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A NOTE ON FRACTIONAL INTEGRAL OPERATORS DEFINED BY WEIGHTS AND NON-DOUBLING MEASURES

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Abstract

Given a metric measure space (X, d, μ), a weight w defined on $(0, \infty)$ and a kernel $k_w(x, y)$ satisfying the standard fractional integral type estimates, we study the boundedness of the operators $K_w f(x) = \int_X k_w(x, y) f(y)\, d\mu(y)$ and $\tilde{K}_w f(x) = \int_X (k_w(x, y) - k_w(x_0, y)) f(y)\, d\mu(y)$ on Lebesgue spaces $L^p(\mu)$ and generalized Lipschitz spaces Lip_ϕ, respectively, for certain range of the parameters depending on the n-dimension of μ and some indices associated to the weight w.

1. Introduction

It is well known that a basic assumption in the classical Calderón-Zygmund theory in R^n is the doubling property of the underlying measure space, i.e. $\mu(B(x, 2r)) \le C\mu(B(x, r))$ for all $x \in \mathsf{R}^n$ and $r > 0$. However, it has been recently shown that many results of the theory still hold for general metric spaces X assuming only that $\mu(B(x, r)) \le Cr^n$ for all $x \in X$ and $r > 0$. The reader is referred to [8], [9], [26] for results on vector-valued inequalities and weights and to [13], [19], [34], [35] for results on classical spaces such as H^1 and *BMO* in the setting of non-doubling measures.

The aim of this note is to analyze the boundedness of the fractional integral-type operators defined on non-doubling measure spaces acting on Lebesgue spaces and generalized Lipschitz spaces. This study was initiated in the work of J. García-Cuerva and A. E. Gatto (see [6], [7], [10]) for the classical fractional integral operators and Lipschitz spaces, which had been previously developed in the setting of spaces of homogeneous type in [11], [12]. In this paper we are able to extend some of their results, including weights more general than the potential ones, and to see that a similar theory can be applied to operators defined with kernels more general than the fractional integral ones.

The action of the fractional integral operator

$$I_\alpha(f)(x) = \int_{\mathsf{R}^n} \frac{f(y)}{|x - y|^{n-\alpha}}\, dy$$

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on Hölder spaces goes back to the work of Hardy-Littlewood in [14]. Since then, many different extensions have been considered. Similar results for power weights were proved in [27], [28] and later, extended to other classes of weights, including power-logarithmic type ones, in [21]. On a different direction some development of the theory in the setting of generalized Lipschitz spaces and spaces of homogeneous type was initiated in [17], [18] and continued in [11]. More recently there are several studies of potential operators in generalized Lipschitz that have been initiated (see [4], [16], [31]).

In [22] E. Nakai introduces the "generalized fractional integral"

$$I_\rho(f)(x) = \int_{\mathbb{R}^n} f(y) \frac{\rho(|x-y|)}{|x-y|^n}\, dy$$

for a given $\rho : (0,\infty) \to (0,\infty)$ with certain properties and studies its boundedness properties on Lebesgue spaces. Also in [23] he considers, in the setting of n-homogeneous spaces (X, d, μ) such that $\mu(B(x,r)) \approx r^n$, the operator

$$I_\rho(f)(x) = \int_X f(y) \frac{\rho(d(x,y))}{d(x,y)^n}\, d\mu(y)$$

and extends the boundedness results even to Orlicz spaces.

The reader is referred to further boundedness results of the generalized fractional integrals in other settings to the papers [25], [24], [5].

Our aim is to study these type of operators in the setting of non-doubling measures and to see how the boundedness results in Lebesgue and Lipschitz type spaces can be described in term of certain well-known indices associated to the weight function defining the operators.

Throughout the paper (X, d, μ) will be a metric measure space, that is a metric space (X, d) equipped with a Borel measure μ such that

$$\mu(B(x,r)) \leq Cr^n \tag{1}$$

for every ball $B(x,r) = \{y \in X : d(x,y) < r\}$, where $n > 0$ is some fixed constant and C is independent of x and r. We shall deal, for simplicity, only with the case $\mathrm{diam}(X) = \infty$.

For us a weight w on an interval $I \subset (0,\infty)$ will always be a continuous function $w : I \to (0,\infty)$. We shall use weights defined on $(0,\infty)$ but we shall relate them with the known theory for weights defined on $(0,1]$. Given $w : (0,\infty) \to (0,\infty)$ we denote by $w_0(t) = w(t)$ and $w_\infty(t) = w(1/t)$ for $0 < t \leq 1$.

We consider the indices $m(w)$, $M(w)$, $m_\infty(w)$ and $M_\infty(w)$ introduced by N. G. Samko in the case of weights defined on the finite interval $(0,1]$ (see [29]) or by N. G. et al. in the case $[1,\infty)$ (see [32]) (which actually were motivated by the Matuszewska-Orlicz indices first introduced in [20]). We shall also work in the class of weights $\widetilde{W}$ such that there exists $a, b \in \mathsf{R}$ such that $t^a w(t)$ is almost increasing in $(0, 1]$, $t^b w(t)$ is almost decreasing in $[1, \infty)$ and $-\infty < M(w), m_\infty(w) < +\infty$.

In the paper we shall consider $\mathscr{B}(X) \times \mathscr{B}(X)$-measurable functions $k_w : X \times X \to \mathsf{C}$ that satisfy the following conditions:

$$|k_w(x, y)| \leq C \frac{w(d(x, y))}{d(x, y)^n}, \qquad x, y \in X, x \neq y \tag{2}$$

and there exists $\varepsilon > 0$ such that

$$|k_w(x, z) - k_w(y, z)| \leq C \left(\frac{d(x, y)}{d(x, z)} \right)^\varepsilon \frac{w(d(x, z))}{d(x, z)^n}, \quad d(x, z) \geq 2d(x, y) > 0. \tag{3}$$

This extends the definition of fractional kernels of order α and regularity ε introduced in [6] for $w(t) = t^\alpha$ and also the case I_ρ introduced in [22], [23].

For such kernels we define the operators

$$K_w f(x) = \int_X k_w(x, y) f(y)\, d\mu(y)$$

and

$$\tilde{K}_w f(x) = \int_X (k_w(x, y) - k(x_0, y)) f(y)\, d\mu(y)$$

and study their boundedness on Lebesgue spaces and generalized Lipschitz spaces. As pointed out above operators of such a fashion have been previously considered in [22], [23], [25] in the setting of homogeneous spaces and also there their boundedness in Lebesgue and Orlicz spaces have been studied.

Our considerations are inspired by those developed in the case $w(t) = t^\alpha$ corresponding to the classical fractional integrals. However we will explore the connections between the weight w and the measure μ that still allow the operators K_w and $\tilde{K}_w$ to be well defined for functions in $L^p(\mu)$ and will find the dependence between their boundedness on some spaces and the indices of the weight w. We shall find a Hardy-Littlewood-Sobolev type inequality for K_w in our setting in Theorem 3.2. We will study the boundedness of $\tilde{K}_w$ from $L^p(\mu)$ into Lip_ϕ for $\phi(t) = t^{-n/p} w(t)$ in Theorem 4.6 and from Lip_ϕ into Lip_ψ, where ψ depends on ϕ and w in some special fashion, in Theorem 4.9. Our results recover those obtained in [6] for the fractional integral operator (corresponding to $w(t) = t^\alpha$) and classical Lipschitz classes (corresponding to $\phi(t) = t^\beta$).

The paper is divided into three sections. In the first one we prove the basic lemmas on weights to be used in the paper. Section 3 is devoted to get conditions on the weights for the operator K_w to be defined on $L^p(\mu)$ for some values on p. Section 4 contains the results on $\tilde{K}_w$ and its boundedness on the generalized Lipschitz classes.

As usual $A \approx B$ means that $K^{-1}A \leq B \leq KA$ for some $K > 1$, C denotes a constant that may vary from line to line and p' stands for the conjugate exponent, $1/p + 1/p' = 1$.

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2. Admisible weights

In what follows we shall use the following indices introduced by N. G. Samko for weights defined on (0, 1] (see [29, Def. 2.3], see also [20]) or by N. G. Samko et al. for weights defined on $[1, \infty)$ (see [32, Pag. 566], see also [20]). We write

$$(4) \qquad m(w) = \sup_{x>1} \frac{\log\left(\underline{\lim}_{h\to 0} \frac{w(xh)}{w(h)}\right)}{\log x}, \quad M(w) = \inf_{x>1} \frac{\log\left(\overline{\lim}_{h\to 0} \frac{w(xh)}{w(h)}\right)}{\log x},$$

$$(5) \quad m_\infty(w) = \sup_{x>1} \frac{\log\left(\underline{\lim}_{h\to \infty} \frac{w(xh)}{w(h)}\right)}{\log x}, \quad M_\infty(w) = \inf_{x>1} \frac{\log\left(\overline{\lim}_{h\to \infty} \frac{w(xh)}{w(h)}\right)}{\log x}.$$

DEFINITION 2.1. We shall say that a weight on $(0, \infty)$ belongs to the class $\widetilde{W}$ if there exist $a, b \in \mathbb{R}$ such that $t^a w(t)$ is almost increasing in (0, 1] (i.e. there exists $C \geq 1$ such that $t^a w(t) \leq C s^a w(s)$ for $0 < t \leq s \leq 1$), $t^b w(t)$ is almost decreasing in $[1, \infty)$ (i.e. there exists $C \geq 1$ such that $s^b w(s) \leq C t^b w(t)$ for $1 \leq t \leq s < \infty$) and $-\infty < M(w), m_\infty(w) < +\infty$.

For a weight $w \in \widetilde{W}$, we use the notation $m_w = \min\{m(w), m_\infty(w)\}$ and $M_w = \max\{M(w), M_\infty(w)\}$.

DEFINITION 2.2. Given $-\infty < \sigma_1, \sigma_2 < \infty$, we say that a weight w on $(0, \infty)$ belongs to $\Delta(\sigma_1, \sigma_2)$ if $t^{\sigma_1} w(t)$ is almost increasing in $(0, \infty)$ and $t^{\sigma_2} w(t)$ is almost decreasing in $(0, \infty)$.

REMARK 2.3. Observe that if $w \in \Delta(\sigma_1, \sigma_2)$ then there exists $C \geq 1$ such that, for $0 < s < \infty$,

$$(6) \qquad C^{-1} x^{-\sigma_2} w(s) \leq w(xs) \leq C x^{-\sigma_1} w(s), \qquad 0 < x \leq 1,$$

$$(7) \qquad C^{-1} x^{-\sigma_1} w(s) \leq w(xs) \leq C x^{-\sigma_2} w(s), \qquad 1 \leq x.$$

Hence it follows immediately that if $w \in \Delta(\sigma_1, \sigma_2)$ then $\sigma_2 \leq \sigma_1$.

Our first objective is to show that the class $\widetilde{W}$ can be described as $\widetilde{W} = \bigcup_{\sigma_1,\sigma_2} \Delta(\sigma_1, \sigma_2)$.

To such a purpose, let us first recall some classical weights considered by Zygmund, Bari and Stechkin (see [1]) which play an important role in extending results valid for $w(t) = t^\alpha$ to more general weights and that will be connected with our class of weights.

Let $-\infty < \beta, \gamma < \infty$ and let w be a weight on $(0, 1]$. w is said to belong to $\mathscr{Z}^\beta([0, 1])$ if there exists $C > 0$ such that

$$\int_0^h \frac{w(t)}{t^{1+\beta}}\, dt \leq C \frac{w(h)}{h^\beta}, \qquad h < 1. \tag{8}$$

w is said to belong to $\mathscr{Z}_\gamma([0, 1])$ if there exists $C > 0$ such that

$$\int_h^1 \frac{w(t)}{t^{1+\gamma}}\, dt \leq C \frac{w(h)}{h^\gamma}, \qquad h \leq 1. \tag{9}$$

w is said to belong to $\widetilde{W}_0([0, 1])$ if there exists $a \in \mathsf{R}$ such that

$$t^a u(t) \text{ is almost increasing.} \tag{10}$$

The class of weights in $\mathscr{Z}^\beta([0, 1]) \cap \mathscr{Z}_\gamma([0, 1]) \cap \widetilde{W}_0([0, 1])$ is called the generalized Zygmund-Bari-Stechkin class in [15]. These classes of weights have been used by many authors and under different names (see [2], [3] for the notation d_ϵ and b_δ and references therein).

We have the following connection between the Zygmund-Bari-Steckin classes and the former indices (see [29, Pg. 125], [15, Thm 3.1 and Thm 3.2], [32, Thm 2.4]).

THEOREM 2.4. *Let $w \in \widetilde{W}_0([0, 1])$ and $-\infty < \beta, \gamma < \infty$. The following are equivalent.*

(a) $w \in \mathscr{Z}^\beta([0, 1])$ *(resp.* $w \in \mathscr{Z}_\gamma([0, 1])$*).*

(b) $m(w) > \beta$ *(resp.* $M(w) < \gamma$*).*

(c) *For all $m(w) > \delta > \beta$ one has that $\frac{w(t)}{t^\delta}$ is almost increasing in* $(0, 1]$ (*resp. for all $M(w) < \delta < \gamma$ one has that $\frac{w(t)}{t^\delta}$ is almost decreasing in* $(0, 1]$).

We now collect in the following result several facts which easily follow from the definition and the previously mentioned results.

THEOREM 2.5. *Let w be a weight on $(0, \infty)$. The following are equivalent.*

(i) $w \in \bigcup_{\sigma_1,\sigma_2} \Delta(\sigma_1, \sigma_2)$.

(ii) $w \in \widetilde{W}$.

(iii) *There exist $u, v \in \widetilde{W}_0([0, 1])$ such that $u(1) = v(1)$, $M(u), M(v) \in \mathsf{R}$ and*

$$w(t) = \begin{cases} u(t), & 0 < t \leq 1; \\ v(1/t), & 1 \leq t < \infty. \end{cases}$$

For examples in the class $\widetilde{W}$ we refer to [30].

It is not difficult to see that $m(w) \leq M(w)$ when $w \in \widetilde{W}_0([0, 1])$ (see [30, (2,4)–(2.5)]). Let us mention the following useful result given in terms of the indices previously defined.

PROPOSITION 2.6. *Let $w \in \widetilde{W}$ and $\beta < m_w \leq M_w < \gamma$. Then $w \in \Delta(-\beta_1, -\gamma_1)$ for any $\beta < \beta_1 < m_w$ and $M_w < \gamma_1 < \gamma$.*

PROOF. Using Theorem 2.4 applied to w_0 and w_∞, since $m(w_0) = m(w) > \beta$ and $M(w_\infty) = -m_\infty(w) < -\beta$, we have $t^{-\beta_1} w(t)$ and $t^{\beta_1} w_\infty(t)$ are almost increasing and decreasing in $(0, 1]$ respectively. This shows that $t^{-\beta_1} w(t)$ is almost increasing in $(0, \infty)$.

Similarly we get the corresponding result for γ_1.

We shall start by proving a couple of basic lemmas that will be used in the sequel.

LEMMA 2.7. *Let $w \in \widetilde{W}$ and $\varepsilon \in \mathsf{R}$. Then there exists $C > 0$ such that, for all $x \in X$ and $r > 0$,*

$$\int_{B(x,r)} \frac{w(d(x, y))}{d(x, y)^{n-\varepsilon}}\, d\mu(y) \leq C \int_0^r t^\varepsilon w(t) \frac{dt}{t}. \tag{11}$$

PROOF. Assume $w \in \Delta(\sigma_1, \sigma_2)$. Define, for $j = 0, 1, \ldots,$

$$B_j = \{y \in B(x, r) : 2^{-(j+1)} r \leq d(x, y) < 2^{-j} r\}.$$

Note that (6) gives

$$C^{-1} w(2^{-j} r) \leq w(d(x, y)) \leq C w(2^{-j} r),\ y \in B_j. \tag{12}$$

Observe that $\bigcup_j B_j = B(x,r) \setminus \{x\}$ and $\mu(\{x\}) = 0$. Now, using condition (1), we have

$$\begin{aligned}
\int_{B(x,r)} \frac{w(d(x,y))}{d(x,y)^{n-\varepsilon}}\, d\mu(y) &= \sum_{j=0}^{\infty} \int_{B_j} \frac{w(d(x,y))}{d(x,y)^{n-\varepsilon}}\, d\mu(y) \\
&\approx \sum_{j=0}^{\infty} w(2^{-j}r)(2^{-j}r)^{\varepsilon-n} \int_{B_j} d\mu(y) \\
&\le C \sum_{j=0}^{\infty} w(2^{-j}r)(2^{-j}r)^{\varepsilon-n} \mu(B(x, 2^{-j}r)) \\
&\le C \sum_{j=0}^{\infty} (2^{-j}r)^{\varepsilon} w(2^{-j}r) \\
&\le C \sum_{j=0}^{\infty} (2^{-j}r)^{\varepsilon} \int_{2^{-(j+1)}r}^{2^{-j}r} w(t) \frac{dt}{t} \\
&\le C \sum_{j=0}^{\infty} \int_{2^{-(j+1)}r}^{2^{-j}r} t^{\varepsilon} w(t) \frac{dt}{t} \\
&= C \int_0^r t^{\varepsilon} w(t) \frac{dt}{t}.
\end{aligned}$$

COROLLARY 2.8. *Let $w \in \widetilde{W}$ and $-\varepsilon < m_w$. Then there exists $C > 0$ such that, for all $x \in X$ and $r > 0$,*

$$\int_{B(x,r)} \frac{w(d(x,y))}{d(x,y)^{n-\varepsilon}}\, d\mu(y) \le C r^{\varepsilon} w(r). \tag{13}$$

PROOF. From Proposition 2.6 one obtains $w \in \Delta(\sigma_1, \sigma_2)$ for some $\varepsilon > \sigma_1$. Invoking Lemma 2.7 and using (6) we have

$$\int_0^r t^{\varepsilon} w(t) \frac{dt}{t} = r^{\varepsilon} \int_0^1 s^{\varepsilon} w(rs) \frac{ds}{s} \le C r^{\varepsilon} w(r) \int_0^1 s^{\varepsilon-\sigma_1} \frac{ds}{s} \le C r^{\varepsilon} w(r).$$

REMARK 2.9. If $\gamma > 0$ and $\beta \in \mathsf{R}$ then (see [6, Lemma 2.1] for $\beta = 0$)

$$\int_{B(x,r)} \frac{\big(1 + |\log(d(x,y))|\big)^{\beta}}{d(x,y)^{n-\gamma}}\, d\mu(y) \le C r^{\gamma} (1 + |\log r|)^{\beta}, \quad 0 < r < \infty. \tag{14}$$

To obtain (14) for $0 < r \leq 1$ apply Corollary 2.8 for $\varepsilon = 0$ to $w(t) = w^{\gamma,\beta}(t)$ which belongs to $\Delta(\sigma_1, \sigma_2)$ whenever $-\sigma_1 < \gamma < -\sigma_2$. The case $r > 1$ follows similarly using $w^{\gamma,-\beta}$.

LEMMA 2.10. *Let $w \in \widetilde{W}$ and $\delta \in \mathbb{R}$. Then there exists $C > 0$ such that, for all $x \in X$ and $r > 0$,*

$$\int_{X\setminus B(x,r)} \frac{w(d(x,y))}{d(x,y)^{n+\delta}}\, d\mu(y) \leq C \int_r^\infty \frac{w(t)}{t^\delta} \frac{dt}{t}. \tag{15}$$

PROOF. Assume again $w \in \Delta(\sigma_1, \sigma_2)$ and now consider for $j = 0, 1, \ldots$

$$A_j = \{y \in X : 2^j r \leq d(x,y) < 2^{j+1} r\}.$$

As above

$$C^{-1} w(2^j r) \leq w(d(x,y)) \leq C w(2^j r), \qquad y \in A_j. \tag{16}$$

Using again (1) we have

$$\begin{aligned}
\int_{X\setminus B(x,r)} \frac{w(d(x,y))}{d(x,y)^{n+\delta}}\, d\mu(y) &= \sum_{j=0}^{\infty} \int_{A_j} \frac{w(d(x,y))}{d(x,y)^{n+\delta}}\, d\mu(y) \\
&\approx C \sum_{j=0}^{\infty} (2^j r)^{-\delta-n} w(2^j r) \int_{A_j} d\mu(y) \\
&\leq C \sum_{j=0}^{\infty} (2^j r)^{-\delta-n} w(2^j r) \mu(B(x, 2^{j+1} r)) \\
&\leq C \sum_{j=0}^{\infty} (2^j r)^{-\delta} w(2^j r) \\
&\approx C \sum_{j=0}^{\infty} (2^j r)^{-\delta} \int_{2^j r}^{2^{j+1} r} w(t) \frac{dt}{t} \\
&\leq C \sum_{j=0}^{\infty} \int_{2^j r}^{2^{j+1} r} \frac{w(t)}{t^\delta} \frac{dt}{t} = C \int_r^\infty \frac{w(t)}{t^\delta} \frac{dt}{t}.
\end{aligned}$$

COROLLARY 2.11. *Let $w \in \widetilde{W}$ and $M_w < \delta$. Then there exists $C > 0$ such that, for all $x \in X$ and $r > 0$,*

$$\int_{X\setminus B(x,r)} \frac{w(d(x,y))}{d(x,y)^{n+\delta}}\, d\mu(y) \leq C \frac{w(r)}{r^\delta}. \tag{17}$$

PROOF. From Proposition 2.6 one obtains $w \in \Delta(\sigma_1, \sigma_2)$ for some $\delta > -\sigma_2$ Invoking Lemma 2.10 and (7) we get the estimate

$$\int_r^\infty \frac{w(t)}{t^\delta}\frac{dt}{t} = \frac{1}{r^\delta}\int_1^\infty \frac{w(rs)}{s^\delta}\frac{ds}{s} \le C\frac{w(r)}{r^\delta}\int_1^\infty s^{-\sigma_2-\delta}\frac{ds}{s} \le C\frac{w(r)}{r^\delta}.$$

REMARK 2.12. If $\gamma > 0$ and $\beta \in \mathsf{R}$ then (see [6, Lemma 2.2] for $\beta = 0$)

(18)
$$\int_{X\setminus B(x,r)} \frac{\big(1+|\log(d(x,y))|\big)^\beta}{d(x,y)^{n+\gamma}}\,d\mu(y) \le C\frac{1}{r^\gamma}(1+|\log r|)^\beta, \quad 0 < r < \infty,$$

To obtain (18) for $0 < r \le 1$ we use Corollary 2.11 with $\delta = 0$ applied to $w^{-\gamma,\beta}$, which belongs to $\Delta(\sigma_1, \sigma_2)$ for $\sigma_2 < \gamma < \sigma_1$. The case $r > 1$ follows similarly using the weight $w^{-\gamma,-\beta}$.

3. The weighted fractional kernels

DEFINITION 3.1. Let $w \in \widetilde{W}$. A $\mathscr{B}(X) \times \mathscr{B}(X)$-measurable function $k_w : X \times X \to \mathsf{C}$ is said to be a *w-fractional kernel* if

$$|k_w(x,y)| \le C\frac{w(d(x,y))}{d(x,y)^n}, \qquad x, y \in X, x \ne y. \tag{19}$$

Denote by K_w the operator given by

$$K_w f(x) = \int_X k_w(x,y) f(y)\,d\mu(y), \qquad x \in X.$$

Note that if $\int_0^1 \frac{w(t)}{t} < \infty$, in particular if $w \in \Delta(\sigma_1, \sigma_2)$ with $\sigma_1 < 0$, then K_w is well defined on bounded functions f with bounded support (due to Lemma 2.7), or if $w \in \widetilde{W}$ and $w(t) \le Ct^n$ for $0 < t < \infty$ then K_w is well defined on integrable functions f.

Let us extend the definition of such operator to more general functions depending on the properties of w.

In [6, Theorem 3.2] it was shown that for $w(t) = t^\alpha$ and $1 \le p < n/\alpha$ the operator K_α maps $L^p(\mu)$ into $L^{q,\infty}(\mu)$ for $1/q = 1/p - \alpha/n$ extending to the non-doubling setting the Hardy-Littlewood-Sobolev inequality which holds for R^n and the Lebesgue measure (see [33]). The reader is referred to [25, Thm 1.3] for the boundedness of I_ρ from L^p into some Orlicz space under certain conditions of ρ and in the setting of Q-homogeneous spaces and to [22, Thm 3.1] for the boundedness of I_ρ from $L^\Phi(\mathsf{R}^n)$ into $L^\Psi(\mathsf{R}^n)$.

Here we present a “weak type” result which can be achieved in the non-doubling setting.

THEOREM 3.2. *Let* $w \in \widetilde{W}$ *with* $0 < m_w \leq M_w < n$ *and let* k_w *be a* w*-fractional kernel. If* $1 \leq p < n/M_w$, $0 < \varepsilon < m_w$ *and* $0 < \delta < n - M_w$ *then there exists* $A > 0$ *such that, for* $1/q_1 = 1/p - (m_w - \varepsilon)/n$ *and* $1/q_2 = 1/p - (M_w + \delta)/n$*, we have for every* f *with* $\|f\|_{L^p(\mu)} = 1$

$$\mu\{x : |K_w(f)(x)| > \lambda\} \leq \frac{C}{\lambda^{q_2}}, \qquad 0 < \lambda \leq A, \tag{20}$$

$$\mu\{x : |K_w(f)(x)| > \lambda\} \leq \frac{C}{\lambda^{q_1}}, \qquad \lambda \geq A. \tag{21}$$

PROOF. From Proposition 2.6 we have $w \in \Delta(\sigma_1, \sigma_2)$ for all $0 < -\sigma_1 < m_w \leq M_w < -\sigma_2 < n$. Put $\sigma_1 = \varepsilon - m_w$ and $\sigma_2 = -M_w - \delta$. Now, let $1 < p < n/M_w$, $f \in L^p(\mu)$ and $r > 0$ and define

$$I_r(f, x) = \int_{B(x,r)} |K_w(x, y)||f(y)|\, d\mu(y), \qquad x \in X,$$

$$II_r(f, x) = \int_{X \setminus B(x,r)} |K_w(x, y)||f(y)|\, d\mu(y), \qquad x \in X.$$

On the one hand, using Hölder's inequality and Lemma 2.7, we have

$$\begin{aligned}
&I_r(f, x)\\
&= \int_{B(x,r)} |K_w(x, y)||f(y)|\, d\mu(y)\\
&\leq C \int_{B(x,r)} \frac{w(d(x, y))}{d(x, y)^n} |f(y)|\, d\mu(y)\\
&\leq C \left(\int_{B(x,r)} \frac{w(d(x, y))}{d(x, y)^n} |f(y)|^p\, d\mu(y) \right)^{1/p} \left(\int_{B(x,r)} \frac{w(d(x, y))}{d(x, y)^n}\, d\mu(y) \right)^{1/p'}.
\end{aligned}$$

Now, using that $m_w > 0$ in Corollary 2.8, we obtain

$$I_r(f, x) \leq C w(r)^{1/p'} \left(\int_{B(x,r)} \frac{w(d(x, y))}{d(x, y)^n} |f(y)|^p\, d\mu(y) \right)^{1/p} \tag{22}$$

Now, using Fubini's theorem and Corollary 2.8 again, we have

$$\begin{aligned}
&\int_X I_r(f, x)^p\, d\mu(x)\\
&\qquad \leq C w(r)^{p/p'} \int_X \left(\int_{B(y,r)} \frac{w(d(x, y))}{d(x, y)^n}\, d\mu(x) \right) |f(y)|^p\, d\mu(y)\\
&\qquad \leq C w(r)^p \int_X |f(y)|^p d\mu(y).
\end{aligned}$$

On the one hand

$$
\begin{aligned}
&II_r(f,x)\\
&\quad = \int_{X\setminus B(x,r)} |K_w(x,y)||f(y)|\,d\mu(y)\\
&\quad \le C\int_{X\setminus B(x,r)} \frac{w(d(x,y))}{d(x,y)^n}|f(y)|\,d\mu(y)\\
&\quad \le C\biggl(\int_{X\setminus B(x,r)} |f(y)|^p\,d\mu(y)\biggr)^{1/p}\biggl(\int_{X\setminus B(x,r)} \frac{w^{p'}(d(x,y))}{d(x,y)^{np'}}\,d\mu(y)\biggr)^{1/p'}.
\end{aligned}
$$

and now using that $M_{w^{p'}} = p'M_w < (p'-1)n$ and Corollary 2.11, we have

$$
II_r(f,x) \le Cr^{-n/p}w(r)\biggl(\int_{X\setminus B(x,r)} |f(y)|^p\,d\mu(y)\biggr)^{1/p}.
$$

Now, for each $\|f\|_p = 1$, the estimates (6) and (7) allow us to write

$$
II_r(f,x) \le C_0 r^{-n/p}\max\{r^{-\sigma_1}, r^{-\sigma_2}\} = \phi(r).
$$

Denoting

$$
\phi(r) = \begin{cases} C_0 r^{-n/p-\sigma_1}, & 0 < r \le 1;\\ C_0 r^{-n/p-\sigma_2}, & 1 \le r < \infty, \end{cases}
$$

we have that ϕ is continuous, decreasing in $(0,\infty)$, $\lim_{r\to 0}\phi(r) = \infty$ and $\lim_{r\to\infty}\phi(r) = 0$. Hence for any $\lambda > 0$ there is a unique $0 < r < \infty$ such that $\phi(r) = \lambda/2$ and $II_r(f,x) \le \lambda/2$ for all $x \in X$. Hence we have

$$
\begin{aligned}
\mu\{x : |K_w(f)(x)| > \lambda\} &\le \mu\{x : I_r(f,x) > \lambda/2\}\\
&\le C\lambda^{-p}\|I_r(f,.)\|_p^p\\
&\le C\lambda^{-p}w(r)^p\\
&\le C\lambda^{-p}r^n\phi(r)^p\\
&= C[\phi^{-1}(\lambda/2)]^n.
\end{aligned}
$$

To finish the proof observe that if $\lambda \ge 2C_0$ then $\phi^{-1}(\lambda/2) = C_1\lambda^{-q_1/n}$ where $n/q_1 = n/p + \sigma_1$ and that if $0 < \lambda \le 2C_0$ then $\phi^{-1}(\lambda/2) = C_2\lambda^{-q_2/n}$ where $n/q_2 = n/p + \sigma_2$.

The case $p = 1$ is similar with the obvious modifications.

4. Boundedness in Lipschitz spaces

DEFINITION 4.1. Let $\phi : (0, \infty) \to (0, \infty)$ be a continuous function. A function $f : X \to \mathbb{C}$ is said to satisfy a ϕ-Lipschitz condition if

$$(23) \qquad |f(x) - f(y)| \le C\phi(d(x, y)), \qquad x, y \in X, \quad x \neq y.$$

The smallest constant satisfying (23) will be denoted $\|f\|_{\mathrm{Lip}(\phi)}$. It is easy to see that $\|\cdot\|_{\mathrm{Lip}(\phi)}$ is a norm on the linear space of all ϕ-Lipschitz functions, modulo constants, and $\mathrm{Lip}(\phi)$ is complete under this norm.

REMARK 4.2. If $\lim_{t\to 0^+} \phi(t) = 0$ then functions in Lip_ϕ are continuous.

REMARK 4.3. Assume that there exist constants $C > 1$ and $K > 1$ so that $K^{-1}\phi(t) \le \phi(s) \le K\phi(t)$ whenever $C^{-1}t \le s \le Ct$. In this case $\mathrm{Lip}(\phi)$ defines the same space for all equivalent distances in X and with equivalent norms.

DEFINITION 4.4. Let k_w be a w-fractional kernel. We say that k_w has regularity $\varepsilon > 0$ if it satisfies

$$(24) \quad |k_w(x, z) - k_w(y, z)| \le C\left(\frac{d(x, y)}{d(x, z)}\right)^{\varepsilon} \frac{w(d(x, z))}{d(x, z)^n}, \qquad d(x, z) \ge 2d(x, y) > 0.$$

For a given $x_0 \in X$ define

$$(25) \qquad \tilde{K}_w f(x) = \int_X \big(k_w(x, y) - k_w(x_0, y)\big) f(y)\, d\mu(y).$$

Note that, from Lemma 2.10, if f is bounded with $\mathrm{supp}(f) \cap B(x_0, 2R) = \emptyset$ then $K_w f(x)$ is well defined for any $x \in B(x_0, R)$.

EXAMPLE 4.5. Let $k_w(x, y) = \frac{w(d(x,y))}{d(x,y)^n}$ where $w \in \widetilde{W}$ is differentiable and

$$\sup_{t>0} \left| \frac{tw'(t)}{w(t)} - n \right| < \infty.$$

Then k_w has regularity 1.

PROOF. Consider $w_1(t) = \frac{w(t)}{t^n}$. By the mean value theorem

$$|w_1(t) - w_1(s)| \le |w_1'((1 - \theta)s + \theta t)||t - s|.$$

Hence, setting $t(\theta, x, y, z) = t_0 = (1-\theta)d(x,z) + \theta d(y,z)$ then

$$\begin{aligned} |k_w(x,z) - k_w(y,z)| &\le |w_1'(t_0)||d(x,z) - d(y,z)| \\ &\le \frac{|t_0 w'(t_0) - n w(t_0)|}{t_0^{n+1}} d(x,y) \\ &\le C \frac{w(t_0)}{t_0^{n+1}} d(x,y). \end{aligned}$$

Let $x, y, z \in X$ such that $d(x,z) \ge 2d(x,y)$, i.e. $d(x,z) - d(x,y) \ge d(x,y)$. It is elementary to see that

$$\frac{3}{2} d(x,z) \ge d(y,z) \ge \frac{1}{2} d(x,z) \ge d(x,y).$$

This shows that

$$\frac{1}{2} d(x,z) \le t(\theta, x, y, z) \le \frac{3}{2} d(x,z),$$

and allows to conclude that

$$|k_w(x,z) - k_w(y,z)| \le C \frac{w(d(x,z))}{d(x,z)^{n+1}} d(x,y).$$

THEOREM 4.6. *Let $w \in \widetilde{W}$ with $m_w > 0$. Assume that k_w is a w-fractional kernel with regularity $0 < \varepsilon < M_w$ and*

$$\max\{n/m_w, 1\} < p < n/(M_w - \varepsilon).$$

Then $\tilde{K}_w$ is bounded from $L^p(\mu)$ to $\mathrm{Lip}(\phi)$ for $\phi(t) = t^{-n/p} w(t)$.

PROOF. We have $n/p < m_w \le M_w < n/p + \varepsilon$. Let $f \in L^p(\mu)$, $x, y \in X$ with $x \ne y$ and $r = d(x,y)$. Then

$$\begin{aligned} |\tilde{K}_w f(x) - \tilde{K}_w f(y)| &\le \int_X |k_w(x,z) - k_w(y,z)||f(z)|\, d\mu(z) \\ &\le \int_{B(x,2r)} |k_w(x,z)||f(z)|\, d\mu(z) \\ &\quad + \int_{B(x,2r)} |k_w(y,z)||f(z)|\, d\mu(z) \\ &\quad + \int_{X \setminus B(x,2r)} |k_w(x,z) - k_w(y,z)||f(z)|\, d\mu(z). \end{aligned}$$

First, using Hölder's inequality and Corollary 2.8 (because $m_{w^{p'}} = p'm_w > n(p'-1)$), we estimate

$$
\begin{aligned}
\int_{B(x,2r)} &|k_w(x,z)||f(z)|\,d\mu(z) \\
&\le C\int_{B(x,2r)} \frac{w(d(x,z))}{d(x,z)^n}|f(z)|\,d\mu(z) \\
&\le C\left(\int_{B(x,2r)} \frac{w^{p'}(d(x,z))}{d(x,z)^{np'}}\,d\mu(z)\right)^{1/p'}\left(\int_{B(x,2r)} |f(z)|^p\,d\mu(z)\right)^{1/p} \\
&\le C\frac{w(2r)}{r^{n/p}}\|f\|_{L^p(\mu)}.
\end{aligned}
$$

The second term is estimated similarly using $B(x,2r)\subset B(y,3r)$,

$$
\int_{B(x,2r)} |k_w(y,z)||f(z)|\,d\mu(z) \le C\frac{w(3r)}{r^{n/p}}\|f\|_{L^p(\mu)}.
$$

Finally we use (24) and Corollary 2.11 (since $M_{w^{p'}} = p'M_w < n(p'-1)+\varepsilon p'$) to obtain

$$
\begin{aligned}
\int_{X\setminus B(x,2r)} &|k_w(x,z)-k_w(y,z)||f(z)|\,d\mu(z) \\
&\le Cd(x,y)^{\varepsilon}\int_{X\setminus B(x,2r)} \frac{w(d(x,z))}{d(x,z)^{n+\varepsilon}}|f(z)|\,d\mu(z) \\
&\le Cd(x,y)^{\varepsilon}\left(\int_{X\setminus B(x,2r)} \frac{w^{p'}(d(x,z))}{d(x,z)^{(n+\varepsilon)p'}}\,d\mu(z)\right)^{1/p'} \\
&\qquad\cdot\left(\int_{X\setminus B(x,2r)} |f(z)|^p\,d\mu(z)\right)^{1/p} \\
&\le C\frac{w(2r)}{r^{n/p}}\|f\|_{L^p(\mu)}.
\end{aligned}
$$

Therefore, using that $w(r)\approx w(2r)\approx w(3r)$ and $r=d(x,y)$ one gets

$$
|\tilde{K}_w f(x) - \tilde{K}_w f(y)| \le C\frac{w(d(x,y))}{d(x,y)^{n/p}}\|f\|_p.
$$

We write k_α for k_w in the case $w=t^\alpha$.

COROLLARY 4.7 (See [6, Theorem 5.2]). *Let $0 < \alpha < n$ and k_α be a w-fractional kernel with regularity $\varepsilon > 0$. If $n/\alpha < p \leq \infty$ and $\alpha - n/p < \varepsilon$, then $\tilde{K}_\alpha$ maps boundedly $L^p(\mu)$ into* $\mathrm{Lip}(\alpha - n/p)$.

REMARK 4.8. The reader is referred to [22, Thm 3.3] for similar result for I_ρ and even its extension to Orlicz spaces.

Let us now analyze the boundedness of $\tilde{K}_w$ on Lipschitz spaces.

THEOREM 4.9. *Assume that $u, w \in \widetilde{W}$ with $m_w > 0$, $m_u > 0$ and $M_{uw} < \varepsilon$. Let k_w be a w-fractional kernel with regularity ε. Then $\tilde{K}_w(1) = 0$ if and only if $\tilde{K}_w$ maps continuously* $\mathrm{Lip}(u)$ *into* $\mathrm{Lip}(uw)$.

PROOF. Assume $\tilde{K}_w(1) = 0$. Equivalently

$$\int_X (k_w(x,z) - k_w(y,z))\, d\mu(z) = 0, \qquad x, y \in X.$$

If $f \in \mathrm{Lip}(u)$, $x \neq y$ and $r = d(x,y)$ then we can write

$$\begin{aligned}
|\tilde{K}_w f(x) - \tilde{K}_w(y)| &= \left| \int_X (k_w(x,z) - k_w(y,z))(f(z) - f(x))\, d\mu(z) \right| \\
&\leq \int_{B(x,2r)} |k_w(x,z)||f(z) - f(x)|\, d\mu(z) \\
&\quad + \int_{B(x,2r)} |k_w(y,z)||f(z) - f(x)|\, d\mu(z) \\
&\quad + \int_{X \setminus B(x,2r)} |k_w(x,z) - k_w(y,z)||f(z) - f(x)|\, d\mu(z).
\end{aligned}$$

Now, since $m_{uw} > 0$ (see Proposition 2.6), one gets

$$\begin{aligned}
&\int_{B(x,2r)} |k_w(x,z)||f(z) - f(x)|\, d\mu(z) \\
&\qquad \leq C \int_{B(x,2r)} \frac{w(d(x,z))}{d(x,z)^n} u(d(x,z))\, d\mu(z) \\
&\qquad \leq C u(2r) w(2r)
\end{aligned}$$

by virtue of Corollary 2.11.

Using, as above, the fact that $B(x,2r) \subset B(y,3r)$ one also gets

$$\begin{aligned}
&\int_{B(x,2r)} |k_w(y,z)||f(z)-f(x)|\,d\mu(z)\\
&\quad\le \int_{B(y,3r)} |k_w(y,z)|(|f(z)-f(y)|+|f(y)-f(x)|)\,d\mu(z)\\
&\quad\le C\int_{B(y,3r)} \frac{w(d(y,z))}{d(y,z)^n}u(d(y,z))\,d\mu(z)\\
&\qquad + Cu(d(x,y))\int_{B(y,3r)} \frac{w(d(y,z))}{d(y,z)^n}\,d\mu(z).
\end{aligned}$$

Since $w(3t) \approx w(2t) \approx w(t)$ and $u(3t) \approx u(2t) \approx u(t)$, Corollary 2.8 implies that

$$\int_{B(y,3r)} \frac{w(d(y,z))u(d(y,z))}{d(y,z)^n}\,d\mu(z) + u(d(x,y))\int_{B(y,3r)} \frac{w(d(y,z))}{d(y,z)^n}\,d\mu(z) \le Cu(r)w(r).$$

Finally, we have

$$\begin{aligned}
&\int_{X\setminus B(x,2r)} |k_w(x,z)-k_w(y,z)||f(z)-f(x)|\,d\mu(z)\\
&\qquad \le Cd(x,y)^\varepsilon \int_{X\setminus B(x,2r)} \frac{w(d(x,z))}{d(x,z)^{n+\varepsilon}}u(d(x,z))\,d\mu(z).
\end{aligned}$$

Also using Corollary 2.11 we have $\int_{X\setminus B(x,2r)} \frac{w(d(x,z))u(d(x,z))}{d(x,z)^{n+\varepsilon}}\,d\mu(z) \le C\frac{w(2r)u(2r)}{r^\varepsilon}$. Hence, the previous estimates imply

$$|\tilde{K}_w f(x) - \tilde{K}_w f(x)| \le Cu(r)w(r).$$

Conversely, if we assume that $\tilde{K}_w$ is bounded from $\mathrm{Lip}(u)$ to $\mathrm{Lip}(uw)$ then $\tilde{K}(1)$ should have norm zero in $\mathrm{Lip}(uw)$, that is $\tilde{K}(1)$ is constant, but since $\tilde{K}_w(1)(x_0) = 0$ the constant should be zero.

Applying the previous result for $w(t) = t^\alpha$ and $u(t) = t^\beta$ we recover the following theorem.

Corollary 4.10 (See [6, Theorem 5.3]). *Let $\alpha, \beta > 0$ and k_α be a fractional kernel with regularity $\varepsilon > 0$ with $\alpha+\beta < \varepsilon$. Then $\tilde{K}_\alpha$ maps boundedly* $\mathrm{Lip}(\beta)$ *into* $\mathrm{Lip}(\alpha+\beta)$ *if and only if $\tilde{K}_\alpha(1) = 0$.*

REMARK 4.11. The reader is referred to [22, Thm 3.4] and [24, Thm 3.6] for similar results for $\tilde{I}_\rho$ and even its extension to Orlicz spaces, where

$$\tilde{I}_\rho(f)(x) = \int_{\mathsf{R}^n} f(y)\left(\frac{\rho(|x-y|)}{|x-y|^n} - \frac{\rho(y)(1-\chi_{B_0}(y))}{|y|^n}\right) dy.$$

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