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THE NONCOMMUTATIVE CHOQUET BOUNDARY III: OPERATOR SYSTEMS IN MATRIX ALGEBRAS

WILLIAM ARVESON

Abstract

We classify operator systems $S \subseteq \mathcal{B}(H)$ that act on finite dimensional Hilbert spaces H by making use of the noncommutative Choquet boundary. S is said to be *reduced* when its boundary ideal is $\{0\}$. In the category of operator systems, that property functions as semisimplicity does in the category of complex Banach algebras.

We construct explicit examples of reduced operator systems using sequences of "parameterizing maps" $\Gamma_k : \mathbf{C}^r \to \mathcal{B}(H_k)$, $k = 1, \dots, N$. We show that every reduced operator system is isomorphic to one of these, and that two sequences give rise to isomorphic operator systems if and only if they are "unitarily equivalent" parameterizing sequences.

Finally, we construct nonreduced operator systems S that have a given boundary ideal K and a given reduced image in $C^*(S)/K$, and show that these constructed examples exhaust the possibilities.

1. Introduction

This paper continues the series [2] and [3] by addressing the problem of classifying operator spaces that generate finite dimensional C^*-algebras. While this is a restricted class of operator spaces in which many subtle topological obstructions disappear, one can also argue that it contains all of the noncommutativity. Correspondingly, our classification results will make essential use of the noncommutative Choquet boundary.

An *operator space* is a norm closed linear subspace of the algebra $\mathcal{B}(H)$ of bounded operators on a Hilbert space H. Operator spaces are the objects of a category that refines the classical category of Banach spaces in a significant way, the refinement being that morphisms in the category of operator spaces are completely bounded linear maps rather than bounded linear maps. In the operator space category, maps are typically endowed with their completely bounded norm. If one restricts to the smaller category of operator spaces with *completely contractive* linear maps, one obtains a noncommutative refinement of the category of Banach spaces in which the term *classification* means classifying operator spaces up to completely isometric isomorphism. As we have

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said above, this paper addresses (somewhat indirectly) the problem of classifying operator spaces that can be realized as subspaces of $\mathscr{B}(H)$ where H is a *finite dimensional* Hilbert space.

We say indirectly because we do not deal directly with operator spaces below, but rather with operator *systems*. An operator system is an operator space that is closed under the $*$-operation of $\mathscr{B}(H)$ and which contains the identity operator. Throughout the matrix hierarchy over an operator system S there are enough positive operators to generate the space of self adjoint matrices over S. Correspondingly, the morphisms of the category of operator systems are the unit-preserving completely positive (UCP) linear maps. An *isomorphism* of operator systems S_1, S_2 is a UCP map $\phi : S_1 \to S_2$ that has a UCP inverse $\phi^{-1} : S_2 \to S_1$; and it is known that this is equivalent to the existence of a completely isometric linear map of S_1 on S_2 that carries the unit of S_1 to the unit of S_2.

Paulsen's device (see p. 104 of [8] or p. 21 of [6]) allows one to associate an operator system $\tilde{S} \subseteq \mathscr{B}(H \oplus H)$ with an arbitrary operator space $S \subseteq \mathscr{B}(H)$ in the following way

$$\tilde{S} = \left\{ \begin{pmatrix} \lambda \cdot \mathbf{1} & s \\ t^* & \mu \cdot \mathbf{1} \end{pmatrix} : s, t \in S,\ \lambda, \mu \in \mathbb{C} \right\},$$

and this association of $\tilde{S}$ with S is functorial in that completely contractive maps of operator spaces S give rise to UCP maps of operator systems $\tilde{S}$. In this way, many if not most results about operator systems lead directly to results in the somewhat broader category of operator spaces. Consequently, we shall work exclusively with operator systems throughout this paper.

The fundamental fact about general operator systems is that there is a *largest* closed two-sided ideal K in the C^*-subalgebra $C^*(S)$ generated by S such that the natural map $x \in C^*(S) \mapsto \dot{x} \in C^*(S)/K$ is completely isometric on S. In this paper we will follow [3] by referring to this ideal K as the *boundary ideal* for S. The associated embedding $\dot{S} \subseteq C^*(S)/K$ is called the C^*-envelope of S.

DEFINITION 1.1. An operator system $S \subseteq C^*(S)$ is said to be *reduced* if its boundary ideal is $\{0\}$.

We have found it useful to view the property of Definition 1.1 as the proper counterpart for operator systems of the semisimplicity property of complex Banach algebras. Thus, the boundary ideal functions for operator systems in much the same way as the radical does for Banach algebras.

Our main results address the problem of classifying reduced operator systems $S \subseteq \mathscr{B}(H)$ that act on a finite dimensional Hilbert space H, and can be summarized as follows. The C^*-algebra generated by such an operator system decomposes uniquely into a central direct sum of matrix algebras

$$C^*(S) = A_1 \oplus \cdots \oplus A_N, \qquad A_k \cong \mathcal{B}(H_k) \tag{1.1}$$

and it has several integer invariants associated with it: the dimension d of S itself, the number N of mutually inequivalent irreducible representations of its generated C^*-algebra $\pi_k : C^*(S) \to \mathcal{B}(H_k)$, and the dimensions $n_k = \dim H_k$, $1 \le k \le N$ of these representations. These numbers satisfy only one obvious constraint, namely

$$\dim \pi_k(S) \le n_k^2, \qquad k = 1, \ldots, N, \tag{1.2}$$

and consequently $d \le n_1^2 + \cdots + n_N^2$.

Given a set of numbers $d, N, n_1, \ldots, n_N$, we first show how one constructs examples of reduced operator systems S of dimension d that have these integer invariants. By a $*$-vector space we mean a finite dimensional complex vector space that has been endowed with a distinguished conjugation (an antilinear map $z \mapsto z^*$ satisfying $z^{**} = z$), and a distinguished "unit" $\mathbf{1}$ – a nonzero self adjoint element of Z. Nothing is lost if one thinks of Z as the $*$-vector space $\mathbb{C}^d$ with involution $(z_1, \ldots, z_d)^* = (\bar{z}_1, \ldots, \bar{z}_d)$ and unit $\mathbf{1} = (1, 1, \ldots, 1)$. For every $k = 1, \ldots, N$ let H_k be a Hilbert space of dimension n_k, and let

$$\Gamma_k : Z \to \mathcal{B}(H_k)$$

be a linear map that preserves the $*$-operation and maps the unit of Z to the identity operator of $\mathcal{B}(H_k)$. We assume further that these maps Γ_k satisfy the following three requirements:

(i) *Irreducibility*: For each $k = 1, \ldots, N$, $\Gamma_k(Z)$ is an irreducible space of operators in $\mathcal{B}(H_k)$.

(ii) *Faithfulness*: $\ker \Gamma_1 \cap \cdots \cap \ker \Gamma_N = \{0\}$.

(iii) *Strong separation*: For every $k = 1, \ldots, N$, there is a positive integer $p = p(k)$ and a $p \times p$ matrix $(z_{ij}) = (z_{ij}(k)) \in M_p(Z)$ with entries in Z such that

$$\|(\Gamma_k(z_{ij}))\| > \max_{j \ne k} \|(\Gamma_j(z_{ij}))\|, \qquad k = 1, 2, \ldots, N. \tag{1.3}$$

Property (i) simply means that $C^*(\Gamma_k(Z)) = \mathcal{B}(H_k)$, and property (ii) can be arranged in general by replacing $A = \mathbb{C}^d$ with a suitably smaller parameter space, if necessary. Property (iii) is critical, and as we shall see, it connects with the noncommutative Choquet boundary in an essential way.

The N-tuple of operator mappings $\mathbf{\Gamma} = (\Gamma_1, \dots, \Gamma_N)$ gives rise to an operator system $S_{\mathbf{\Gamma}}$ acting on $H_1 \oplus \cdots \oplus H_N$ as follows

$$S_{\mathbf{\Gamma}} = \{\Gamma_1(z) \oplus \cdots \oplus \Gamma_N(z) : z \in Z\}, \tag{1.4}$$

and by property (ii), we have $\dim(S_{\mathbf{\Gamma}}) = \dim Z = d$.

THEOREM 1.2. *The operator system $S_{\mathbf{\Gamma}}$ is reduced, the center of $C^*(S_{\mathbf{\Gamma}})$ is $\mathbb{C}^N$, and the numbers $n_1, \dots, n_N$ are given by $n_k = \dim H_k$ for $k = 1, \dots, N$. Conversely, every reduced operator system with the same integer invariants is isomorphic to an $S_{\mathbf{\Gamma}}$ constructed in this way from an N-tuple of linear maps $\Gamma_k : Z \to \mathscr{B}(\tilde{H}_k)$, $k = 1, \dots, N$ satisfying properties (i), (ii), (iii).*

Theorem 1.2 reduces the classification problem for reduced operator systems acting on finite dimensional Hilbert spaces to the problem of determining when two sequences of parameterizing maps give rise to isomorphic operator systems. Notice that the operator space $S_{\mathbf{\Gamma}}$ does not change if we compose each map of the sequence $\mathbf{\Gamma} = (\Gamma_1, \dots, \Gamma_N)$ with a single automorphism of the unital $*$ structure of the parameter space Z. Moreover, it is easy to check that the *isomorphism class* of $S_{\mathbf{\Gamma}}$ does not change if we replace each Γ_k with a unitarily equivalent map $\tilde{\Gamma}_k$, or if we permute the component maps $\Gamma_1, \dots, \Gamma_N$. Thus we say that two parameterizing sequences are *equivalent* if they have the same length $\mathbf{\Gamma} = (\Gamma_1, \dots, \Gamma_N)$ and $\tilde{\mathbf{\Gamma}} = (\tilde{\Gamma}_1, \dots, \tilde{\Gamma}_N)$, there is a unit-preserving $*$-automorphism $\theta : Z \to Z$ of the parameter space, a permutation σ of $\{1, 2, \dots, N\}$, and a sequence of unitary operators $U_k : H_k \to \tilde{H}_{\sigma(k)}$, $k = 1, \dots, N$, such that

$$\tilde{\Gamma}_{\sigma(k)}(\theta(z)) = U_k \Gamma_k(z) U_k^{-1}, \qquad z \in Z, \quad k = 1, \dots, N.$$

The following result completes the classification picture:

THEOREM 1.3. *Let S and $\tilde{S}$ be two reduced operator systems of the same dimension d and having the same set of integer invariants. Let $Z = \mathbb{C}^d$ and let $\mathbf{\Gamma}$ and $\tilde{\mathbf{\Gamma}}$ be two N-tuples of maps satisfying* (i)–(iii) *such that $S \cong S_{\mathbf{\Gamma}}$ and $\tilde{S} \cong S_{\tilde{\mathbf{\Gamma}}}$. Then S and $\tilde{S}$ are isomorphic operator systems iff the two parameterizing sequences $\mathbf{\Gamma}$ and $\tilde{\mathbf{\Gamma}}$ are equivalent.*

REMARK 1.4 (About the term "classification"). The combined statements of Theorems 1.2 and 1.3 amount to a classification of reduced operator systems that generate finite dimensional C^*-algebras. It seems appropriate to offer some support for that claim, since the invariants of this classification, namely equivalence classes of parameterizing sequences $\mathbf{\Gamma} = (\Gamma_1, \dots, \Gamma_N)$ that satisfy properties (i), (ii) and (iii), are somewhat unusual.

Consider first the simplest case $N = 1$. Here we have a single self adjoint linear map Γ from the parameter space $Z = \mathbb{C}^d$ to operators on a finite dimensional Hilbert space H such that $\Gamma(Z)$ is an irreducible set of operators, and which carries the unit of Z to the identity operator. Such a map Γ automatically satisfies property (i), property (ii) can obviously be arranged, and property (iii) is satisfied vacuously. Given a second map $\tilde{\Gamma} : Z \to \mathcal{B}(\tilde{H})$ with that property, then Γ and $\tilde{\Gamma}$ are equivalent iff there is a unitary operator $U : H \to \tilde{H}$ and an automorphism θ of the unital $*$-structure of Z such that

$$\tilde{\Gamma}(\theta(z)) = U\Gamma(z)U^{-1}, \qquad z \in Z.$$

Thus, in the case $N = 1$, Theorems 1.2 and 1.3 make the assertion that *two irreducible operator systems are isomorphic as operator systems iff they are unitarily equivalent*. One can appreciate the content of that statement by considering that it has the following implication for two dimensional operator spaces: Given two irreducible $n \times n$ matrices a and b, the map

$$\lambda \cdot \mathbf{1} + \mu \cdot a \mapsto \lambda \cdot \mathbf{1} + \mu \cdot b, \qquad \lambda, \mu \in \mathbb{C}$$

is completely isometric iff the operators a and b are unitarily equivalent.

In more complex situations where $N \geq 2$, the assertion is that a) every operator system is made up of irreducible ones with $N = 1$, and b) there is no obstruction to assembling the irreducible pieces into a larger operator system other than the requirements of the strong separation property (iii) above. In particular, the notion of *equivalence* for parameter sequences $\mathbf{\Gamma} = (\Gamma_1, \ldots, \Gamma_N)$ primarily involves conditions on the individual coordinate maps Γ_k, with no interaction between different coordinates.

Needless to say, this paper does not represent the only attempt to look seriously at operator systems in matrix algebras; indeed, experience with completely positive maps of operator systems shows that corresponding finite dimensional issues arise frequently and naturally. For example, see Theorem 4.2 of [5]. On the other hand, the structure and classification of such operator systems has not been addressed before, and Theorems 1.2 and 1.3 are new.

2. Brief on the noncommutative Choquet boundary

A *boundary representation* for an operator system $S \subseteq C^*(S)$ is an irreducible representation $\pi : C^*(S) \to \mathcal{B}(H)$ with the property that the only UCP map $\phi : C^*(S) \to \mathcal{B}(H)$ that agrees with π on S is π itself. It was shown in [2] that every separable operator system has sufficiently many boundary representations; and when compact operators are involved, it was shown in [3] that boundary representations are the noncommutative counterparts of peak points of function systems. In this section we summarize that material in a form suitable for the analysis of operator systems in matrix algebras, referring the reader to [2] and [3] for more detail and history. While it is true that these results can be simplified very substantially for operator systems in matrix algebras, much of the subtlety persists even in the finite dimensional context (see Remark 2.4 below). Consequently, we have not attempted to make the following discussion of matrix algebras and their operator systems self-contained. We now summarize some general results of [1], [2] and [3] in the form we require.

Much of the discussion to follow rests on the main result of [2] (Theorem 7.1), which we repeat here for reference:

THEOREM 2.1. *Every separable operator system $S \subseteq C^*(S)$ has sufficiently many boundary representations in the sense that for every $n \geq 1$ and every $n \times n$ matrix (s_{ij}) with components $s_{ij} \in S$, one has*

$$\|(s_{ij})\| = \sup_{\pi} \|(\pi(s_{ij}))\|, \tag{2.1}$$

the supremum on the right taken over all boundary representations π for S.

In Theorem 2.2.3 of [1], it was shown that in all cases where sufficiently many boundary representations exist, the boundary ideal is the intersection of the kernels of all boundary representations. This leads to the following characterization of reduced operator systems in matrix algebras:

COROLLARY 2.2. *An operator system $S \subseteq \mathscr{B}(H)$ acting on a finite dimensional Hilbert space H is reduced iff every irreducible representation of $C^*(S)$ is a boundary representation for S.*

PROOF. If every irreducible representation is a boundary representation for S, then the result from [1] cited above implies that the boundary ideal is trivial.

Conversely, assume that the boundary ideal for S is trivial and consider the decomposition (1.1)

$$C^*(S) = A_1 \oplus \cdots \oplus A_N, \qquad A_k \cong \mathscr{B}(H_k) \tag{2.2}$$

of the C^*-algebra generated by S into a direct sum of full matrix algebras. If one of the associated irreducible representations, say $\pi_k : C^*(S) \to \mathscr{B}(H_k)$, $1 \leq k \leq N$, were *not* a boundary representation, then every boundary representation would annihilate the ideal

$$K = \bigcap_{j \neq k} \ker \pi_j \neq \{0\}$$

and therefore would correspond to an irreducible representation of the quotient C^*-algebra $C^*(S)/K$. In that event, Theorem 2.1 above implies that the quotient map $x \in C^*(S) \mapsto \dot{x} \in C^*(S)/K$ restricts to a completely isometric map on S, contradicting the hypothesis that S is reduced.

COROLLARY 2.3. *Every irreducible operator system $S \subseteq \mathcal{B}(H)$ acting on a finite dimensional Hilbert space is reduced in the sense of Definition 1.1, and the identity representation of $\mathcal{B}(H)$ is a boundary representation for S.*

PROOF. Since S is an irreducible self adjoint family of operators that contains $\mathbf{1}$, the double commutant theorem implies $C^*(S) = \mathcal{B}(H)$. Every operator system that generates a simple C^*-algebra must be reduced, since the only candidate for the boundary ideal is $\{0\}$. In this case, the identity representation of $C^*(S)$ is, up to equivalence, the only irreducible representation, so that Theorem 2.1 implies that it must be a boundary representation.

REMARK 2.4 (Fixed points of UCP maps on matrix algebras). Corollary 2.3 is equivalent to the following assertion: If $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$ is a UCP map on a matrix algebra that fixes an irreducible set S of operators, then ϕ is the identity map. When $S = \{a\}$ consists of a single irreducible operator a, for example, there appears to be no direct route to a proof of the assertion. In the special case where ϕ preserves the tracial state of $\mathcal{B}(H)$, a straightforward application of the Schwarz inequality implies that the operator system $S = \{x \in \mathcal{B}(H) : \phi(x) = x\}$ is closed under operator multiplication, and that implies ϕ is the identity map when S is irreducible.

But in general, the best one can say is that there is a UCP idempotent E with the same fixed elements – recall that the set of accumulation points of the sequence ϕ^k, $k = 1, 2, \dots$ contains a unique idempotent E. Once one has such an E one can introduce the Choi-Effros multiplication [7] on S to make it into a C^*-algebra. However, since the Choi-Effros multiplication need not be the ambient multiplication in $\mathcal{B}(H)$, this fails to address the issue. Thus, Theorem 2.1 and Corollary 2.3 make significant assertions even for operator systems in matrix algebras.

Finally, we recall the key property of reduced operator systems in general, which follows from Theorem 2.2.5 of [1] together with Theorem 2.1:

THEOREM 2.5. *Let $S_1 \subseteq C^*(S_1)$ and $S_2 \subseteq C^*(S_2)$ be two reduced separable operator systems. Then every isomorphism of operator systems $\theta : S_1 \to S_2$ extends uniquely to a $*$-isomorphism of C^*-algebras $\tilde{\theta} : C^*(S_1) \to C^*(S_2)$.*

3. Peaking and boundary representations

Let X be a compact Hausdorff space. A *function system* is a linear subspace of $C(X)$ that separates points, is closed under complex conjugation, and contains the constants. A *peak point* for a function system $S \subseteq C(X)$ is a point $x \in X$ with the property that there is a "peaking function" $f \in S$ (which of course depends on x) such that

$$|f(x)| > |f(y)|, \qquad \forall\, y \in X \setminus \{x\}.$$

Since function systems are spanned by their real functions, one can reformulate this condition so as to get rid of absolute values; but the form given is the one that we choose for the following discussion. For separable function algebras, a theorem of Bishop and de Leeuw [4] asserts that the peak points are exactly the points of the Choquet boundary, while for the more general separable function systems, the peak points are dense in the Choquet boundary (see [9] or [10], pp. 39–40 and Corollary 8.4).

We now recall the definition of peaking representations from [3]. Given an operator system $S \subseteq C^*(S)$, every representation π of $C^*(S)$ on a Hilbert space K gives rise to a representation of the matrix hierarchy over $C^*(S)$, in which for an $n \times n$ matrix (x_{ij}) of elements of $C^*(S)$, the $n \times n$ operator matrix $(\pi(x_{ij}))$ represents an operator on the direct sum $n \cdot K$ of n copies of K, and the map $(x_{ij}) \mapsto (\pi(x_{ij})) \in \mathscr{B}(n \cdot K)$ is a representation of $M_n(C^*(S))$ on $n \cdot K$.

Definition 3.1. An irreducible representation $\pi : C^*(S) \to \mathscr{B}(K)$ is said to be *peaking* for S if there is an $n = 1, 2, \ldots$ and an $n \times n$ matrix $(s_{ij}) \in M_n(S)$ with entries in S that has the following property: For every irreducible representation σ of $C^*(S)$ that is inequivalent to π, one has

$$\|(\pi(s_{ij}))\| > \|(\sigma(s_{ij}))\|. \tag{3.1}$$

Such an operator matrix $(s_{ij}) \in M_n(S)$ is called a *peaking operator* for π.

Actually, in [3] it was necessary to employ a stronger variant of this concept, called *strong peaking*. That stronger property will not be required here since the C^*-algebras of this paper are finite dimensional, with only a finite number of inequivalent irreducible representations. We will make use the following special case of a general result in [3]:

Theorem 3.2. *Let S be an operator system that generates a finite dimensional C^*-algebra $C^*(S)$. An irreducible representation of $C^*(S)$ is a boundary representation for S iff it is peaking for S.*

Proof. The decomposition (2.2) of $C^*(S)$ into a central direct sum of matrix algebras implies that there are exactly N mutually inequivalent irreducible representations $\pi_1, \ldots, \pi_N$ of $C^*(S)$. So for example, π_1 is peaking for S iff there is a $p \geq 1$ and a $p \times p$ matrix (s_{ij}) over S such that

$$\|(\pi_1(s_{ij}))\| > \max_{2 \leq k \leq N} \|(\pi_k(s_{ij}))\|.$$

Theorem 6.2 of [3] implies that this assertion is equivalent to the assertion that π_1 is a boundary representation for S.

4. Proof of Theorem 1.2

In this section we prove Theorem 1.2. Choose an N-tuple of unit preserving self adjoint linear maps $\Gamma_k : Z \to \mathscr{B}(H_k)$, $k = 1, \dots, N$, satisfying properties (i)–(iii) of Section 1, let $\mathbf{\Gamma} = (\Gamma_1, \dots, \Gamma_N)$ and consider the operator system $S_{\mathbf{\Gamma}} \subseteq \mathscr{B}(H_1 \oplus \cdots \oplus H_N)$ defined by

$$S_{\mathbf{\Gamma}} = \{\Gamma_1(z) \oplus \cdots \oplus \Gamma_n(z) : z \in Z\}.$$

We will show that

$$C^*(S_{\mathbf{\Gamma}}) = \mathscr{B}(H_1) \oplus \cdots \oplus \mathscr{B}(H_N), \tag{4.1}$$

that each of the visible irreducible representations $\pi_k : C^*(S) \to \mathscr{B}(H_k)$

$$\pi_k(x_1 \oplus \cdots \oplus x_k) = x_k, \qquad k = 1, \dots, N$$

is a boundary representation for $S_{\mathbf{\Gamma}}$, and we will deduce that $S_{\mathbf{\Gamma}}$ is a reduced operator system with the asserted integer invariants.

To prove (4.1), note first that each of the representations $\pi_1, \dots, \pi_N$ of $C^*(S_{\mathbf{\Gamma}})$ is irreducible, since $\pi_k(C^*(S_{\mathbf{\Gamma}}))$ contains the irreducible space of operators $\Gamma_k(Z) \subseteq \mathscr{B}(H_k)$. Moreover, the strong separation property (iii) implies that the representations π_k are mutually inequivalent. It follows that they are mutually disjoint; and since the identity representation of $C^*(S_{\mathbf{\Gamma}})$ is the direct sum $\pi_1 \oplus \cdots \oplus \pi_N$, it follows that the projection p_k of $H_1 \oplus \cdots \oplus H_N$ onto each summand H_k belongs to $C^*(S_{\mathbf{\Gamma}})$. Formula (4.1) follows.

Finally, let $L : Z \to \mathscr{B}(H_1 \oplus \cdots \oplus H_N)$ be the linear map

$$L(z) = \Gamma_1(z) \oplus \cdots \oplus \Gamma_N(z), \qquad z \in Z.$$

We have $S_{\mathbf{\Gamma}} = L(Z)$ by definition of $S_{\mathbf{\Gamma}}$, and moreover

$$\Gamma_k(z) = \pi_k(L(z)), \qquad z \in Z, \quad k = 1, \dots, N.$$

Using the latter formula, one finds that the strong separation property (iii) implies that each representation π_k is peaking for $S_{\mathbf{\Gamma}}$. Theorem 3.2 implies that each π_k is a boundary representation for S, and since $\{\pi_1, \dots, \pi_N\}$ is a complete list of the irreducible representations of $C^*(S_{\mathbf{\Gamma}})$ up to equivalence, Corollary 2.2 implies that $S_{\mathbf{\Gamma}}$ is reduced.

To prove the converse assertion of Theorem 1.2, let $S \subseteq \mathscr{B}(H)$ be a reduced operator system acting on a finite dimensional Hilbert space H. We have to show that there is an N-tuple $\mathbf{\Gamma} = (\Gamma_1, \dots, \Gamma_N)$ of linear maps with the stated properties such that S and $S_{\mathbf{\Gamma}}$ are isomorphic operator systems.

To do that, consider the natural decomposition of $C^*(S)$

$$C^*(S) = A_1 \oplus \cdots \oplus A_N, \qquad A_k \cong \mathscr{B}(H_k), \quad k = 1, \dots, N$$

into finite dimensional subfactors. We first exhibit an appropriate parameterization for S in terms of the parameter space $Z = \mathbb{C}^d$, $d = \dim S$. For that we claim that there is a linear basis $s_1, \dots, s_d$ for S consisting of self adjoint operators s_k that satisfies $s_1 + \cdots + s_d = \mathbf{1}$. Indeed, if we choose a $d-1$ dimensional subspace S_0 of S such that $S_0^* = S_0$ and $\mathbf{1} \notin S_0$, then we obtain such a basis by choosing a linear basis $s_1, \dots, s_{d-1}$ for S_0 consisting of self adjoint operators and setting $s_d = \mathbf{1} - (s_1 + \cdots + s_{d-1})$. Set $Z = \mathbb{C}^d$. Then we have arranged that the linear map $L : Z \to S$ defined by

$$L(z_1, \dots, z_d) = z_1 \cdot s_1 + \cdots + z_d \cdot s_d$$

is linear isomorphism of vector spaces that preserves the $*$ operation and carries $(1, 1, \dots, 1)$ to the identity operator of S.

Let $\pi_k : C^*(S) \to \mathscr{B}(H_k)$ be the representation associated with the kth term in the decomposition (4.1) and define $\Gamma_k : Z \to \mathscr{B}(H_k)$ by

$$\Gamma_k(z) = \pi_k(L(z)), \qquad z \in Z. \tag{4.2}$$

Each Γ_k is a linear map that preserves the $*$-operation and maps the vector $(1, 1, \dots, 1) \in Z$ to the operator $\mathbf{1}_{H_k}$. We have $\Gamma_k(Z) = \pi_k(S)$, so that $C^*(\Gamma_k(Z)) = \pi_k(C^*(S)) = \mathscr{B}(H_k)$, and hence $\Gamma_k(Z)$ is an irreducible operator system in $\mathscr{B}(H_k)$, $k = 1, \dots, N$. Note too that these maps $\Gamma_1, \dots, \Gamma_N$ satisfy property (ii) of Section 1 since if $\Gamma_k(z) = 0$ for every k, then $\pi_k(L(z)) = 0$ for every $k = 1, \dots, N$, hence $L(z) = 0$ since $\ker \pi_1 \cap \cdots \cap \ker \pi_N = \{0\}$, and finally $z = 0$ since L is an isomorphism of vector spaces.

We claim that the N-tuple $\mathbf{\Gamma} = (\Gamma_1, \dots, \Gamma_N)$ satisfies property (iii) of Section 1. Indeed, by Corollary 2.2, every representation $\pi_k : C^*(S) \to \mathscr{B}(H_k)$ is a boundary representation for S which, by Theorem 3.2, is a peaking representation for S. Property (iii) now follows after one unravels that assertion through each of the parameterizations $\Gamma_k(z) = \pi_k(L(z))$, $k = 1, \dots, N$, in terms of the basic map $L : Z \to S$.

Thus if we form the operator system associated with the N-tuple $\mathbf{\Gamma} = (\Gamma_1, \dots, \Gamma_N)$

$$S_{\mathbf{\Gamma}} = \{\Gamma_1(z) \oplus \cdots \oplus \Gamma_N(z) : z \in Z\} \subseteq \mathcal{B}(H_1 \oplus \cdots \oplus H_N),$$

then formula (4.1) implies that

$$C^*(S_{\mathbf{\Gamma}}) = \mathcal{B}(H_1) \oplus \cdots \oplus \mathcal{B}(H_N).$$

Now for each $k = 1, \dots, N$, the irreducible representation $\pi_k : A_k \to \mathcal{B}(H_k)$ is an isomorphism of C^*-algebras, so we can define a $*$-isomorphism of C^*-algebras $\pi : C^*(S) \to C^*(S_{\mathbf{\Gamma}})$ by way of

$$\pi(x_1 \oplus \cdots \oplus x_N) = \pi_1(x_1) \oplus \cdots \oplus \pi_N(x_N), \quad x_k \in A_k, \quad k = 1, \dots, N.$$

This isomorphism satisfies $\pi(L(z)) = \Gamma_1(z) \oplus \cdots \oplus \Gamma_N(z)$ for all $z \in Z$, and therefore $\pi(S) = S_{\mathbf{\Gamma}}$. Hence S and $S_{\mathbf{\Gamma}}$ are isomorphic operator systems.

5. Proof of Theorem 1.3

Let $\mathbf{\Gamma} = (\Gamma_1, \dots, \Gamma_N)$ and $\tilde{\mathbf{\Gamma}} = (\tilde{\Gamma}_1, \dots, \tilde{\Gamma}_N)$ be two parameterizing sequences of linear maps defined on $Z = \mathbb{C}^d$ of the same length that satisfy (i), (ii), (iii) of Section 1, and assume that $S_{\mathbf{\Gamma}}$ and $S_{\tilde{\mathbf{\Gamma}}}$ are isomorphic operator systems. We will show that $\mathbf{\Gamma}$ and $\tilde{\mathbf{\Gamma}}$ are equivalent parameter sequences. Since $S_{\mathbf{\Gamma}}$ and $S_{\tilde{\mathbf{\Gamma}}}$ are reduced operator systems, Theorem 2.5 implies that there is a $*$-isomorphism $\theta : C^*(S_{\mathbf{\Gamma}}) \to C^*(S_{\tilde{\mathbf{\Gamma}}})$ such that $\theta(S_{\mathbf{\Gamma}}) = S_{\tilde{\mathbf{\Gamma}}}$. After consideration of the central decompositions of these C^*-algebras into full matrix algebras

$$C^*(S_{\mathbf{\Gamma}}) = A_1 \oplus \cdots \oplus A_n, \qquad C^*(S_{\tilde{\mathbf{\Gamma}}}) = \tilde{A}_1 \oplus \cdots \oplus A_N,$$

$A_k \cong \mathcal{B}(H_k)$, $\tilde{A}_k \cong \mathcal{B}(\tilde{H}_k)$, it follows that there is a permutation σ of $\{1, 2, \dots, N\}$ such that $\theta(A_k) = \tilde{A}_{\sigma(k)}$, $k = 1, \dots, N$. For each k, the restriction of θ to $A_k \cong \mathcal{B}(H_k)$ can be viewed as a $*$-isomorphism of $\mathcal{B}(H_k)$ onto $\mathcal{B}(\tilde{H}_{\sigma(k)})$, which is implemented by a unitary operator $U_k : H_k \to \tilde{H}_{\sigma(k)}$

$$\theta(a) = U_k a U_k^{-1}, \qquad a \in A_k, \quad k = 1, \dots, N.$$

We conclude that

$$\begin{aligned}\theta(S_{\mathbf{\Gamma}}) &= \{U_1 \Gamma_1(z) U_1^{-1} \oplus \cdots \oplus U_N \Gamma_N(z) U_N^{-1} : z \in Z\} \\ &= S_{\tilde{\mathbf{\Gamma}}} = \{\tilde{\Gamma}_{\sigma(1)}(z) \oplus \cdots \oplus \tilde{\Gamma}_{\sigma(N)}(z) : z \in Z\}.\end{aligned}$$

Since $\ker\Gamma_1 \cap \cdots \cap \ker\Gamma_N = \ker\tilde{\Gamma}_1 \cap \cdots \cap \ker\tilde{\Gamma}_N = \{0\}$, the equality of these two spaces of operators implies that for every $z \in Z$ there is a unique vector $\alpha(z) \in Z$ such that

$$\tilde{\Gamma}_{\sigma(k)}(\alpha(z)) = U_k\Gamma_k(z)U_k^{-1}, \qquad z \in Z, \quad k = 1, \ldots, N. \tag{5.1}$$

Moreover, since all of the maps Γ_k and $\tilde{\Gamma}_k$ are complex linear, preserve the $*$ operation and map the unit of Z to the corresponding identity operator, (5.1) implies that α must in fact be a unit preserving automorphism of the $*$-vector space structure of Z. Hence (5.1) shows that the two parameter sequences $\mathbf{\Gamma}$ and $\tilde{\mathbf{\Gamma}}$ are equivalent.

The proof of the converse is a straightforward reversal of this argument.

6. Structure of nonreduced operator systems

We conclude with a discussion of how one constructs nonreduced operator systems using these methods. We will outline the construction – giving precise definitions but no proofs – sketching the proof of only a single key lemma to illustrate the technique.

Let S be an operator system that generates a finite dimensional C^*-algebra $C^*(S)$, let K be the boundary ideal for S, let $\dot{S} \subseteq C^*(S)/K$ be the corresponding reduced operator system in the C^*-envelope of S, and consider the central decomposition of $C^*(S)$ into factors

$$C^*(S) = A_1 \oplus \cdots \oplus A_s, \qquad A_k \cong \mathscr{B}(H_k).$$

A complete list of irreducible representations $\pi_k : C^*(S) \to \mathscr{B}(H_k)$ of $C^*(S)$ is associated with the minimal central projections. Some of these representations are boundary representations and others are not. Let us relabel so as to collect the boundary representations together with the first N of the summands $A_1, \ldots, A_N$ and the others as the remaining $n - N = M$ terms $A_{N+1}, \ldots, A_{N+M}$. It follows that the boundary ideal is $\ker\pi_1 \cap \cdots \cap \ker\pi_N$,

$$K = 0 \oplus \cdots \oplus 0 \oplus A_{N+1} \oplus \cdots \oplus A_{N+M},$$

and the quotient $C^*(S)/K$ is, in this finite dimensional setting, isomorphic to the remaining summand

$$C^*(S)/K \cong A_1 \oplus \cdots \oplus A_N \oplus 0 \oplus \cdots \oplus 0,$$

the quotient map being identified with the obvious homomorphism of $C^*(S)$ on this initial segment. We conclude from these observations that the most general nonreduced operator system is obtained in the following way: Let

$A \oplus K$ be a direct sum of two finite dimensional C^*-algebras, and let S be a linear subspace of $A \oplus K$ with the following properties:

(i) $A \oplus K$ is the C^*-algebra generated by S.

(ii) Every irreducible representation of A is a boundary representation for S.

(iii) No irreducible representation of K is a boundary representation for S.

Then as we have argued above, the boundary ideal for S is K and A is identified with the C^*-envelope of S.

Of course, we have not spelled out explicitly how one constructs all such configurations, but it is not hard to do so. To sketch the details, let $Z = \mathbb{C}^d$ be a unital $*$-vector space as in Section 1, and let

$$\mathbf{\Gamma} = (\Gamma_1, \dots, \Gamma_N), \qquad \mathbf{\Omega} = (\Omega_1, \dots, \Omega_M)$$

be two tuples of unit-preserving $*$-preserving linear maps from the parameter space Z to $\mathcal{B}(H_1), \dots, \mathcal{B}(H_N)$ and $\mathcal{B}(K_1), \dots, \mathcal{B}(K_M)$ respectively, such that the range of each of the $M + N$ maps is an irreducible operator system. One can use $\mathbf{\Gamma} = (\Gamma_1, \dots, \Gamma_N)$ to construct the C^*-envelope of an operator system and $\mathbf{\Omega} = (\Omega_1, \dots, \Omega_M)$ to construct its boundary ideal in the following way. The required properties are:

(a) (Subordination of $\mathbf{\Omega}$ to $\mathbf{\Gamma}$) For every $r = 1, \dots, M$, every $p = 1, 2, \dots$, and every $p \times p$ matrix (z_{ij}) over Z,

$$\|(\Omega_r(z_{ij}))\| \le \max_{1 \le k \le N} \|(\Gamma_k(z_{ij}))\|.$$

(b) (Strong separation in the components of $\mathbf{\Gamma}$) For every $k = 1, \dots, N$, there is a $p = p(k) = 1, 2, \dots$ and a $p \times p$ matrix $(z_{ij}) = (z_{ij}(k))$ with components in Z such that

$$\|(\Gamma_k(z_{ij}))\| > \max_{l \ne k} \|(\Gamma_l(z_{ij}))\|.$$

(c) (Weak separation in all components) For every $1 \le r < s \le M$, there is a $p = p(r, s) = 1, 2, \dots$ and a $p \times p$ matrix $(z_{ij}) = (z_{ij}(r, s))$ over Z such that

$$\|(\Omega_r(z_{ij}))\| \ne \|(\Omega_s(z_{ij}))\|,$$

and for every $1 \le r \le M$ and every $1 \le k \le N$, there is a $p = p(r, k) = 1, 2, \dots$ and a $p \times p$ matrix $(z_{ij}) = (z_{ij}(r, k))$ over Z such that

$$\|(\Omega_r(z_{ij}))\| \ne \|(\Gamma_k(z_{ij}))\|,$$

We can assemble the component maps of $\mathbf{\Gamma}$ and $\mathbf{\Omega}$ to define an operator system (6.1). The following result shows that the C^*-algebra generated by this operator system is the expected direct sum of matrix algebras:

LEMMA 6.1. *Let* $\mathbf{\Gamma} = (\Gamma_1, \dots, \Gamma_N)$ *and* $\mathbf{\Omega} = (\Omega_1, \dots, \Omega_M)$ *be two sequences of self adjoint unit preserving linear maps*

$$\Gamma_k : Z \to \mathscr{B}(H_k), \qquad \Omega_r : Z \to \mathscr{B}(K_r)$$

such that each $\Gamma_k(Z)$ *and each* $\Omega_r(Z)$ *is an irreducible operator system, and which together satisfy the weak separation property (c). Let* $S_{\mathbf{\Gamma},\mathbf{\Omega}}$ *be the associated operator system in* $\mathscr{B}(H_1 \oplus \cdots \oplus H_N \oplus K_1 \oplus \cdots \oplus K_M)$ *defined by*

$$S_{\mathbf{\Gamma},\mathbf{\Omega}} = \{\Gamma_1(z) \oplus \cdots \oplus \Gamma_N(z) \oplus \Omega_1(z) \oplus \cdots \oplus \Omega_M(z) : z \in Z\}. \tag{6.1}$$

Then $C^*(S_{\mathbf{\Gamma},\mathbf{\Omega}}) = \mathscr{B}(H_1) \oplus \cdots \oplus \mathscr{B}(H_M) \oplus \mathscr{B}(K_1) \oplus \cdots \oplus \mathscr{B}(K_M)$.

SKETCH OF PROOF. For every $r = 1, \dots, M + N$, let π_r be the representation of $C^*(S_{\mathbf{\Gamma},\mathbf{\Omega}})$ defined by

$$\pi_r(x_1 \oplus \cdots \oplus x_{M+N}) = x_r, \qquad 1 \le r \le M + N. \tag{6.2}$$

π_r is an irreducible representation because its range contains the irreducible operator system $\Gamma_r(Z)$ if $1 \le r \le N$ or $\Omega_{r-N}(Z)$ if $N < r \le M + N$. The hypothesis (c) implies that $\pi_1, \dots, \pi_{M+N}$ are mutually inequivalent, so by irreducibility, they are mutually disjoint. It follows that for every r, the projection corresponding to the rth summand of the C^*-algebra

$$\mathscr{B}(H_1) \oplus \cdots \oplus \mathscr{B}(H_N) \oplus \mathscr{B}(K_1) \oplus \cdots \oplus \mathscr{B}(K_M)$$

belongs to the center of $C^*(S_{\mathbf{\Gamma},\mathbf{\Omega}})$. The assertion is now immediate.

Now assume that properties (a), (b) and (c) are satisfied, let $S_{\mathbf{\Gamma},\mathbf{\Omega}}$ be the operator system (6.1) and let A and K be the summands of $C^*(S_{\mathbf{\Gamma},\mathbf{\Omega}})$ defined by

$$A = \mathscr{B}(H_1) \oplus \cdots \oplus \mathscr{B}(H_N) \oplus 0 \oplus \cdots \oplus 0$$

$$K = 0 \oplus \cdots \oplus 0 \oplus \mathscr{B}(K_1) \oplus \cdots \oplus \mathscr{B}(K_M),$$

and let $\pi_1, \dots, \pi_{M+N}$ be the list of irreducible representations (6.2). Then minor variations of arguments already given show that $\pi_1, \dots, \pi_N$ are all peaking representations for $S_{\mathbf{\Gamma},\mathbf{\Omega}}$, whereas none of $\{\pi_{N+1}, \dots, \pi_{N+M}\}$ is peaking for $S_{\mathbf{\Gamma},\mathbf{\Omega}}$. It follows from Corollary 2.2 and Theorem 3.2 that K is the boundary ideal for $S_{\mathbf{\Gamma},\mathbf{\Omega}}$ and that A is identified with the C^*-envelope of $S_{\mathbf{\Gamma},\mathbf{\Omega}}$ in the manner described above. Moreover, every (nonreduced) operator system which generates a finite dimensional C^*-algebra is isomorphic to one obtained from the above construction. We omit those details.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA
BERKELEY, CA 94720
U.S.A.
E-mail: arveson@math.berkeley.edu