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SHELLABLE CACTUS GRAPHS

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Abstract

In this paper a new class of vertex decomposable graphs are determined. Moreover, all shellable and sequentially Cohen-Macaulay cactus graphs (i.e., connected graphs in which each edge belongs to at most one cycle) are characterized.

1. Introduction

Assume that G is a finite simple graph with vertex set $V(G) = \{1, \dots, n\}$ and edge set $E(G)$. Let K be an arbitrary field and $R = K[x_1, \dots, x_n]$. The ideal $I(G) \subset R$ which is generated by all monomials $x_i x_j$ such that $\{i, j\} \in E(G)$ is called the edge ideal of G. The simplicial complex Δ_G of a graph G is defined by

$$\Delta_G = \{A \subseteq V(G) \mid A \text{ is an independent set of } G\},$$

where A is an independent set of G if none of its elements are adjacent. In fact Δ_G is precisely the Stanley-Reisner simplicial complex of $I(G)$. A graded R-module M is called *sequentially Cohen-Macaulay* (over K) if there exists a finite filtration of graded R-modules

$$0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$$

such that each M_i/M_{i-1} is Cohen-Macaulay, and the Krull dimensions of the quotients are increasing:

$$\dim(M_1/M_0) < \dim(M_2/M_1) < \cdots < \dim(M_r/M_{r-1}).$$

A graph G is said to be (sequentially) Cohen-Macaulay if the ring $K[x_1, \dots, x_n]/I(G)$ is a (sequentially) Cohen-Macaulay ring.

In [16] Stanley showed that every shellable simplicial complex is sequentially Cohen-Macaulay. Here we mean the non-pure definition of shellability as introduced by Björner and Wachs [1]. However, the notion of a pure shellable

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complex was studied earlier in [15], [13]. In [18] Van Tuyl and Villarreal introduced the notion of a *shellable graph*. A graph G is called shellable if Δ_G is a shellable simplicial complex. Also, Dochtermann and Engström [5] and Woodroofe [20] studied the vertex decomposable graphs.

Studying vertex decomposable, shellable or (sequentially) Cohen-Macaulay graphs has attracted significant attention of researchers working in the borderline of combinatorial commutative algebra and algebraic combinatorics, (see [5], [8], [10], [17], [19], [20]). In [10] Herzog, Hibi, and Zheng classified all Cohen-Macaulay chordal graphs. Recently Woodroofe [20] showed that all 5-chordal graphs with no chordless 4-cycles are vertex decomposable.

We are interested in determining the families of shellable graphs. Since every shellable simplicial complex is sequentially Cohen-Macaulay, by identifying shellable graphs we are in fact identifying some of the sequentially Cohen-Macaulay graphs. A *cactus graph* (sometimes called a *cactus tree*) is a connected graph in which any two simple cycles have at most one vertex in common. Equivalently, every edge in such a graph may belong to at most one cycle. Cactus graphs were first studied under the name of Husimi trees [11]. In fact a cactus can be constructed from a tree by replacing some set of edges with cycles of arbitrary size. Note that every pseudo-tree (i.e., a graph containing exactly one cycle C_n for some $n \geqslant 3$) is a cactus graph.

In this paper we determine a class of vertex decomposable graphs in Theorem 2.3. Motivated by Francisco, H$\grave{a}$ and Villarreal's works in [8], [19], we study the effect of adding whiskers, ears and cycles C_3 or C_5 to a graph. Theorem 2.3 gives us a criteria to construct more vertex decomposable graphs by making some modification on graphs, (see Corollary 2.5).

Next we characterize all vertex decomposable, shellable and sequentially Cohen-Macaulay cactus graphs, (see Theorem 2.8). Moreover, it is shown that a cactus graph is vertex decomposable if and only if it is sequentially Cohen-Macaulay.

2. Shellable and sequentially Cohen-Macaulay cactus graphs

Vertex decomposability was introduced by Provan and Billera [14] in the pure case, and extended to the non-pure case by Björner and Wachs [1], [2]. We will use the following definition of vertex decomposable graph which is an interpretation of the definition of vertex decomposability for the independence complex of a graph studied first in [5], [20]. Let $N(u)$ be the set of all adjacent vertices of u.

DEFINITION 2.1. The independence complex of G is recursively defined to be *vertex decomposable* if G is a totally disconnected graph (with no edges), or if

- $G \setminus \{u\}$ and $G \setminus (\{u\} \cup N(u))$ are both vertex decomposable, and
- No independent set in $G \setminus (\{u\} \cup N(u))$ is a maximal independent set in $G \setminus \{u\}$.

A vertex u which satisfies in the second condition is called a *shedding vertex*.

Shellability was initially considered only for pure complexes, (see [14], [15]) and then extended to non-pure complexes by Björner and Wachs in [1] as follows.

DEFINITION 2.2. A simplicial complex Δ is *shellable* if the facets (maximal faces) of Δ can be ordered $F_1, \ldots, F_s$ such that for all $1 \leqslant i < j \leqslant s$, there exists some $v \in F_j \setminus F_i$ and some $l \in \{1, \ldots, j-1\}$ with $F_j \setminus F_l = \{v\}$. We call an ordering $F_1, \ldots, F_s$ of the facets of Δ satisfying this condition a shelling of Δ.

A graph G is called vertex decomposable (shellable) if the independence complex Δ_G is vertex decomposable (shellable). By [2, Theorem 11.3], vertex decomposability implies shellability and it is shown first by Stanley [16], that shellability implies sequentially Cohen-Macaulayness.

Let H be an induced subgraph of G. For any vertex v in $V(G)$, define $d(v, H)$ as $d(v, H) = \min\{d(v, u) \mid u \in V(H)\}$, where $d(v, u)$ is the length of shortest path between two vertices v and u in G. If there exists no path between v and u, then $d(v, u)$ is infinite.

In the following theorem we find a class of vertex decomposable graphs including chordal graphs and graphs considered by Woodroofe in [20].

THEOREM 2.3. *The graph G is vertex decomposable/shellable/sequentially Cohen-Macaulay if for any chordless cycle C_m, $m \neq 3, 5$, one of the following holds:*

(i) *There is a vertex of degree one adjacent to C_m.*

(ii) *There is a cycle C_3 such that $V(C_3) \cap V(C_m) \neq \emptyset$ and* $\deg_G(v) = 2$ *for some $v \in V(C_3)$.*

(iii) *There is a cycle C_5 such that $V(C_5) \cap V(C_m) = \{u\}$ for some vertex u and* $\deg_G(v) = \deg_G(w) = 2$*, where $N_{C_5}(u) = \{v, w\}$.*

PROOF. We do a proof by induction on $|V(G)|$. If $|V(G)| \leq 3$, then the result is obvious. Suppose $|V(G)| \geq 4$ and the result holds for any graph with fewer vertices than G. If G does not have any chordless cycle C_m, $m \neq 3, 5$, then by [20, Theorem 1.1] the result holds. Now suppose that G has at least one chordless cycle C_m for $m \neq 3, 5$. First we show that $G' = G \setminus (\{u\} \cup N_G(u))$ fulfills the induction hypothesis for any $u \in V(G)$. Let C_m for $m \neq 3, 5$ be a cycle of G'. If there is a vertex v of degree one adjacent to C_m in G, then

v is in G' too. If C_m satisfies the condition (ii) in G, then it has the same property in G', when its joint cycle C_3 was not removed. Otherwise the vertex v in (ii) is a vertex of degree one adjacent to C_m. Let C_m obeys the condition (iii) in G. If C_5 does not appear in G', then $\deg_{G'}(v) = 1$ or $\deg_{G'}(w) = 1$ which are some adjacent vertices to C_n. Thus G' is a graph which fulfills the induction hypothesis and so it is vertex decomposable. A similar argument shows that $G \setminus \{u\}$ satisfies the condition of the theorem, where u is on any C_m for $m \neq 3, 5$. In the following we find a shedding vertex of G in each case. In all cases the graphs $G \setminus (\{u\} \cup N_G(u))$ and $G \setminus \{u\}$ are vertex decomposable by the above argument and so G is vertex decomposable by induction hypothesis.

Case (i). Let v be a vertex of degree one adjacent to C_m for $m \neq 3, 5$ and let u be the adjacent vertex to v. Any maximal independent set of G which does not contain u, contains v. Hence an independent set of $G \setminus (\{u\} \cup N_G(u))$ is not a maximal independent set of $G \setminus \{u\}$ and so u is a shedding vertex of G.

Case (ii). Let $u \in V(C_m) \cap V(C_3)$ and $\deg_G(v) = 2$ for some $v \in V(C_3)$. For any independent set A of $G \setminus (\{u\} \cup N_G(u))$, $A \cup \{v\}$ is an independent set of $G \setminus \{u\}$ and so u is a shedding vertex of G.

Case (iii). Any maximal independent set of G which does not contain u, contains either v or w. Thus any independent set of $G \setminus (\{u\} \cup N(u))$ is not a maximal independent set of $G \setminus \{u\}$. It follows that u is a shedding vertex of G.

The idea of adding some vertices and edges to a graph in order to get a (sequentially) Cohen-Macaulay graph is studied widely in [5], [7], [19]. For a graph G, adding a *whisker* to G which means adding a new vertex to G and joining it to a vertex in G, is considered in [7], [19] and adding an *ear* to G (adding a new vertex to G and joining it to two adjacent vertices in G) is studied in [5], [8]. Also, by adding a cycle C_5 or C_3 to G, we mean to add a cycle C_5 or C_3 to G which is adjacent to exactly one vertex of G.

REMARK 2.4. Our proof of Theorem 2.3 implies that for a vertex decomposable graph G, by adding a whisker, or an ear, or a cycle C_5 or C_3, we get a vertex decomposable graph. Also, when G is shellable, the constructed graph by adding a whisker, an ear or a cycle C_3 or C_5 is again shellable. The shelling order of the new graph is that of $G \setminus \{u\}$, followed by the shelling order of $G \setminus (\{u\} \cup N(u))$ with u added to each facet, where u is the shedding vertex as found in each case.

As an immediate consequence of the proof of Theorem 2.3 we have

COROLLARY 2.5. *Let G be a graph and G' be a graph constructed by adding a whisker, or a cycle C_3 or C_5 at every vertex of G. Then G' is vertex decomposable.*

The next result has been considered previously in [5, Theorem 4.4] for adding a whisker in any vertex of graph, (see also [8], [19]).

COROLLARY 2.6. *Let G be a graph and G' be a graph constructed by adding a whisker, or a cycle C_3 or a cycle C_5 at every vertex of G. Then the independence complex of G' is pure and vertex decomposable.*

PROOF. Let F be a facet of the independence complex of G'. For any vertex $u \in G$, if there is an adjacent vertex v to u of degree one, then $u \in F$ or $v \in F$. Suppose that there is an adjacent cycle C_3 to u in G'. It means that the vertices $v, w \in C_3$ and the edges $\{u, v\}, \{u, w\}, \{v, w\}$ are added to G. So F contains one of the vertices u, v or w. In the case that there is an adjacent cycle C_5 to u in G', the vertices $v, x, y, w \in C_5$ and the edges $\{u, v\}, \{v, x\}, \{x, y\}, \{y, w\}, \{w, u\}$ are added to G'. Since F contains the two vertices u, x, u, y, w, x, w, v or v, y, the independence complex of G' is pure and so Corollary 2.5 completes the proof.

Recall that the *link* of a face F in Δ is defined as

$$\operatorname{link}_\Delta(F) = \{G \in \Delta; G \cup F \in \Delta, G \cap F = \emptyset\}.$$

The following lemma has been studied in [2, Proposition 10.14] with respect to shellability and in [3] for the sequentially Cohen-Macaulay version.

LEMMA 2.7. *Let Δ be a sequentially Cohen-Macaulay complex. Then for any face F in Δ,* $\operatorname{link}_\Delta(F)$ *is also sequentially Cohen-Macaulay.*

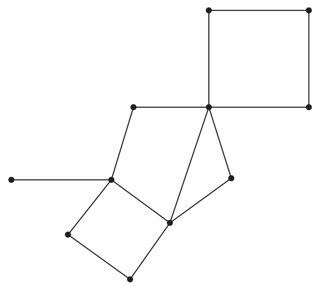
PROOF. Let $F \in \Delta$ and let G be a face in $\Delta' = \operatorname{link}_\Delta(F)$. It is easy to check that $\operatorname{link}_{\Delta'}(G) = \operatorname{link}_\Delta(F \cup G)$. Thus [3, Definition 1.2(i)] shows that Δ' is sequentially Cohen-Macaulay.

It is shown that in bipartite graphs, three concepts vertex decomposability, shellability and sequentially Cohen-Macaulayness are equivalent, see [17, Theorem 2.10]. Using Lemma 2.7 we have the same property in cactus graphs. For any graph G and a subset A of $V(G)$, by a maximal independent subset A' of A, we mean an independent set of G which can not be extended to another independent set contained in A. Hence for any $u \in A \setminus A'$, there is a vertex $v \in A'$ adjacent to u.

THEOREM 2.8. *Let G be a cactus graph. Then G is sequentially Cohen-Macaulay if and only if G satisfies the condition of Theorem 2.3. In particular, the following are equivalent:*
(i) *G is sequentially Cohen-Macaulay.*
(ii) *G is shellable.*
(iii) *G is vertex decomposable.*

PROOF. It is enough to show that any sequentially Cohen-Macaulay graph satisfies the condition of Theorem 2.3. Let G be a sequentially Cohen-Macaulay graph. By contradiction assume that there is a cycle C_m for $m \neq 3, 5$, such that it does not obey the condition of Theorem 2.3. Let $A = \{v \in V(G); d(v, C_m) = 2\}$. By Theorem 2.3 (iii), for any cycle $C_5 : u, v, x, y, w, u$ with $V(C_m) \cap V(C_5) = \{u\}$, we can assume that $\deg_G(v) > 2$ and $\{v, z\} \in E(G)$ for some vertex z. Consider a maximal independent subset A' of A such that for any cycle C_5 adjacent to C_m with above indices, $z, y \in A'$. Thus A' is an independent set of G such that for any vertex v adjacent to C_m, there is a vertex $y \in A'$ adjacent to v. Therefore one of the connected components of $G \setminus (A' \cup N_G(A'))$ is C_m which is not sequentially Cohen-Macaulay by [8, Proposition 4.1]. On the other hand, by Lemma 2.7 the independent complex of $G \setminus (A' \cup N_G(A'))$, $\mathrm{link}_{\Delta_G}(A')$, is sequentially Cohen-Macaulay which is a contradiction.

From Theorem 2.3 one can get several examples of vertex decomposable graphs which are not trees, chordal or bipartite. For example, the following graph obeys the condition of Theorem 2.3 and so is vertex decomposable.



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REFERENCES

1. Björner, A., and Wachs, M. L., *Shellable nonpure complexes and posets I*, Trans. Amer. Math. Soc. 348 (1996), 1299–1327.
2. Björner, A., and Wachs, M. L., *Shellable nonpure complexes and posets II*, Trans. Amer. Math. Soc. 349 (1997), 3945–3975.

3. Björner, A., Wachs, M. L., and Welker, V., *On sequentially Cohen-Macaulay complexes and posets,* Israel J. Math. 169 (2009), 295–316.

4. Bruns, W., and Herzog, J., *Cohen-Macaulay Rings*, Cambridge Studies in Adv. Math. 39, Cambridge University Press, Cambridge 1993.

5. Dochtermann, A., and Engström, A., *Algebraic properties of edge ideals via combinatorial topology,* Electron. J. Combin. 16:2 (2009), #R2.

6. Faridi, S., *Simplicial trees are sequentially Cohen-Macaulay*, J. Pure Appl. Algebra 190 (2004), 121–136.

7. Francisco, C. A., and Hà, H. T., *Whiskers and sequentially Cohen-Macaulay graphs*, J. Combin. Theory (A) 115 (2008), 304–316.

8. Francisco, C. A., and Tuyl, A. V., *Sequentially Cohen-Macaulay edge ideals*, Proc. Amer. Math. Soc. 135 (2007), 2327–2337.

9. Herzog, J., and Hibi, T., *Componentwise linear ideals*, Nagoya Math. J. 153 (1999), 141–153

10. Herzog, J., Hibi, T., and Zheng, X., *Cohen-Macaulay chordal graphs*, J. Combin. Theory (A) 113 (2006), 911–916.

11. Harary, F., and Uhlenbeck, G. E., *On the number of Husimi trees I*, Proc. Nat. Acad. Sci. U. S. A. 39 (1953), 315–322.

12. Hochster, M., *Rings of invariants of tori, Cohen-Macaulay rings generated by monomials, and polytopes*, Ann. of Math. (2) 96 (1972), 318–337.

13. McMullen, P., *The maximum numbers of faces of a convex polytope*, Mathematika 17 (1970), 179–184.

14. Provan, J., and Billera, L., *Decompositions of simplicial complexes related to diameters of convex polyhedra*, Math. Oper. Res. 5 (1980), 576–594.

15. Rudin, M. E., *An unshellable triangulation of a tetrahedron*, Bull. Amer. Math. Soc. 64 (1958), 90–91.

16. Stanley, R. P., *Combinatorics and Commutative Algebra*, Second edition. Progress in Mathematics 41. Birkhäuser, Boston, MA 1996.

17. Van Tuyl, A., *Sequentially Cohen-Macaulay bipartite graphs: vertex decomposability and regularity*, (2009) Preprint, arXiv:0906.0273v1.

18. Van Tuyl, A., and Villarreal, R., *Shellable graphs and sequentially Cohen-Macaulay bipartite graphs*, J. Combin. Theory (A) 115 (2008), 799–814.

19. Villarreal, R. H., *Cohen-Macaulay graphs*, Manuscripta Math. 66 (1990), 277–293.

20. Woodroofe, R., *Vertex decomposable graphs and obstruction to shellability*, Proc. Amer. Math. Soc. 137 (2009), 3235–3246.

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