

UNIFORM APPROXIMATION ON IDEALS OF MULTILINEAR MAPPINGS

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Abstract

For each ideal of multilinear mappings \mathcal{M} we explicitly construct a corresponding ideal ${}^a\mathcal{M}$ such that multilinear forms in ${}^a\mathcal{M}$ are exactly those which can be approximated, in the uniform norm, by multilinear forms in \mathcal{M} . This construction is then applied to finite type, compact, weakly compact and absolutely summing multilinear mappings. It is also proved that the correspondence $\mathcal{M} \mapsto {}^a\mathcal{M}$ is Aron-Berner stability preserving.

Introduction

The theory of ideals of multilinear mappings (multi-ideals) between Banach spaces (see [6], [7], [8], [9], [10], [11], [16], [17], [18], [19], [24], [25] and references therein) studies, as in the theory of linear operator ideals, plenty of non-closed ideals. So the question of describing their closures in the uniform norm is quite natural. Our main aim is to construct an approximation scheme for multi-ideals similar to the H. Jarchow and A. Pełczyński description of the closed injective hull of an operator ideal, which can be found in [21, Section 20.7].

Given a multi-ideal \mathcal{M} , we construct a corresponding multi-ideal ${}^a\mathcal{M}$ such that every multilinear form in ${}^a\mathcal{M}$ can be approximated, in the uniform norm, by multilinear forms in \mathcal{M} . For multilinear mappings taking values in a Banach space F the approximation takes place, like in the linear case, in the larger space $\ell_\infty(B_{F^*})$. We do not only prove the existence of such multi-ideal ${}^a\mathcal{M}$, we give it an explicit description that roughly speaking means that a multilinear mapping A can be approximated by elements in \mathcal{M} if the norm of any sum of images of A is almost “dominated” by the norm of the sum of the corresponding images by some element in \mathcal{M} , (see Definition 2.1). Relying on such description we obtain several properties of ${}^a\mathcal{M}$. This is done in Section 2.

It should be noted that, as usual, there are several *a priori* possibilities to transpose the Jarchow-Pełczyński construction to the multilinear case. The dir-

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ect and most obvious transposition simply does not work. So an important step was to identify, among all possible multilinear generalizations of the Jarchow-Pełczyński construction, the one that performs the desired approximation. The results we prove show that we have selected the correct definition.

The rest of paper is organized as follows. Section 1 provides the basic notions and fixes the notation. In Section 3 we apply the approximation theorem to finite type and compact mappings. Denoting the multi-ideal of multilinear mappings of finite type by \mathcal{L}_f , we prove that its corresponding ideal ${}^a(\mathcal{L}_f)$ coincides, modulo the approximation property, with the extensively studied (see [3] or [15] for instance) class of multilinear mappings which are weakly continuous on bounded sets. The case of weakly compact and absolutely summing mappings is studied in Section 4, where we prove multilinear counterparts of some important linear results. In Section 5 we show that the correspondence $\mathcal{M} \mapsto {}^a\mathcal{M}$ preserves the stability of \mathcal{M} with respect to Aron-Berner extensions of multilinear mappings to the bidual spaces; this will extend to the multilinear setting known results about bitranspose linear operators. A contribution to the linear theory is also obtained.

1. Background and notation

Throughout n is a positive integer, $E, E_1, \dots, E_n, F, G_1, \dots, G_n$ and H are (real or complex) Banach spaces. $\mathcal{L}(E; F)$ denotes the Banach space, endowed with the usual sup norm, of bounded linear operators from E to F and $\mathcal{L}(E_1, \dots, E_n; F)$ the Banach space, endowed with the usual sup norm, of continuous n -linear mappings from $E_1 \times \dots \times E_n$ to F . If F is the scalar field we simply write E^* and $\mathcal{L}(E_1, \dots, E_n)$. If $E_1 = \dots = E_n = E$ we write $\mathcal{L}({}^n E; F)$ and $\mathcal{L}({}^n E)$. Linear combinations of mappings of the form $A(x_1, \dots, x_n) = \varphi_1(x_1) \cdots \varphi_n(x_n)b$, where $\varphi_j \in E_j^*$ and $b \in F$, are called *n -linear mappings of finite type*. The space of all such mappings is denoted by $\mathcal{L}_f(E_1, \dots, E_n; F)$. The mappings belonging to its closure $\overline{\mathcal{L}_f}$ are called *approximable*. For each $A \in \mathcal{L}(E_1, \dots, E_n; F)$, we denote by $A_L \in \mathcal{L}(E_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi E_n; F)$ its linearization on the completed n -fold projective tensor product that is defined by

$$A_L(x_1 \otimes \cdots \otimes x_n) = A(x_1, \dots, x_n) \quad \text{for all } x_j \in E_j.$$

For the general theory of multilinear mappings between Banach spaces we refer to S. Dineen [15]

DEFINITION 1.1 (Ideals of multilinear mappings or multi-ideals). *An ideal of n -linear mappings (or n -ideal) \mathcal{M} is a subclass of the class of all continuous n -linear mappings between Banach spaces such that for Banach spaces*

E_1, \dots, E_n and F , the components $\mathcal{M}(E_1, \dots, E_n; F) := \mathcal{L}(E_1, \dots, E_n; F) \cap \mathcal{M}$ satisfy:

- (i) $\mathcal{M}(E_1, \dots, E_n; F)$ is a linear subspace of $\mathcal{L}(E_1, \dots, E_n; F)$ which contains the n -linear mappings of finite type.
- (ii) The ideal property: if $A \in \mathcal{M}(E_1, \dots, E_n; F), u_j \in \mathcal{L}(G_j; E_j)$ for $j = 1, \dots, n$ and $t \in \mathcal{L}(F; H)$, then $t \circ A \circ (u_1, \dots, u_n)$ is in $\mathcal{M}(G_1, \dots, G_n; H)$.

If there is a function $\|\cdot\|_{\mathcal{M}}: \mathcal{M} \rightarrow \mathbb{R}^+$ satisfying

- (i') There is $0 < p \leq 1$ such that $\|\cdot\|_{\mathcal{M}}$ restricted to $\mathcal{M}(E_1, \dots, E_n; F)$ is a p -norm for all Banach spaces E_1, \dots, E_n and F ,
- (ii') $\|A: \mathbb{K}^n \rightarrow \mathbb{K}: A(\lambda_1, \dots, \lambda_n) = \lambda_1 \dots \lambda_n\|_{\mathcal{M}} = 1$ for all n ,
- (iii') If $A \in \mathcal{M}(E_1, \dots, E_n; F), u_j \in \mathcal{L}(G_j; E_j)$ for $j = 1, \dots, n$ and $t \in \mathcal{L}(F; H)$, then $\|t \circ A \circ (u_1, \dots, u_n)\|_{\mathcal{M}} \leq \|t\| \|A\|_{\mathcal{M}} \|u_1\| \dots \|u_n\|$,

then \mathcal{M} is called a *quasi-normed (normed if $p = 1$) n -ideal*. *Quasi-Banach (Banach if $p = 1$) n -ideals* are defined in the obvious way. A *(Banach) multi-ideal* is a sequence $(\mathcal{M}_n)_{n=1}^\infty$ where each \mathcal{M}_n is a (Banach) n -ideal.

When $n = 1$ we recover the classical theory of operator ideals, for which the reader is referred to [13].

A Banach n -ideal \mathcal{M} is said to be *closed* if each component $\mathcal{M}(E_1, \dots, E_n; F)$ is a (sup-norm) closed subspace of $\mathcal{L}(E_1, \dots, E_n; F)$. A Banach multi-ideal $(\mathcal{M}_n)_{n=1}^\infty$ is closed if each \mathcal{M}_n is closed.

An n -ideal \mathcal{M} is *injective* if whenever $A \in \mathcal{L}(E_1, \dots, E_n; F), i: F \rightarrow G$ is an isometric embedding and $i \circ A \in \mathcal{M}(E_1, \dots, E_n; G)$, then $A \in \mathcal{M}(E_1, \dots, E_n; F)$.

2. The approximation scheme

DEFINITION 2.1. Let \mathcal{M} be an n -ideal. A mapping $A \in \mathcal{L}(E_1, \dots, E_n; F)$ is said to be *\mathcal{M} -approximable*, in symbols $A \in {}^a\mathcal{M}(E_1, \dots, E_n; F)$, if there are a Banach space G and an n -linear mapping $B \in \mathcal{M}(E_1, \dots, E_n; G)$ such that for every $\varepsilon > 0$ there is $K_\varepsilon > 0$ such that

$$\left\| \sum_{i=1}^k A(x_i^1, \dots, x_i^n) \right\| \leq K_\varepsilon \left\| \sum_{i=1}^k B(x_i^1, \dots, x_i^n) \right\| + \varepsilon \sum_{i=1}^k \|x_i^1\| \dots \|x_i^n\|,$$

for every $k \in \mathbb{N}$ and any $x_i^j \in E_j, j = 1, \dots, n, i = 1, \dots, k$.

REMARK 2.2. (a) It is obvious from the definition that, for every n -ideal \mathcal{M} , ${}^a\mathcal{M}$ is injective. It is also obvious that ${}^a\mathcal{M} \subseteq {}^a\mathcal{N}$ if $\mathcal{M} \subseteq \mathcal{N}$.

(b) Let us see that ${}^a(\mathcal{M}) = {}^a\mathcal{M}$: it is clear that ${}^a\mathcal{M}$ contains \mathcal{M} , so ${}^a\mathcal{M} \subseteq {}^a({}^a\mathcal{M})$. For the converse, given $A \in {}^a({}^a\mathcal{M})(E_1, \dots, E_n; F)$, there are a Banach space G and an n -linear mapping $B \in {}^a\mathcal{M}(E_1, \dots, E_n; G)$ such that for every $\varepsilon > 0$ there is K_ε such that

$$\left\| \sum_{i=1}^k A(x_i^1, \dots, x_i^n) \right\| \leq K_\varepsilon \left\| \sum_{i=1}^k B(x_i^1, \dots, x_i^n) \right\| + \varepsilon \sum_{i=1}^k \|x_i^1\| \dots \|x_i^n\|,$$

for every $k \in \mathbb{N}$ and any $x_i^j \in E_j$, $j = 1, \dots, n$, $i = 1, \dots, k$. Since $B \in {}^a\mathcal{M}$, there are a Banach space G and an n -linear mapping $B \in {}^a\mathcal{M}(E_1, \dots, E_n; G)$ such that for every $\varepsilon > 0$ there is N_ε such that

$$\left\| \sum_{i=1}^k B(x_i^1, \dots, x_i^n) \right\| \leq N_\varepsilon \left\| \sum_{i=1}^k C(x_i^1, \dots, x_i^n) \right\| + \varepsilon \sum_{i=1}^k \|x_i^1\| \dots \|x_i^n\|,$$

for every $k \in \mathbb{N}$ and any $x_i^j \in E_j$, $j = 1, \dots, n$, $i = 1, \dots, k$. Given $\varepsilon > 0$, one easily checks that for every $k \in \mathbb{N}$ and any $x_i^j \in E_j$, $j = 1, \dots, n$, $i = 1, \dots, k$,

$$\left\| \sum_{i=1}^k A(x_i^1, \dots, x_i^n) \right\| \leq L_\varepsilon \left\| \sum_{i=1}^k C(x_i^1, \dots, x_i^n) \right\| + \varepsilon \sum_{i=1}^k \|x_i^1\| \dots \|x_i^n\|,$$

where L_ε may be chosen as $K_\varepsilon \cdot N_{\frac{\varepsilon}{2}} \cdot N_{\frac{\varepsilon}{2K_\varepsilon}}$. Thus, $A \in {}^a\mathcal{M}(E_1, \dots, E_n; F)$.

(c) For operator ideals, this is the Jarchow-Pełczyński description of the closed injective hull of a given operator ideal (see [21, Theorem 20.7.3]). In particular, when $n = 1$ and \mathcal{M} is the ideal Π_p of absolutely p -summing operators, $1 \leq p < +\infty$, we get ${}^a\mathcal{M} = \mathcal{H}$, where \mathcal{H} is the ideal of absolutely continuous linear operators (see [14, Chapter 15]).

PROPOSITION 2.3. *If \mathcal{M} is a Banach n -ideal, then ${}^a\mathcal{M}$ is a closed injective n -ideal containing \mathcal{M} .*

PROOF. Since ${}^a\mathcal{M}$ contains \mathcal{M} , it also contains the n -linear mappings of finite type.

Given $A_1, A_2 \in {}^a\mathcal{M}(E_1, \dots, E_n; F)$, let $B_j \in \mathcal{M}(E_1, \dots, E_n; G_j)$, $j = 1, 2$, for which the definition holds. Define $G := G_1 \oplus G_2$ and $B \in \mathcal{L}(E_1, \dots, E_n; G)$ by $B(x_1, \dots, x_n) = (B_1(x_1, \dots, x_n), B_2(x_1, \dots, x_n))$. B belongs to \mathcal{M} as $B = i_1 \circ B_1 + i_2 \circ B_2$, where i_j are the canonical inclusions. It is a routine computation to verify that such B fulfills the conditions of the definition for $A_1 + A_2$. Thus $A_1 + A_2 \in {}^a\mathcal{M}(E_1, \dots, E_n; F)$.

The ideal property is easily checked.

Let $(A_j)_{j=1}^\infty \subseteq {}^a\mathcal{M}(E_1, \dots, E_n; F)$, $A_j \rightarrow A \in \mathcal{L}(E_1, \dots, E_n; F)$ in norm. For each $j \in \mathbb{N}$, take G_j and $0 \neq B_j \in \mathcal{M}(E_1, \dots, E_n; G_j)$ associated to A_j according to the definition. Define $G := (\bigoplus_{j=1}^\infty G_j)_1$ and for each j consider the canonical inclusion $i_j: G_j \rightarrow G$. It is clear that each $i_j \circ B_j$ belongs to \mathcal{M} . Since

$$\sum_{j=1}^\infty \frac{1}{2^j \|B_j\|_{\mathcal{M}}} \|i_j \circ B_j\|_{\mathcal{M}} \leq \sum_{j=1}^\infty \frac{1}{2^j \|B_j\|_{\mathcal{M}}} \|i_j\| \|B_j\|_{\mathcal{M}} = \sum_{j=1}^\infty \frac{1}{2^j} < +\infty,$$

and $(\mathcal{M}(E_1, \dots, E_n; G), \|\cdot\|_{\mathcal{M}})$ is a Banach space, the series $\sum_{j=1}^\infty \frac{1}{2^j \|B_j\|_{\mathcal{M}}} i_j \circ B_j$ converges in this space. Call

$$B := \sum_{j=1}^\infty \frac{1}{2^j \|B_j\|_{\mathcal{M}}} i_j \circ B_j \in \mathcal{M}(E_1, \dots, E_n; G).$$

From $\|\cdot\| \leq \|\cdot\|_{\mathcal{M}}$ (see [9, Satz 2.2.5]) it follows that such series is pointwise convergent. Given $\varepsilon > 0$, let $j_0 \in \mathbb{N}$ be such that $\|A_{j_0} - A\| < \frac{\varepsilon}{2}$. Take K such that for every $k \in \mathbb{N}$ and any $x_i^j \in E_j$, $j = 1, \dots, n$, $i = 1, \dots, k$,

$$\left\| \sum_{i=1}^k A_{j_0}(x_i^1, \dots, x_i^n) \right\| \leq K \left\| \sum_{i=1}^k B_{j_0}(x_i^1, \dots, x_i^n) \right\| + \frac{\varepsilon}{2} \sum_{i=1}^k \|x_i^1\| \dots \|x_i^n\|.$$

Then, for every $k \in \mathbb{N}$ and any $x_i^j \in E_j$, $j = 1, \dots, n$, $i = 1, \dots, k$,

$$\begin{aligned} & \left\| \sum_{i=1}^k A(x_i^1, \dots, x_i^n) \right\| \\ & \leq \left\| \sum_{i=1}^k (A - A_{j_0})(x_i^1, \dots, x_i^n) \right\| + \left\| \sum_{i=1}^k A_{j_0}(x_i^1, \dots, x_i^n) \right\| \\ & \leq K \left\| \sum_{i=1}^k B_{j_0}(x_i^1, \dots, x_i^n) \right\| + \varepsilon \sum_{i=1}^k \|x_i^1\| \dots \|x_i^n\| \\ & = K 2^{j_0} \|B_{j_0}\|_{\mathcal{M}} \left\| \sum_{i=1}^k \frac{B_{j_0}(x_i^1, \dots, x_i^n)}{2^{j_0} \|B_{j_0}\|_{\mathcal{M}}} \right\| + \varepsilon \sum_{i=1}^k \|x_i^1\| \dots \|x_i^n\| \\ & \leq K 2^{j_0} \|B_{j_0}\|_{\mathcal{M}} \sum_{j=1}^\infty \left\| \sum_{i=1}^k \frac{B_j(x_i^1, \dots, x_i^n)}{2^j \|B_j\|_{\mathcal{M}}} \right\| + \varepsilon \sum_{i=1}^k \|x_i^1\| \dots \|x_i^n\| \\ & \leq K 2^{j_0} \|B_{j_0}\|_{\mathcal{M}} \left\| \sum_{i=1}^k B(x_i^1, \dots, x_i^n) \right\| + \varepsilon \sum_{i=1}^k \|x_i^1\| \dots \|x_i^n\|. \end{aligned}$$

It results that A belongs to ${}^a\mathcal{M}$, completing the proof.

For every Banach space E , by i_E we mean the canonical isometric embedding $E \rightarrow \ell_\infty(B_{E^*})$. Besides of performing the desired approximation scheme, next result shows that ${}^a\mathcal{M}$ could have been defined by a condition which seems to be less demanding at first glance.

THEOREM 2.4. *Let \mathcal{M} be an n -ideal. The following are equivalent for an n -linear mapping $A \in \mathcal{L}(E_1, \dots, E_n; F)$:*

- (a) $A \in {}^a\mathcal{M}(E_1, \dots, E_n; F)$.
- (b) For every $\varepsilon > 0$ there is an n -linear mapping $C \in \mathcal{M}(E_1, \dots, E_n; \ell_\infty(B_{F^*}))$ such that $\|i_F \circ A - C\| < \varepsilon$.
- (c) There is a set Γ and an isometric embedding $i_F^\Gamma: F \rightarrow \ell_\infty(\Gamma)$ such that for every $\varepsilon > 0$ there is an n -linear mapping $C \in \mathcal{M}(E_1, \dots, E_n; \ell_\infty(\Gamma))$ such that $\|i_F^\Gamma \circ A - C\| < \varepsilon$.
- (d) For every $\varepsilon > 0$ there are a Banach space G_ε and an n -linear mapping $B_\varepsilon \in \mathcal{M}(E_1, \dots, E_n; G_\varepsilon)$ such that

$$\left\| \sum_{i=1}^k A(x_i^1, \dots, x_i^n) \right\| \leq \left\| \sum_{i=1}^k B_\varepsilon(x_i^1, \dots, x_i^n) \right\| + \varepsilon \sum_{i=1}^k \|x_i^1\| \dots \|x_i^n\|,$$

for every $k \in \mathbb{N}$ and any $x_i^j \in E_j$, $j = 1, \dots, n$, $i = 1, \dots, k$.

PROOF. (d) \implies (b) Let $\varepsilon > 0$ be given. By assumption, there are G_ε and $B_\varepsilon \in \mathcal{M}(E_1, \dots, E_n; G_\varepsilon)$ such that for every $k \in \mathbb{N}$ and any $x_i^j \in E_j$, $j = 1, \dots, n$, $i = 1, \dots, k$,

$$\left\| \sum_{i=1}^k A(x_i^1, \dots, x_i^n) \right\| \leq \left\| \sum_{i=1}^k B_\varepsilon(x_i^1, \dots, x_i^n) \right\| + \varepsilon \sum_{i=1}^k \|x_i^1\| \dots \|x_i^n\|.$$

So we have for their respective linearizations A_L and $(B_\varepsilon)_L$,

$$\left\| A_L \left(\sum_{i=1}^k x_i^1 \otimes \dots \otimes x_i^n \right) \right\| \leq \left\| (B_\varepsilon)_L \left(\sum_{i=1}^k x_i^1 \otimes \dots \otimes x_i^n \right) \right\| + \varepsilon \sum_{i=1}^k \|x_i^1\| \dots \|x_i^n\|,$$

for every $\sum_{i=1}^k x_i^1 \otimes \dots \otimes x_i^n \in E_1 \otimes_\pi \dots \otimes_\pi E_n$. Taking the infimum over all representations of a tensor $\theta \in E_1 \otimes_\pi \dots \otimes_\pi E_n$ in the form $\theta = \sum_{i=1}^k x_i^1 \otimes \dots \otimes x_i^n$ it results that $\|A_L(\theta)\| \leq \|(B_\varepsilon)_L(\theta)\| + \varepsilon \pi(\theta)$ for every $\theta \in E_1 \otimes_\pi \dots \otimes_\pi E_n$. Since A_L and $(B_\varepsilon)_L$ are continuous, this inequality also holds on $E_1 \widehat{\otimes}_\pi \dots \widehat{\otimes}_\pi E_n$. Consider the following linear operator:

$$a_\varepsilon: E_1 \otimes_\pi \dots \otimes_\pi E_n \rightarrow (G_\varepsilon \oplus (E_1 \otimes_\pi \dots \otimes_\pi E_n))_1, \quad a_\varepsilon(\theta) = ((B_\varepsilon)_L(\theta), \varepsilon\theta).$$

The fact that a_ε is injective allows us to define a linear operator $b_\varepsilon: \text{Range}(a_\varepsilon) \rightarrow F$ by $b_\varepsilon(a_\varepsilon(\theta)) = A_L(\theta)$. From

$$\|b_\varepsilon(a_\varepsilon(\theta))\| = \|A_L(\theta)\| \leq \|(B_\varepsilon)_L(\theta)\| + \varepsilon\pi(\theta) = \|a_\varepsilon(\theta)\|,$$

we find that $\|b_\varepsilon\| \leq 1$. As a continuous linear operator from the normed space $\text{Range}(a_\varepsilon)$ into the injective Banach space $\ell_\infty(B_{F^*})$, $i_F \circ b_\varepsilon$ can be extended to a continuous linear operator $\overline{i_F \circ b_\varepsilon}: (G_\varepsilon \oplus (E_1 \otimes_\pi \cdots \otimes_\pi E_n))_1 \rightarrow \ell_\infty(B_{F^*})$, $\|\overline{i_F \circ b_\varepsilon}\| \leq 1$. Define $C \in \mathcal{L}(E_1, \dots, E_n; \ell_\infty(B_{F^*}))$ by

$$C(x_1, \dots, x_n) = \overline{i_F \circ b_\varepsilon}(B_\varepsilon(x_1, \dots, x_n), 0).$$

If $j_\varepsilon: G_\varepsilon \rightarrow (G_\varepsilon \oplus (E_1 \otimes_\pi \cdots \otimes_\pi E_n))_1$ is given by $j_\varepsilon(y) = (y, 0)$, then $C = \overline{i_F \circ b_\varepsilon} \circ j_\varepsilon \circ B_\varepsilon$, hence $C \in \mathcal{M}(E_1, \dots, E_n; \ell_\infty(B_{F^*}))$. For $(x_1, \dots, x_n) \in E_1 \times \cdots \times E_n$,

$$\begin{aligned} i_F \circ A_L(x_1 \otimes \cdots \otimes x_n) &= i_F(b_\varepsilon(a_\varepsilon(x_1 \otimes \cdots \otimes x_n))) \\ &= \overline{i_F \circ b_\varepsilon}(((B_\varepsilon)_L(x_1 \otimes \cdots \otimes x_n), \varepsilon x_1 \otimes \cdots \otimes x_n)) \\ &= C(x_1, \dots, x_n) + \overline{i_F \circ b_\varepsilon}((0, \varepsilon x_1 \otimes \cdots \otimes x_n)). \end{aligned}$$

Thus,

$$\begin{aligned} \|\overline{i_F \circ A}(x_1, \dots, x_n) - C(x_1, \dots, x_n)\| &= \|\overline{i_F \circ b_\varepsilon}((0, \varepsilon x_1 \otimes \cdots \otimes x_n))\| \\ &\leq \|\overline{i_F \circ b_\varepsilon}\| \varepsilon \pi(x_1 \otimes \cdots \otimes x_n) \\ &\leq \varepsilon \|x_1\| \cdots \|x_n\|, \end{aligned}$$

which proves that $\|\overline{i_F \circ A} - C\| \leq \varepsilon$.

(c) \implies (a) The assumption assures that $i_F^\Gamma \circ A$ belongs to the sup-norm closure of $\mathcal{M}(E_1, \dots, E_n; \ell_\infty(\Gamma))$. It follows easily from the definition of ${}^a\mathcal{M}$ that $\overline{\mathcal{M}} \subseteq {}^a\mathcal{M}$, therefore $i_F^\Gamma \circ A \in {}^a\mathcal{M}(E_1, \dots, E_n; \ell_\infty(\Gamma))$. The injectivity of ${}^a\mathcal{M}$ gives $A \in {}^a\mathcal{M}(E_1, \dots, E_n; F)$.

The proof is complete as (a) \implies (d) and (b) \implies (c) are obvious.

REMARK 2.5. Given an n -ideal \mathcal{M} , it follows immediately from Theorem 2.4 that, for every E_1, \dots, E_n and F ,

$$\begin{aligned} {}^a\mathcal{M}(E_1, \dots, E_n; F) &= \{A \in \mathcal{L}(E_1, \dots, E_n; F) : i_F \circ A \in \overline{\mathcal{M}(E_1, \dots, E_n; \ell_\infty(B_{F^*}))}\} \\ &= \{A \in \mathcal{L}(E_1, \dots, E_n; F) : i_F^\Gamma \circ A \in \overline{\mathcal{M}(E_1, \dots, E_n; \ell_\infty(\Gamma))} \\ &\quad \text{for some } \Gamma \text{ and some isometric embedding } i_F^\Gamma: F \rightarrow \ell_\infty(\Gamma)\}. \end{aligned}$$

In the proof of Theorem 2.4, if F is injective there is no need to go to the larger space $\ell_\infty(B_{F^*})$.

COROLLARY 2.6. *Let \mathcal{M} be an n -ideal. If F is an injective Banach space, then ${}^a\mathcal{M}(E_1, \dots, E_n; F) = \overline{\mathcal{M}(E_1, \dots, E_n; F)}$ for every E_1, \dots, E_n . In particular, ${}^a\mathcal{M}(E_1, \dots, E_n) = \overline{\mathcal{M}(E_1, \dots, E_n)}$ for every E_1, \dots, E_n .*

THEOREM 2.7. *Let \mathcal{M} be a Banach n -ideal. $\mathcal{M} = {}^a\mathcal{M}$ if and only if \mathcal{M} is closed and injective. Furthermore, ${}^a\mathcal{M}$ is the smallest closed injective n -ideal containing \mathcal{M} .*

PROOF. Assume that \mathcal{M} is closed and injective. Given A in ${}^a\mathcal{M}(E_1, \dots, E_n; F)$, Theorem 2.4 yields $i_F \circ A \in \overline{\mathcal{M}(E_1, \dots, E_n; \ell_\infty(B_{F^*}))}$. But \mathcal{M} is closed, so $i_F \circ A \in \mathcal{M}(E_1, \dots, E_n; \ell_\infty(B_{F^*}))$. From the injectivity of \mathcal{M} it follows that $A \in \mathcal{M}(E_1, \dots, E_n; F)$, so $\mathcal{M} = {}^a\mathcal{M}$. The converse follows from the fact that ${}^a\mathcal{M}$ is closed and injective (Proposition 2.3). Concerning the last assertion, we know that ${}^a\mathcal{M}$ is a closed injective n -ideal containing \mathcal{M} . Let \mathcal{N} be a closed injective n -ideal containing \mathcal{M} . Then, ${}^a\mathcal{M} \subseteq {}^a\mathcal{N} = \mathcal{N}$ by the first assertion.

We thus have that $\mathcal{M} \neq {}^a\mathcal{M}$ if \mathcal{M} fails either to be closed or to be injective. Concrete nonlinear examples will be given in both the closed non-injective case (Example 3.5) and the injective non-closed case (Example 4.2).

3. Finite type and compact mappings

In the linear case, the closed injective hull of the ideal \mathcal{F} of finite rank operators coincides with the ideal \mathcal{K} of compact operators, that is ${}^a\mathcal{F} = \mathcal{K}$ (see [21, Proposition 19.2.3]). Denoting by $\mathcal{L}_{\mathcal{K}}$ the closed multi-ideal of compact multilinear mappings (bounded sets are sent onto relatively compact sets), it is obvious to ask if the equality ${}^a(\mathcal{L}_f) = \mathcal{L}_{\mathcal{K}}$ holds true. We begin this section by giving a negative answer.

EXAMPLE 3.1. Let $A \in \mathcal{L}({}^2\ell_2)$ be given by $A((x_j)_{j=1}^\infty, (y_j)_{j=1}^\infty) = \sum_{j=1}^\infty x_j y_j$. Considering the canonical unit vectors we see that A is not weakly sequentially continuous, hence A is not approximable. As $\overline{\mathcal{L}_f}$ is closed, by Corollary 2.6 we find that $A \notin {}^a(\overline{\mathcal{L}_f})({}^2\ell_2)$, hence $A \notin {}^a(\mathcal{L}_f)({}^2\ell_2)$. On the other hand, it is obvious that $A \in \mathcal{L}_{\mathcal{K}}({}^2\ell_2)$.

Once we know that ${}^a(\mathcal{L}_f) \neq \mathcal{L}_{\mathcal{K}}$, it is natural to look for another multi-ideal which generalizes the compact operators and coincides with ${}^a(\mathcal{L}_f)$. We will accomplish this task almost entirely. First we need some terminology.

The notation $E_1, \hat{\cdot}, E_n$ means that E_i is omitted, that is: $(E_1, \hat{\cdot}, E_n) = (E_1, \dots, E_{i-1}, E_{i+1}, \dots, E_n)$, the same for $(x_1, \hat{\cdot}, x_n)$. For $i = 1, \dots, n$, consider the isometric isomorphism $I_i: \mathcal{L}(E_1, \dots, E_n; F) \longrightarrow \mathcal{L}(E_i; \mathcal{L}(E_1, \hat{\cdot}, E_n; F))$,

$$I_i(A)(x_i)(x_1, \hat{\cdot}, x_n) = A(x_1, \dots, x_n).$$

For the case $n = 1$ to make sense, we consider $I_1(A) = A$ for $A \in \mathcal{L}(E; F)$.

Given Banach operator ideals $\mathcal{I}_1, \dots, \mathcal{I}_n$, an n -linear mapping $A \in \mathcal{L}(E_1, \dots, E_n; F)$ is said to be

- of type $[\mathcal{I}_1, \dots, \mathcal{I}_n]$, and in this case we write $A \in [\mathcal{I}_1, \dots, \mathcal{I}_n](E_1, \dots, E_n; F)$, if $I_i(A) \in \mathcal{I}_i(E_i; \mathcal{L}(E_1, \dots, E_n; F))$, for every $i = 1, \dots, n$.
- of type $\mathcal{L}(\mathcal{I}_1, \dots, \mathcal{I}_n)$, and in this case we write $A \in \mathcal{L}(\mathcal{I}_1, \dots, \mathcal{I}_n)(E_1, \dots, E_n; F)$, if there are Banach spaces G_1, \dots, G_n , linear operators $u_j \in \mathcal{I}_j(E_j; G_j)$, $j = 1, \dots, n$, and $B \in \mathcal{L}(G_1, \dots, G_n; F)$ such that $A = B \circ (u_1, \dots, u_n)$.

If $\mathcal{I}_1 = \dots = \mathcal{I}_n = \mathcal{I}$ we simply write $[\mathcal{I}]$ and $\mathcal{L}(\mathcal{I})$.

It is well known that $[\mathcal{I}_1, \dots, \mathcal{I}_n]$ and $\mathcal{L}(\mathcal{I}_1, \dots, \mathcal{I}_n)$ are (closed, if $\mathcal{I}_1, \dots, \mathcal{I}_n$ are closed) n -ideals (see [6]). It is clear that $\mathcal{L}(\mathcal{I}_1, \dots, \mathcal{I}_n) \subseteq [\mathcal{I}_1, \dots, \mathcal{I}_n]$. If $\mathcal{I}_1, \dots, \mathcal{I}_n$ are closed and injective, then $\mathcal{L}(\mathcal{I}_1, \dots, \mathcal{I}_n) = [\mathcal{I}_1, \dots, \mathcal{I}_n]$ ([10], [20]). In particular, $\mathcal{L}(\mathcal{K}) = [\mathcal{K}]$.

Our next aim is to show that ${}^a(\mathcal{L}_f) = [\mathcal{K}]$ modulo the approximation property. Recall that $[\mathcal{K}]$ coincides with the class of multilinear mappings which are weakly continuous on bounded sets [3].

LEMMA 3.2. *Let $\mathcal{I}, \dots, \mathcal{I}_n$ be operator ideals. If each \mathcal{I}_j is injective, then so are $\mathcal{L}(\mathcal{I}_1, \dots, \mathcal{I}_n)$ and $[\mathcal{I}_1, \dots, \mathcal{I}_n]$.*

PROOF. Let $A \in \mathcal{L}(E_1, \dots, E_n; F)$ and $i: F \rightarrow G$ be an isometric embedding. If $i \circ A \in \mathcal{L}(\mathcal{I}_1, \dots, \mathcal{I}_n)(E_1, \dots, E_n; G)$, we can find G_1, \dots, G_n , $u_j \in \mathcal{I}_j(E_j; G_j)$, $j = 1, \dots, n$, and $B \in \mathcal{L}(G_1, \dots, G_n; G)$ such that $i \circ A = B \circ (u_1, \dots, u_n)$. For $j = 1, \dots, n$, define $u_j^R: E_j \rightarrow \overline{\text{Range}(u_j)}$ by $u_j^R(x_j) = u_j(x_j)$, and let $i_j: \overline{\text{Range}(u_j)} \rightarrow G_j$ be the formal inclusion. Thus $i_j \circ u_j^R = u_j$ belongs to \mathcal{I}_j . The injectivity of \mathcal{I}_j yields that each u_j^R belongs to \mathcal{I}_j . Define $C \in \mathcal{L}(\text{Range}(u_1), \dots, \text{Range}(u_n); F)$ by $C(u_1(x), \dots, u_n(x)) = A(x_1, \dots, x_n)$ and extend it continuously to a $\overline{C} \in \mathcal{L}(\overline{\text{Range}(u_1)}, \dots, \overline{\text{Range}(u_n)}; F)$. So $A = \overline{C} \circ (u_1^R, \dots, u_n^R)$, which shows that $A \in \mathcal{L}(\mathcal{I}_1, \dots, \mathcal{I}_n)(E_1, \dots, E_n; F)$.

Suppose now that $i \circ A \in [\mathcal{I}_1, \dots, \mathcal{I}_n](E_1, \dots, E_n; G)$. Let $j \in \{1, \dots, n\}$. We know that $I_j(i \circ A) \in \mathcal{I}_j(E_j; \mathcal{L}(E_1, \dots, E_n; G))$. Defining

$$J_j: \mathcal{L}(E_1, \dots, E_n; F) \rightarrow \mathcal{L}(E_1, \dots, E_n; G), \quad J_j(B) = i \circ B,$$

it is clear that J is an isometric embedding. For $x_j \in E_j$ and $(x_1, \dots, x_n) \in$

$E_1 \times \cdots \times E_n$, we have

$$\begin{aligned} [(J_j \circ I_j(A))(x_j)](x_1, \dots, x_n) &= [J_j(I_j(A)(x_j))](x_1, \dots, x_n) \\ &= [i \circ (I_j(A)(x_j))](x_1, \dots, x_n) \\ &= i((I_j(A)(x_j))(x_1, \dots, x_n)) \\ &= i(A(x_1, \dots, x_n)) \\ &= (i \circ A)(x_1, \dots, x_n) \\ &= [(I_j(i \circ A))(x_j)](x_1, \dots, x_n). \end{aligned}$$

This shows that $J_j \circ I_j(A) = I_j(i \circ A)$. We have \mathcal{I}_j injective, $J_j \circ I_j(A) \in \mathcal{I}_j$ and J_j is an isometric embedding. It follows that $I_j(A)$ belongs to \mathcal{I}_j , proving that $A \in [\mathcal{I}_1, \dots, \mathcal{I}_n](E_1, \dots, E_n; F)$.

THEOREM 3.3. *Suppose that E_1^*, \dots, E_n^* have the approximation property. Then ${}^a(\mathcal{L}_f)(E_1, \dots, E_n; F) = [\mathcal{K}](E_1, \dots, E_n; F)$.*

PROOF. It is well known that $[\mathcal{K}]$ is a closed (because \mathcal{K} is closed) multi-ideal. It is injective by Lemma 3.2 as \mathcal{K} is injective, hence $[\mathcal{K}] = {}^a[\mathcal{K}]$ by Theorem 2.7. From $\mathcal{L}_f \subseteq [\mathcal{K}]$ we conclude that ${}^a(\mathcal{L}_f) \subseteq {}^a[\mathcal{K}] = [\mathcal{K}]$. As mentioned earlier, $[\mathcal{K}]$ coincides with the class of multilinear mappings which are weakly continuous on bounded sets [3, Theorem 2.9]. Supposing that E_1^*, \dots, E_n^* have the approximation property, [3, Corollary 2.11] gives $[\mathcal{K}](E_1, \dots, E_n; F) = \overline{\mathcal{L}_f}(E_1, \dots, E_n; F)$. The proof is complete as $\overline{\mathcal{L}_f} \subseteq {}^a(\mathcal{L}_f)$.

The next example shows that Theorem 3.3 does not hold true without the approximation property.

EXAMPLE 3.4. Examining the proof of [2, Theorem 4.5] we find a Banach space E lacking the approximation property and a compact symmetric non-approximable linear operator $u: E \rightarrow E^*$. Defining $A \in \mathcal{L}({}^2E)$ by $A(x, y) = u(x)(y)$ it is immediate that $A \in [\mathcal{K}]({}^2E)$ as $I_1(A) = I_2(A) = u$. Suppose that $A \in \overline{\mathcal{L}_f}({}^2E)$. Given $\varepsilon > 0$, there is $B \in \mathcal{L}({}^2E)$ of finite type such that $\|A - B\| < \varepsilon$. Say $B = \sum_{j=1}^k \varphi_j \psi_j$, $\varphi_1, \dots, \varphi_k, \psi_1, \dots, \psi_k \in E^*$. Defining $v \in \mathcal{L}(E; E^*)$ by $v(x) = \sum_{j=1}^k \varphi_j(x) \psi_j$ we have that v is a finite rank operator. Furthermore,

$$\|u(x)(y) - v(x)(y)\| = \|A(x, y) - B(x, y)\| \leq \|A - B\| \|x\| \|y\| < \varepsilon \|x\| \|y\|,$$

for every $x, y \in E$. It results that $\|u - v\| \leq \varepsilon$, which shows that u is approximable, a contradiction. So, $A \notin \overline{\mathcal{L}_f}({}^2E)$, and by Corollary 2.6 we have $A \notin {}^a(\mathcal{L}_f)({}^2E)$.

Thus far we know that ${}^a(\mathcal{L}_f) = \overline{\mathcal{L}_f}$ if either the range space is injective (Corollary 2.6) or the duals of the domain spaces have the approximation property (Theorem 3.3). In the linear case, we have already mentioned that ${}^a\mathcal{F} = \mathcal{H}$, so any compact non-approximable linear operator assures that ${}^a\mathcal{F} \neq \overline{\mathcal{F}}$. Let us see a nonlinear example.

EXAMPLE 3.5. Let $u: E \rightarrow F$ be a compact non-approximable linear operator, for example the operator from Example 3.4 (actually, for every Banach space E without the approximation property there is a Banach space F and a compact non-approximable operator u from E to F). Fix $\varphi \in E^*$, $\|\varphi\| = 1$, and $a \in E$ with $\varphi(a) = 1$. Define

$$A: E \times E \rightarrow F, \quad A(x, y) = \varphi(x)u(y).$$

Suppose that A belongs to $\overline{\mathcal{L}_f}$. Given $\varepsilon > 0$, there are $\varphi_1, \dots, \varphi_k, \psi_1, \dots, \psi_k \in E^*$ and $b_1, \dots, b_k \in F$ such that $\|A - \sum_{j=1}^k \varphi_j \psi_j b_j\| < \frac{\varepsilon}{\|a\|}$. For every $y \in E$ we have

$$\left\| u(y) - \sum_{j=1}^k \varphi_j(a) \psi_j(y) b_j \right\| = \left\| A(a, y) - \sum_{j=1}^k \varphi_j(a) \psi_j(y) b_j \right\| < \frac{\varepsilon}{\|a\|} \|a\| \|y\|,$$

resulting $\|u - \sum_{j=1}^k \varphi_j(a) \psi_j b_j\| \leq \varepsilon$. But $\sum_{j=1}^k \varphi_j(a) \psi_j b_j$ is a finite rank operator, so u is approximable, a contradiction. Hence $A \notin \overline{\mathcal{L}_f}({}^2E; F)$. In order to show that $A \in {}^a(\mathcal{L}_f)({}^2E; F)$, let $\varepsilon > 0$. Since u is compact, by [21, Proposition 19.2.3] there is a finite rank operator $v: E \rightarrow \ell_\infty(B_{F^*})$ such that $\|i_F \circ u - v\| < \frac{\varepsilon}{\|\varphi\|}$. Defining $B \in \mathcal{L}({}^2E; \ell_\infty(B_{F^*}))$ by $B(x, y) = \varphi(x)v(y)$ we have that B is of finite type and

$$\begin{aligned} \|i_F \circ A(x, y) - B(x, y)\| &= \|i_F(\varphi(x)u(y)) - \varphi(x)v(y)\| \\ &= |\varphi(x)| \|i_F(u(y)) - v(y)\| \\ &\leq \|\varphi\| \|i_F \circ u - v\| \|x\| \|y\| \\ &< \varepsilon \|x\| \|y\|, \end{aligned}$$

for every $x, y \in E$. It follows that $\|i_F \circ A - B\| \leq \varepsilon$, so by Theorem 2.4 we have that $A \in {}^a(\mathcal{L}_f)({}^2E; F)$.

We have seen that, contrary to the linear case, $[\mathcal{H}] \not\subseteq {}^a(\mathcal{L}_f)$. Next we show that this is caused by the fact that multilinear forms are not always of finite type. By $\mathcal{L}_\phi(E_1, \dots, E_n; F)$ we denote the subspace of $\mathcal{L}(E_1, \dots, E_n; F)$ spanned by the mappings of the form $A(x_1, \dots, x_n) = B(x_1, \dots, x_n)b$ where $B \in \mathcal{L}(E_1, \dots, E_n)$ and $b \in F$.

PROPOSITION 3.6. *If $A \in [\mathcal{K}](E_1, \dots, E_n; F)$, then for every $\varepsilon > 0$ there is an n -linear mapping $C \in \mathcal{L}_\phi(E_1, \dots, E_n; \ell_\infty(B_{F^*}))$ such that $\|i_F \circ A - C\| < \varepsilon$.*

PROOF. Since $\mathcal{L}(\mathcal{K}) = [\mathcal{K}]$, let $G_1, \dots, G_n, u_j \in \mathcal{K}(E_j; G_j)$ and $B \in \mathcal{L}(G_1, \dots, G_n; F)$ be such that $A = B \circ (u_1, \dots, u_n)$. It follows that $u_1 \otimes \dots \otimes u_n$ is a compact operator from $E_1 \widehat{\otimes}_\pi \dots \widehat{\otimes}_\pi E_n$ to $G_1 \widehat{\otimes}_\pi \dots \widehat{\otimes}_\pi G_n$ (see [22] 44.6.1). So, $i_F \circ B_L \circ u_1 \otimes \dots \otimes u_n$ is a compact operator from $E_1 \widehat{\otimes}_\pi \dots \widehat{\otimes}_\pi E_n$ to $\ell_\infty(B_{F^*})$. The latter space has the approximation property, hence there are $\varphi_1, \dots, \varphi_k \in (E_1 \widehat{\otimes}_\pi \dots \widehat{\otimes}_\pi E_n)^*$ and $b_1, \dots, b_k \in \ell_\infty(B_{F^*})$ such that

$$\left\| i_F \circ B_L \circ u_1 \otimes \dots \otimes u_n - \sum_{j=1}^k \varphi_j b_j \right\| < \varepsilon.$$

For $j = 1, \dots, k$, take $B_j \in \mathcal{L}(E_1, \dots, E_n)$ such that $(B_j)_L = \varphi_j$. For $(x_1, \dots, x_n) \in E_1 \times \dots \times E_n$ it is clear that $B_L \circ u_1 \otimes \dots \otimes u_n(x_1 \otimes \dots \otimes x_n) = B(u_1(x_1), \dots, u_n(x_n)) = A(x_1, \dots, x_n)$, so $\|i_F \circ A - \sum_{j=1}^k B_j b_j\| < \varepsilon$.

4. Weakly compact and absolutely summing mappings

By \mathcal{W} and $\mathcal{L}_{\mathcal{W}}$ we mean the ideals of weakly compact linear operators and multilinear mappings respectively (bounded sets are sent onto relatively weakly compact sets) and by Π_p the ideal of absolutely p -summing linear operators. In this section we investigate multilinear counterparts of properties of ${}^a\Pi_p$ and their connections with \mathcal{W} . The ideal Π_p has been generalized to the multilinear setting in several ways, and among the most studied ones we find the multi-ideal of dominated mappings:

DEFINITION 4.1. Let $p_1, \dots, p_n \geq 1$. An n -linear mapping $A \in \mathcal{L}(E_1, \dots, E_n; F)$ is (p_1, \dots, p_n) -dominated if there is a constant $C \geq 0$ such that

$$\left(\sum_{j=1}^k \|A(x_j^1, \dots, x_j^n)\|^r \right)^{\frac{1}{r}} \leq C \prod_{i=1}^n \sup_{\varphi \in B_{E_i^*}} \left(\sum_{j=1}^k |\varphi(x_j^i)|^{p_i} \right)^{\frac{1}{p_i}},$$

where $\frac{1}{r} = \frac{1}{p_1} + \dots + \frac{1}{p_n}$, for every $k \in \mathbb{N}$ and any $x_j^i \in E_i, j = 1, \dots, k, i = 1, \dots, n$. In this case we write $A \in \mathcal{L}_{d;p_1, \dots, p_n}(E_1, \dots, E_n; F)$. Denoting the infimum of the constants C working in the inequality by $\|A\|_{d;p_1, \dots, p_n}$ we have that $(\mathcal{L}_{d;p_1, \dots, p_n}, \|\cdot\|_{d;p_1, \dots, p_n})$ is a complete r -normed n -ideal. If $p_1 = \dots = p_n = p$ we say that A is p -dominated and write $A \in \mathcal{L}_{d;p}(E_1, \dots, E_n; F)$.

Before going into the main results of the section we provide the announced nonlinear examples of mappings in ${}^a\mathcal{M}$ but not in \mathcal{M} for injective non-closed

M. The characterization

$$(*) \quad \mathcal{L}_{d;p_1,\dots,p_n} = \mathcal{L}(\Pi_{p_1}, \dots, \Pi_{p_n}),$$

which goes back to [25] (a detailed proof can be found in [23, Corolario 3.23]), shall be useful several times.

EXAMPLE 4.2. Since multilinear forms on c_0 are approximable we have that $\mathcal{L}(^n c_0) = {}^a(\mathcal{L}_{d,p})(^n c_0)$ for every $p \geq n$. On the other hand, $\mathcal{L}(^n c_0) \neq \mathcal{L}_{d,p}(^n c_0)$ for $n \geq 3$ and $p \geq 1$ by [4, Theorem 3.5]. So $\mathcal{L}_{d,p}(^n c_0) \neq {}^a(\mathcal{L}_{d,p})(^n c_0)$ for $p \geq n \geq 3$. Of course this is an implicit example. Let us see an explicit one: let $u \in \mathcal{L}(E; F)$ be a 3-summing non-2-summing linear operator (for example the canonical map $j_3: C[0, 1] \rightarrow L_3[0, 1]$). Fix $\varphi \in E^*$, $\|\varphi\| = 1$, and define $A \in \mathcal{L}(^2 E; F)$ by $A(x, y) = \varphi(x)u(y)$. Since u is 3-summing, u is absolutely continuous, so by [14, page 311] there are a Banach space G and a 2-summing operator $j: E \rightarrow G$ such that for every $\varepsilon > 0$ there is K_ε such that

$$\|u(x)\| \leq K_\varepsilon \|j(x)\| + \varepsilon \|x\| \quad \text{for every } x \in E.$$

Define $B \in \mathcal{L}(^2 E; G)$ by $B(x, y) = \varphi(x)j(y)$. Since $B = C \circ (\varphi, j)$, where $C(\lambda, y) = \lambda y$, it follows from (*) that B is 2-dominated. From

$$\begin{aligned} \left\| \sum_{i=1}^k A(x_i, y_i) \right\| &= \left\| u \left(\sum_{i=1}^k \varphi(x_i) y_i \right) \right\| \\ &\leq K_\varepsilon \left\| j \left(\sum_{i=1}^k \varphi(x_i) y_i \right) \right\| + \varepsilon \left\| \sum_{i=1}^k \varphi(x_i) y_i \right\| \\ &\leq K_\varepsilon \left\| \sum_{i=1}^k B(x_i, y_i) \right\| + \varepsilon \sum_{i=1}^k \|x_i\| \|y_i\|, \end{aligned}$$

we conclude that $A \in {}^a(\mathcal{L}_{d,2})(^2 E; F)$. Suppose that A is 2-dominated. By (*) A can be written as $A = C \circ (v_1, v_2)$ with v_1, v_2 being 2-summing. Choosing $a \in E$ with $\varphi(a) = 1$, for every $x \in E$,

$$u(x) = \varphi(a)u(x) = A(a, x) = C(v_1(a), v_2(x)) = (C \circ (v_1(a), \cdot) \circ v_2)(x),$$

resulting $u = (C \circ (v_1(a), \cdot) \circ v_2)$. This is absurd because v_2 is 2-summing whereas u is not, so A fails to be 2-dominated.

Recall that ${}^a\Pi_p = \mathcal{H}$, the ideal of absolutely continuous operators. Since every absolutely continuous operator is weakly compact and completely continuous [14, Corollary 15.4], it is natural to wonder whether every multilinear

mapping belonging to ${}^a\mathcal{L}_{d;p_1,\dots,p_n}$ is (a) weakly compact and/or (b) weakly sequentially continuous (for multilinear mappings the literature speaks of weakly sequentially continuous mappings rather than completely continuous mappings).

Since there are p -dominated non-weakly compact n -linear mappings [5, Example 1], we have $\mathcal{L}_{d,p} \subseteq {}^a\mathcal{L}_{d,p} \not\subseteq \mathcal{L}_{\mathcal{W}}$, so (a) does not hold in general. This leads us to consider a more suitable standard multilinear generalization of weakly compact operators: the ideal of Arens-regular multilinear mappings [\mathcal{W}].

PROPOSITION 4.3. *Every multilinear mapping belonging to ${}^a\mathcal{L}_{d;p_1,\dots,p_n}$ is weakly sequentially continuous and Arens-regular.*

PROOF. First let us show that dominated multilinear mappings are weakly sequentially continuous. Given $C \in \mathcal{L}_{d;p_1,\dots,p_n}(E_1, \dots, E_n; F)$, applying (*) once more we can write $C = B \circ (u_1, \dots, u_n)$ where each u_j is p_j -summing. Since p_j -summing operators are completely continuous and B is continuous, it follows that C is weakly sequentially continuous. Now consider $A \in {}^a\mathcal{L}_{d;p_1,\dots,p_n}(E_1, \dots, E_n; F)$. By Theorem 2.4,

$$i_F \circ A \in \overline{\mathcal{L}_{d;p_1,\dots,p_n}(E_1, \dots, E_n; \ell_\infty(B_{F^*}))}.$$

Moreover the space of weakly sequentially continuous n -linear mappings from $E_1 \times \dots \times E_n$ to $\ell_\infty(B_{F^*})$ is closed and by the first part of the proof it contains the (p_1, \dots, p_n) -dominated mappings, so $i_F \circ A$ is weakly sequentially continuous. As i_F is an isometric embedding, it follows that A is weakly sequentially continuous.

Recall that p_j -summing operators are weakly compact. Thus $[\Pi_{p_1}, \dots, \Pi_{p_n}] \subseteq [\mathcal{W}]$. Since $[\mathcal{W}]$ is closed and injective and $\mathcal{L}(\Pi_{p_1}, \dots, \Pi_{p_n}) \subseteq [\Pi_{p_1}, \dots, \Pi_{p_n}]$, the second assertion follows from Theorem 2.7.

REMARK 4.4. Although there are Banach spaces F such that $\mathcal{L}(\ell_1; F) = {}^a\Pi_p(\ell_1; F)$, like Hilbert spaces or reflexive quotients of a $C(K)$ space [14, p. 313], such type of coincidence does not hold for multilinear mappings, that is $\mathcal{L}({}^n\ell_1; F) \neq {}^a\mathcal{L}_{d;p_1,\dots,p_n}({}^n\ell_1; F)$ for $n \geq 2$, $p_1, \dots, p_n \geq 1$ and all Banach spaces F . Indeed, were the equality true for some n , p_1, \dots, p_n and some F , combining (*) with [6, Proposition 41] we would have $\mathcal{L}({}^2\ell_1; F) = {}^a\mathcal{L}_{d;p_1,p_2}({}^2\ell_1; F)$. So the same equality would hold true for any complemented subspace of F , thus for the scalar field. However $\mathcal{L}({}^2\ell_1) \neq {}^a\mathcal{L}_{d;p_1,p_2}({}^2\ell_1)$ since the bilinear form on ℓ_1 constructed in [1, p. 83] is not Arens-regular.

As mentioned earlier, ${}^a\Pi_p = \mathcal{H}$, so it follows that ${}^a\Pi_p = {}^a\Pi_q$ for every $p, q \geq 1$ (see also [21, Corollary 20.7.7]). Although we do not know whether

${}^a\mathcal{L}_{d;p_1,\dots,p_n} = {}^a\mathcal{L}_{d;q_1,\dots,q_n}$, we are able to identify another multilinear generalization of the ideal of absolutely summing linear operators in which this phenomenon does occur (as usual, the properties of a given operator ideal are to be found among its several multilinear generalizations, rather than in a specific one):

DEFINITION 4.5 (Composition ideals – see [8], [16]). Let \mathcal{I} be a Banach operator ideal. An n -linear mapping $A \in \mathcal{L}(E_1, \dots, E_n; F)$ belongs to $\mathcal{I} \circ \mathcal{L}$ – in this case we write $A \in \mathcal{I} \circ \mathcal{L}(E_1, \dots, E_n; F)$ – if there are a Banach space G , an n -linear mapping $B \in \mathcal{L}(E_1, \dots, E_n; G)$ and an operator $u \in \mathcal{I}(G; F)$ such that $A = u \circ B$. $\mathcal{I} \circ \mathcal{L}$ is a multi-ideal which becomes a Banach multi-ideal with the norm

$$\|A\|_{\mathcal{I} \circ \mathcal{L}} := \inf\{\|u\|_{\mathcal{I}}\|B\| : A = u \circ B, B \in \mathcal{L}(E_1, \dots, E_n; G), u \in \mathcal{I}(G; F)\}.$$

PROPOSITION 4.6. ${}^a(\mathcal{I} \circ \mathcal{L}) = {}^a\mathcal{I} \circ \mathcal{L}$ for every Banach operator ideal \mathcal{I} . In particular, ${}^a(\Pi_p \circ \mathcal{L}) = {}^a(\Pi_q \circ \mathcal{L})$ for every $p, q \geq 1$.

PROOF. Assume for a while that \mathcal{I} is injective. Let $A \in \mathcal{L}(E_1, \dots, E_n; F)$ and $i: F \rightarrow G$ be an isometric embedding such that $i \circ A \in \mathcal{I} \circ \mathcal{L}(E_1, \dots, E_n; G)$. Notice that according to [8, Proposition 2.2], $(i \circ A)_L \in \mathcal{I}(E_1 \widehat{\otimes}_\pi \dots \widehat{\otimes}_\pi E_n; G)$. Since $i \circ A_L = (i \circ A)_L$, it follows that $i \circ A_L \in \mathcal{I}(E_1 \widehat{\otimes}_\pi \dots \widehat{\otimes}_\pi E_n; G)$. The injectivity of \mathcal{I} gives $A_L \in \mathcal{I}(E_1 \widehat{\otimes}_\pi \dots \widehat{\otimes}_\pi E_n; F)$, so the factorization $A = A_L \circ \sigma_n$, where $\sigma_n: E_1 \times \dots \times E_n \rightarrow E_1 \widehat{\otimes}_\pi \dots \widehat{\otimes}_\pi E_n$ is the canonical n -linear mapping given by $\sigma_n(x_1, \dots, x_n) = x_1 \otimes \dots \otimes x_n$, shows that $A \in \mathcal{I} \circ \mathcal{L}(E_1, \dots, E_n; F)$. Thus far we have proved that $\mathcal{I} \circ \mathcal{L}$ is injective whenever \mathcal{I} is injective. As ${}^a\mathcal{I}$ is closed and injective, ${}^a\mathcal{I} \circ \mathcal{L}$ is closed and injective as well, so ${}^a\mathcal{I} \circ \mathcal{L} = {}^a({}^a\mathcal{I} \circ \mathcal{L})$. From $\mathcal{I} \subseteq {}^a\mathcal{I}$ we get $\mathcal{I} \circ \mathcal{L} \subseteq {}^a\mathcal{I} \circ \mathcal{L}$, hence ${}^a(\mathcal{I} \circ \mathcal{L}) \subseteq {}^a({}^a\mathcal{I} \circ \mathcal{L}) = {}^a\mathcal{I} \circ \mathcal{L}$.

Given $A \in {}^a\mathcal{I} \circ \mathcal{L}(E_1, \dots, E_n; F)$, write $A = u \circ B$ with $B \in \mathcal{L}(E_1, \dots, E_n; G)$ and $u \in {}^a\mathcal{I}(G; F)$. Given $\varepsilon > 0$, there exists an operator $v \in \mathcal{I}(G; \ell_\infty(B_{F^*}))$ such that $\|v - i_F \circ u\| < \frac{\varepsilon}{\|B\|}$. In this fashion $v \circ B \in \mathcal{I} \circ \mathcal{L}(E_1, \dots, E_n; \ell_\infty(B_{F^*}))$ and

$$\|v \circ B - i_F \circ A\| = \|v \circ B - i_F \circ u \circ B\| \leq \|v - i_F \circ u\| \|B\| < \varepsilon.$$

This shows, by Theorem 2.4, that $A \in {}^a(\mathcal{I} \circ \mathcal{L})(E_1, \dots, E_n; F)$.

5. Aron-Berner extensions

In this section we prove that the correspondence $\mathcal{M} \mapsto {}^a\mathcal{M}$ preserves Aron-Berner stability. Recall that the Aron-Berner extension, sometimes called Arens extension or canonical extension, of a multilinear mapping $A \in \mathcal{L}(E_1, \dots, E_n; F)$, denoted $\tilde{A} \in \mathcal{L}(E_1^{**}, \dots, E_n^{**}; F^{**})$, is defined according to

$$\tilde{A}(x_1^{**}, \dots, x_n^{**})(\varphi) = \lim_{i_1} \dots \lim_{i_n} (\varphi \circ A)(x_{i_1}, \dots, x_{i_n}) \quad \text{for all } \varphi \in F^*,$$

where the limits are iterated limits and (x_{i_j}) is a net in E_j weak* converging to $x_j^{**} \in E_j^{**}$. It turns out that $\tilde{\tilde{A}}(x_1^{**}, \dots, x_n^{**}) = \lim_{i_1} \dots \lim_{i_n} A(x_{i_1}, \dots, x_{i_n})$ where the limits are taken in the $w(F^{**}, F^*)$ topology. The mapping

$$A \in \mathcal{L}(E_1, \dots, E_n; F) \mapsto \tilde{A} \in \mathcal{L}(E_1^{**}, \dots, E_n^{**}; F^{**})$$

is a linear into isometry. See [15] for more information about this bidual extension. Depending on the order the iterated limits are taken, A can have several (at most $n!$) different Aron-Berner extensions.

DEFINITION 5.1. An n -ideal \mathcal{M} is said to be *Aron-Berner stable* if all Aron-Berner extensions of any n -linear mapping in \mathcal{M} also belong to \mathcal{M} .

Some multi-ideals are known to be Aron-Berner stable (for example, integral mappings [12, Proposition 8]) and some are known not to be (for example, weakly sequentially continuous mappings [26, Example 1.9]).

THEOREM 5.2. *Let \mathcal{M} be an Aron-Berner stable n -ideal. The following are equivalent for a multilinear mapping $A \in \mathcal{L}(E_1, \dots, E_n; F)$:*

- (a) $A \in {}^a\mathcal{M}(E_1, \dots, E_n; F)$.
- (b) All Aron-Berner extensions of A belong to ${}^a\mathcal{M}(E_1^{**}, \dots, E_n^{**}; F^{**})$.
- (c) Some Aron-Berner extension of A belongs to ${}^a\mathcal{M}(E_1^{**}, \dots, E_n^{**}; F^{**})$.

In particular, ${}^a\mathcal{M}$ is Aron-Berner stable.

PROOF. (a) \implies (b) Let \tilde{A} be an Aron-Berner extension of A . In what follows the notation \tilde{B} means the Aron-Berner extension of B where the iterated limits are taken in the same order as in \tilde{A} . Firstly we remark that the natural embedding

$$I_{F^{**}} : x^{**} \in F^{**} \mapsto (\langle x^{**}, \varphi \rangle)_{\varphi \in B_{F^*}} \in \ell_\infty(B_{F^*}),$$

is a linear isometry that extends the isometric embedding i_F and it is weak*-weak* continuous since it is the transpose of the mapping

$$(a_\varphi)_{\varphi \in B_{F^*}} \in \ell_1(B_{F^*}) \mapsto \sum_{\varphi \in B_{F^*}} a_\varphi \cdot \varphi \in F^*.$$

This remark proves that $i_F \widetilde{\circ} A = I_{F^{**}} \circ \widetilde{A}$. Assume that $A \in {}^a\mathcal{M}(E_1, \dots, E_n; F)$ and fix $\varepsilon > 0$. According to condition (b) in Theorem 2.4, there is an n -linear mapping $C \in \mathcal{M}(E_1, \dots, E_n; \ell_\infty(B_{F^*}))$ such that $\|i_F \circ A - C\| < \varepsilon$. Hence

$$\|I_{F^{**}} \circ \widetilde{A} - \widetilde{C}\| = \|i_F \widetilde{\circ} A - \widetilde{C}\| = \|(i_F \circ \widetilde{A} - C)\| < \varepsilon.$$

The Aron-Berner stability of \mathcal{M} assures that \widetilde{C} belongs to \mathcal{M} , thus we have seen that $I_{F^{**}} \circ \widetilde{A}$ is in the closure of $\mathcal{M}(E_1^{**}, \dots, E_n^{**}; (\ell_\infty(B_{F^*}))^{**})$. Hence $I_{F^{**}} \circ \widetilde{A} \in {}^a\mathcal{M}(E_1^{**}, \dots, E_n^{**}; (\ell_\infty(B_{F^*}))^{**})$, and because of the injectivity of ${}^a\mathcal{M}$, $\widetilde{A} \in {}^a\mathcal{M}(E_1^{**}, \dots, E_n^{**}; F^{**})$.

(b) \implies (c) is obvious.

(c) \implies (a) By J_E we mean the canonical isometric embedding from E into E^{**} . Let \widetilde{A} be an Aron-Berner extension of A belonging to ${}^a\mathcal{M}(E_1^{**}, \dots, E_n^{**}; F^{**})$. The ideal property yields that $J_F \circ A = \widetilde{A} \circ (J_{E_1}, \dots, J_{E_n}) \in {}^a\mathcal{M}(E_1, \dots, E_n; F^{**})$. From the injectivity of ${}^a\mathcal{M}$ it follows that $A \in {}^a\mathcal{M}(E_1, \dots, E_n; F)$.

It is well known that a linear operator u belongs to ${}^a\Pi_p$, that is, u is absolutely continuous, if and only if u^{**} belongs to ${}^a\Pi_p$ as well [14, Corollary 15.5]. It is clear that Theorem 5.2 generalizes this result to multilinear mappings. Moreover, taking $n = 1$ in Theorem 5.2 one sees that even in the linear case [14, Corollary 15.5] is a particular case of a much more general situation:

COROLLARY 5.3. *Let \mathcal{I} be an operator ideal such that $u \in \mathcal{I} \implies u^{**} \in \mathcal{I}$. Given $u \in \mathcal{L}(E; F)$, $u \in {}^a\mathcal{I}(E; F)$ if and only if $u^{**} \in {}^a\mathcal{I}(E^{**}; F^{**})$.*

For instance, maximal Banach operator ideals satisfy the condition $u \in \mathcal{I} \implies u^{**} \in \mathcal{I}$ [13, Corollary 17.8.4] (observe that, for being closed, ${}^a\mathcal{I}$ is maximal only if ${}^a\mathcal{I} = \mathcal{L}$). Let us stress that the above corollary is a genuine generalization of [14, Corollary 15.5]: indeed, on the one hand the proof of [14, Corollary 15.5] works only for operator ideals contained in \mathcal{W} , on the other hand there are maximal operator ideals not contained in \mathcal{W} , for example the ideal of cotype 2 operators (cf. [13, 17.4]).

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