
&\quad\le c\|\nabla_x \Psi; L^2(\Upsilon_R)\|^2.
\end{aligned}$$

We now have

$$\begin{aligned}
|I_2| &\le ca_j \int_{\mathscr{B}^0_{2^j}} r^3 r^{-2} |\nabla_y \overline{\Psi}(y)|\,dy \\
&\le ca_j (\operatorname{mes}_2 \mathscr{B}^0_{2^j})^{1/2} \|h^{1/2}\nabla_y \overline{\Psi}; L^2(\mathsf{B}_R)\| \\
&\le c2^{-j/2} \exp(-2^j). \qquad (57)
\end{aligned}$$

It suffices to examine the term I_1. Integrating by parts it transforms into

$$I_1 = (-\nabla_y \cdot h_0 \nabla_y \Psi_j - \lambda \Psi_j, \overline{\Psi})_{\mathsf{B}_r}. \qquad (58)$$

Since U in (45) solves the equation (38), the function in the first position in the inner product (58) differs from zero only on the set $\mathscr{B}^0_{2^j} \setminus \mathscr{B}^1_{2^j}$. Moreover, performing the differentiation we arrive at

$$\nabla_y \cdot h_0 \nabla_y \Psi_j + \lambda \Psi_j = h_0 U \Delta_y X_j + 2h_0 \nabla_y X_j \cdot \nabla_y U + U \nabla_y h_0 \cdot \nabla_y X_j.$$

By (46), the estimates

$$|\nabla_y^k X_j(-\ln r)| \le c_k r^{-k}, \qquad k \in \mathsf{N}_0,$$

are valid, and, therefore,

$$\begin{aligned}
|I_1| &\le ca_j \int_{\mathscr{B}^0_{2^j} \setminus \mathscr{B}^1_{2^j}} r^{-1} |\overline{\Psi}(y)|\,dy \\
&\le c2^{-j/2} \Big(\int_{\mathscr{B}^0_{2^j} \setminus \mathscr{B}^1_{2^j}} r^{-2} dy \Big)^{1/2} \|\overline{\Psi}; L^2(\mathsf{B}_R)\|. \qquad (59)
\end{aligned}$$

According to (52) and the definition (13) of the inner product in $\mathscr{H}(\Omega; \Lambda) = H^1(\Omega)$ (cf. Remark 3.2), we see that the last norm in (59) does not exceed

$$c\|\Psi(\cdot, 0); L^2(\mathsf{B}_R)\| + c\|\partial_z \Psi; L^2(\Xi_R)\| \le c\|\Psi; H^1(\Omega)\| \le c.$$

In view of the calculation

$$\int_{\mathscr{B}^0_{2^j}\setminus\mathscr{B}^1_{2^j}} r^{-2}\,dy = 2\pi\left(\int_{\exp(-2^j)}^{\exp(1-2^j)}\frac{dr}{r}+\int_{\exp(-2^{j+1}-1)}^{\exp(-2^{j+1})}\frac{dr}{r}\right)$$
$$= 2\pi(2^j-(2^j-1)+(2^{j+1}+1)-2^{j+1}) = 4\pi,$$

we find that I_1 is of the infinitesimal magnitude $\mathscr{O}(2^{-j/2})$ as $j\to+\infty$. Since we have verified above that $I_p = \mathscr{O}(2^{-j/2}\exp(-2^j))$ for $p = 2, 3, 4$, we conlude that the sequence (Ψ_j) is singular for the operator K_Ω at the point

$$\mu = (1+\lambda)^{-1} \geq (1+\lambda_0)^{-1},$$

because of the relations (48) and

$$\|K_\Omega\Psi_j - (1+\lambda)^{-1}\Psi_j;\, H^1(\Omega)\| \to 0 \qquad \text{as} \quad j\to+\infty.$$

This completes the proof of the second assertion in Theorem 2.2.

REFERENCES

1. Agranovich, M. S., Vishik, M. I., *Elliptic problems with a parameter and parabolic problems of general form*, Uspehi Mat. Nauk 19:3 (1964), 53–160. (English transl.: Russian Math. Surveys 19:3 (1964), 53–157.)
2. Birman, M. Sh., Solomyak, M. Z., *Spectral Theory of Self-Adjoint Operators in Hilbert Space*, Reidel, Dordrecht 1986.
3. Campbell, S., *The Loch Ness Monster: The Evidence*, Prometheus Books, Amherst, NY 2004.
4. Daners, D., *A Faber-Krahn inequality for Robin problems in any space dimension*, Math. Ann. 335 (2006), 767–785.
5. Hardy, G. H., Littlewood, J. E., Polya, G., *Inequalities*, Cambridge Univ. Press, New York 1934.
6. Johnson, R. S., *A Modern Introduction to the Mathematical Theory of Water Waves*, Cambridge Texts in Applied Math., Cambridge Univ. Press, Cambridge 1997.
7. Kondratiev, V. A., *Boundary value problems for elliptic problems in domains with conical or angular points*, Trudy Moskov. Mat. Obšč. 16 (1967), 209–292. (English transl.: Trans. Moscow Math. Soc. 16 (1967), 227–313.)
8. Kuznetsov, N., Maz'ya, V., Vainberg, V., *Linear Water Waves. A Mathematical Approach*, Cambridge Univ. Press, Cambridge 2002.
9. Ladyzhenskaya, O. A., *Boundary Value Problems of Mathematical Physics*, Nauka, Moscow 1973. (English transl.: *The Boundary Value Problems of Mathematical Physics*, Applied Math. Sci. 49, Springer, New York 1985.)
10. Maz'ya, V., *Sobolev Spaces*, Springer Series in Soviet Math., Springer, Berlin 1985.
11. Nazarov, S. A., *A general scheme for averaging self-adjoint elliptic systems in multidimensional domains, including thin domains*, Algebra i Analiz. 7:5 (1995), 1–92. (English transl.: St. Petersburg Math. J. 7 (1996), 681–748.)
12. Nazarov, S. A., *Asymptotic Theory of Thin Plates and Rods 1. Dimension Reduction and Integral Estimates* (Russian), Nauchnaya Kniga, Novosibirsk 2001.

13. Nazarov, S. A., *Concentration of the point spectrum on the continuous one in problems of linear water-wave theory*, Zap. Nauchn. Sem. St.-Petersburg Otdel. Mat. Inst. Steklov. 348 (2007), 98–126. (English transl.: J. Math. Sci. (N.Y.) 152 (2008), 674–689.)
14. Nazarov, S. A., *On the essential spectrum of boundary value problems for systems of differential equations in a bounded domain with a peak*, Funkt. Anal. i Prilozhen. 43:1 (2009), 55–67. (English transl.: Funct. Anal. Appl., to appear.)
15. Nazarov, S. A., Taskinen, J., *On the spectrum of the Steklov problem in a domain with a peak*, Vestnik St. Petersburg Univ. 41:1 (2008), 45–52.
16. Nazarov, S. A., Sokolowski, J., Taskinen, J., *Neumann Laplacian on a domain with tangential components in the boundary*, Ann. Acad. Sci. Fenn. Math. 34 (2009), 131–143
17. Stoker, J.J., *Water Waves. The Mathematical Theory with Applications*, Wiley Classics Library, Wiley, New York 1992.

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