MATH. SCAND. 107 (2010), 123–138

A REGULARIZED TRACE FORMULA FOR SECOND ORDER DIFFERENTIAL OPERATOR EQUATIONS

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Abstract

In this paper, we deal with abstract Sturm-Liouville problems when the potential of the differential equation is an operator function in a Hilbert space H. We generalize trace formula obtained by [7], [9] for the classic regular Sturm-Liouville problems. We investigate the spectrum and obtained a regularized trace formula for the Sturm-Liouville operator with an operator coefficient.

1. Introduction

Let H be a separable Hilbert space. In the Hilbert space $H_1 = L_2([0,1], H)$, we consider the self-adjoint operator L generated by the expression

$$l(y) = -y''(x) + Q(x)y(x)$$

with the boundary conditions

$$y(0) = 0, \quad y'(1) + ay(1) = 0, \quad a > 0. \tag{1}$$

Suppose that the operator function $Q(x)$ in the expression $l(y)$ satisfies the following conditions:

(1°) For $\forall x \in [0,1]$, $Q(x) : H \to H$ is a self-adjoint nuclear operator. Moreover, $Q(x)$ has a continuous derivative of second order with respect to the norm in the space $\sigma_1(H)$ in the interval $[0,1]$ and for $x \in [0,1]$, $Q^{(i)}(x) : H \to H$ are self-adjoint operators ($i = 1, 2$).

(2°) $\sup_{x\in[0,1]} \|Q(x)\| < \frac{1}{2}\min_m(\mu_{m+1} - \mu_m)$, where $\mu_1 < \mu_2 < \cdots < \mu_m < \cdots$ are the positive roots of the equation $\sqrt{\lambda}\cos\sqrt{\lambda} + a\sin\sqrt{\lambda} = 0$.

(3°) There is an orthonormal basis $\{\varphi_n\}_{n=1}^{\infty}$ of the space H such that $\sum_{n=1}^{\infty} \|Q(x)\varphi_n\|_{H_1} < \infty$. Here $\sigma_1(H)$ denotes the space of the nuclear operators from H to H, as in Gorbachuk et al [8].

Received 26 August 2008.

Let L_0 be the operator generated by the differential expression $l_0(y) = -y''(x)$ and the boundary conditions (1). The spectrum of the operator L_0 is the set $\{\mu_m\}_{m=1}^{\infty}$, where $\mu_1 < \mu_2 < \cdots < \mu_m < \cdots$ are the positive roots of the equation $\sqrt{\lambda}\cos\sqrt{\lambda} + a\sin\sqrt{\lambda} = 0$. Every number μ_m is eigenvalue of L_0 with infinite multiplicity. The orthonormal eigenfunctions corresponding to the eigenvalue μ_m have the form

$$\psi_{mn}^0 = \alpha_m \sin\sqrt{\mu_m}x \cdot \varphi_n, \qquad n = 1, 2, \ldots, \tag{2}$$

where

$$\alpha_m = \frac{\sqrt{2}}{\sqrt{1 + a^{-1}\cos^2\sqrt{\mu_m}}}. \tag{3}$$

In this paper, we investigate the spectrum and the regularized trace of the operator L. Gelfand and Levitan [7] first obtained a trace formula for the Sturm-Liouville differential equation. After this study several mathematicians were interested in developing trace formulas for different differential operators. The current situation of this subject and studies related to it are presented in the comprehensive survey paper [14].

The trace formulas of the abstract self-adjoint operators with continuous spectrum were first analyzed by Krein [12]. In this work, he also proved the formula mathematically, which had been obtained earlier [13] through physical theories in quantum statistics and crystal theory. The trace formulas related to the Sturm-Liouville problem with bounded self-adjoint operator given an infinite interval and having a continuous spectrum were considered in [1], [2]. Faddeev's study of the regularized trace formula for the Sturm-Liouville equation with the matrix coefficient in [6] has been a precursor for [1], [2].

Note that the trace formulas are used in the inverse problems of spectral analysis of differential equations (see, for example [14]) and have applications in the approximate calculation of eigenvalues of the related operator [4], [5].

Some special cases of the problem under consideration were previously investigated in [3], [4], [7], [9] by different methods.

2. Investigation of the spectrum

Let R_λ^0 and R_λ be the resolvent of the operators L_0 and L, respectively.

LEMMA 1. *If $Q(x)$ satisfies the condition* ($3°$) *and $\lambda \in \rho(L_0)$, then $QR_\lambda^0 : H_1 \to H_1$ is nuclear operator: $QR_\lambda^0 \in \sigma_1(H_1)$.*

PROOF. The system (2) of eigenfunctions of the operator L_0 is an orthonormal basis of the space H_1. Then, as shown in Gorbachuk et al [8], it is sufficient to observe that the series $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\|QR_\lambda^0\psi_{mn}^0\|_{H_1}$ is a convergent series in order to prove that $QR_\lambda^0 \in \sigma_1(H_1)$. By means of the formulas (2) and (3)

$$
\begin{aligned}
&\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\|QR_\lambda^0\psi_{mn}^0\|_{H_1} \\
&\qquad = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}|\mu_m-\lambda|^{-1}\|Q\psi_{mn}^0\|_{H_1} \\
(4)\qquad &\qquad = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}|\mu_m-\lambda|^{-1}\left[\int_0^1 \left(\alpha_m \sin\sqrt{\mu_m}x\right)^2\|Q(x)\varphi_n\|_H^2\,dx\right]^{\frac{1}{2}} \\
&\qquad \le \sqrt{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}|\mu_m-\lambda|^{-1}\cdot\|Q(x)\varphi_n\|_{H_1}
\end{aligned}
$$

is found. From the formula (see, for example [10])

$$
(5)\qquad \mu_m = \left(m-\tfrac{1}{2}\right)\pi + \frac{a+\frac{1}{2}\int_0^1 q(\tau)\,d\tau}{\left(m-\frac{1}{2}\right)\pi} + O\left(\frac{1}{m^2}\right),
$$

and (4) we have

$$
(6)\qquad \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\|QR_\lambda^0\psi_{mn}^0\|_{H_1} \le C_\lambda\sum_{m=1}^{\infty}m^{-2}\sum_{n=1}^{\infty}\|Q(x)\varphi_n\|_{H_1}.
$$

Here C_λ is a positive constant related only to λ. By virtue of the condition (3°) we obtain from (6)

$$
\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\|QR_\lambda^0\psi_{mn}^0\|_{H_1} < \infty.
$$

Lemma 1 is proved.

THEOREM 2. *If $Q(x)$ satisfies conditions* (1°)–(3°), *then the spectrum of the operator L is a subset of the union of the intervals*

$$
\Omega_m = [\mu_m - \|Q\|_{H_1}, \mu_m + \|Q\|_{H_1}], \qquad m = 1, 2, \ldots
$$

which are pairwise disjoint and:

(a) *Every point different from μ_m, belonging to the interval Ω_m of the spectrum of the operator L, is a discrete eigenvalue, whose multiplicity is finite.*

(b) *μ_m may be the eigenvalue, whose multiplicity is finite or infinite of the operator L.*

(c) *The equality* $\lim_{n\to\infty} \lambda_{mn} = \mu_m$ *holds. Here* $\{\lambda_{mn}\}_{n=1}^{\infty}$ *are the eigenvalues, belonging to the interval* Ω_m *of the operator L and each eigenvalue has been repeated according to multiplicity.*

PROOF. The resolvent R_λ of the operator L satisfies the equation

$$R_\lambda^0 - R_\lambda Q R_\lambda^0 = R_\lambda. \tag{7}$$

If $\lambda \in \mathsf{R} \setminus \bigcup_{m=1}^{\infty} \Omega_m$ then by condition (2°) we have

$$|\lambda - \mu_m| > \|Q\|_{H_1}, \qquad m = 1, 2, \dots . \tag{8}$$

For the self-adjoint operator $R_\lambda^0 = (L_0 - \lambda I)^{-1}$, $\|R_\lambda^0\|_{H_1} = \max_m |\lambda - \mu_m|^{-1}$ holds. From here and (8) we obtain

$$\|R_\lambda^0\|_{H_1} < \|Q^{-1}\|_{H_1}.$$

Hence

$$\|QR_\lambda^0\|_{H_1} \le \|Q\|_{H_1} \cdot \|R_\lambda^0\|_{H_1} < 1.$$

Thus, $A(B) = R_\lambda^0 - BQR_\lambda^0$ is a contraction operator from $\mathscr{L}(H_1)$ to $\mathscr{L}(H_1)$. Here $\mathscr{L}(H_1)$ is the linear bounded operators space from H_1 to H_1. According to this, $A(R_\lambda) = R_\lambda$, that is, the equation (7) has a single solution $R_\lambda \in \mathscr{L}(H_1)$. Thus, every point $\lambda \notin \bigcup_{m=1}^{\infty} \Omega_m$ is the regular point of the self-adjoint operator L. So the spectrum of the operator L is $\sigma(L) \subset \bigcup_{m=1}^{\infty} \Omega_m$. From Formula (7) and Lemma 1, for every $\lambda \in \rho(L) \cap \rho(L_0)$, $R_\lambda - R_\lambda^0$ belongs to $\sigma_1(H_1)$, that is, $R_\lambda - R_\lambda^0$ is a nuclear operator. In this case, as it is proved in Kato [11, p. 244], the continuous parts of the spectra of the operators L_0 and L coincide. According to this and since the spectrum of the operator L_0 is continuous, the continuous part of the spectrum of the operator L is the set $\{\mu_m\}_{m=1}^{\infty}$. This also means that the assertions (a), (b) and (c) of Theorem 2 are satisfied.

3. A formula for the regularized trace

Let $\{\psi_{mn}\}_{m,n=1}^{\infty}$ be the orthonormal eigenfunctions corresponding to the eigenvalues $\{\lambda_{mn}\}_{m,n=1}^{\infty}$ of the operator L and

$$\Gamma_p = \left\{ \lambda, |\lambda - \mu_p| = 2^{-1} \min_m (\mu_{m+1} - \mu_m) \right\},$$

$$B_{mn}^0 = (\cdot, \psi_{mn}^0)_{H_1} \psi_{mn}^0, \qquad B_{mn} = (\cdot, \psi_{mn})_{H_1} \psi_{mn},$$

$$L_{0m}^{(r)} = \sum_{n=1}^{\infty} \mu_m^r B_{mn}^0, \quad L_m^{(r)} = \sum_{\substack{n=1 \\ \lambda_{mn} \neq 0}}^{\infty} \lambda_{mn}^r B_{mn}, \qquad r = 1\,,\, -1.$$

THEOREM 3. *If the operator function $Q(x)$ satisfies* (2°) *and* (3°)*, then the series*

$$\sum_{n=1}^{\infty}(\lambda_{pn} - \mu_p), \qquad p = 1, 2, \dots$$

are absolute convergent series.

PROOF. The difference $R_\lambda - R_\lambda^0$ satisfies the following formula

$$R_\lambda - R_\lambda^0 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{B_{mn}}{\lambda_{mn} - \lambda} - \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{B_{mn}^0}{\mu_m - \lambda}. \tag{9}$$

Since except for the eigenvalues μ_p and $\{\lambda_{pn}\}_{n=1}^{\infty}$ of the operators L_0 and L, all their eigenvalues are outside of the circle Γ_p, from the last formula, we have

$$\begin{aligned} &\frac{1}{2\pi i}\int_{\Gamma_p}\lambda(R_\lambda - R_\lambda^0)\,d\lambda \\ &\quad= \sum_{n=1}^{\infty}\left[B_{pn}^0\frac{1}{2\pi i}\int_{\Gamma_p}\frac{\lambda\cdot d\lambda}{\lambda - \mu_p} - B_{pn}\frac{1}{2\pi i}\int_{\Gamma_p}\frac{\lambda\cdot d\lambda}{\lambda - \lambda_{pn}}\right] \\ &\quad= \sum_{n=1}^{\infty}(\mu_p B_{pn}^0 - \lambda_{pn}B_{pn}) = L_{0p}^{(1)} - L_p^{(1)}. \end{aligned} \tag{10}$$

Since the operator function $R_\lambda - R_\lambda^0$ is analytic with respect to the norm in the $\sigma_1(H_1)$ space in the region $\rho(L)$, from (10) we obtain

$$L_p^{(1)} - L_{0p}^{(1)} \in \sigma_1(H_1), \qquad p = 1, 2, \dots \tag{11}$$

Similarly it can be shown that

$$L_p^{(-1)} - L_{0p}^{(-1)} \in \sigma_1(H_1), \qquad p = 1, 2, \dots \tag{12}$$

Since the operator L may only have negative eigenvalues of finite number, in order to prove the theorem, it is necessary to show that

$$\sum_{\substack{n\\ \lambda_{pn}>0}} |\lambda_{pn} - \mu_p| < \infty, \qquad p = 1, 2, \dots$$

For this reason we shall accept in the following that $\lambda_{pn} > 0$, $p, n = 1, 2, \dots$. Since the spectrum of the operator $L_{0p}^{(r)}$ is the set $\{0; \mu_p^r\}$, we have

$$\mu_p^r \geq \left(L_{0p}^{(r)}\psi_{pn}, \psi_{pn}\right)_{H_1}, \qquad \lambda_{pn}^r = \left(L_p^{(r)}\psi_{pn}, \psi_{pn}\right)_{H_1}$$

$$\sum_{\substack{n \\ \lambda_{pn}^r > \mu_p^r}} (\lambda_{pn}^r - \mu_p^r) \le \sum_{\substack{n \\ \lambda_{pn}^r > \mu_p^r}} \big((L_p^{(r)} - L_{0p}^{(r)}) \psi_{pn}, \psi_{pn} \big)_{H_1}$$
$$\le \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \big| \big((L_p^{(r)} - L_{0p}^{(r)}) \psi_{mn}, \psi_{mn} \big)_{H_1} \big|$$
$$\le \big\| L_p^{(r)} - L_{0p}^{(r)} \big\|_{\sigma_1(H_1)}.$$

Using formulas (11) and (12), from the above inequalities, we find

$$\sum_{\substack{n \\ \lambda_{pn} > \mu_p}} (\lambda_{pn} - \mu_p) < \infty,$$

$$\sum_{\substack{n \\ \lambda_{pn} < \mu_p}} (\mu_p - \lambda_{pn}) \le \text{const.} \times \sum_{\substack{n \\ \lambda_{pn} < \mu_p}} (\mu_p - \lambda_{pn}) \mu_p^{-1} \lambda_{pn}^{-1}$$
$$= \text{const.} \times \sum_{\substack{n \\ \lambda_{pn} < \mu_p}} (\lambda_{pn}^{-1} - \mu_p^{-1}) < \infty.$$

From the last relations, we obtain

$$\sum_{n=1}^{\infty} |\lambda_{pn} - \mu_p| < \infty, \qquad p = 1, 2, \ldots$$

Theorem 3 is proved.

Since the operator function $R_\lambda - R_\lambda^0$ belongs to $\sigma_1(H_1)$ for every $\lambda \in \rho(L)$, from the formula (9) and Theorem 3, we have

$$\operatorname{tr}(R_\lambda - R_\lambda^0) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(\frac{1}{\lambda_{mn} - \lambda} - \frac{1}{\mu_m - \lambda} \right).$$

Multiplying both sides of this equality by $\frac{\lambda}{2\pi i}$ and integrating over the circle $|\lambda| = b_p = 2^{-1}(\mu_{p+1} + \mu_p)$, we have

$$\frac{1}{2\pi i} \int_{|\lambda| = b_p} \lambda \operatorname{tr}(R_\lambda - R_\lambda^0)\, d\lambda$$
$$= \frac{1}{2\pi i} \int_{|\lambda| = b_p} \lambda \sum_{m=1}^{p} \sum_{n=1}^{\infty} \left(\frac{1}{\lambda_{mn} - \lambda} - \frac{1}{\mu_m - \lambda} \right) d\lambda \tag{13}$$
$$+ \frac{1}{2\pi i} \int_{|\lambda| = b_p} \lambda \sum_{m=p+1}^{\infty} \sum_{n=1}^{\infty} \left(\frac{1}{\lambda_{mn} - \lambda} - \frac{1}{\mu_m - \lambda} \right) d\lambda.$$

By Theorem 2 and condition (2°)

$$\mu_m - \|Q\|_{H_1} \leq \lambda_{mn} \leq \mu_m + \|Q\|_{H_1} < \frac{\mu_m + \mu_{m+1}}{2} = b_m, \qquad n = 1, 2, \ldots$$

Hence

$$|\lambda_{mn}| < b_p, \quad m \leq p; \quad p \geq 1; \qquad n = 1, 2, \ldots \tag{14}$$

and moreover, for $m > p$,

$$\lambda_{mn} \geq \mu_m - \|Q\|_{H_1} \geq \mu_{p+1} - \|Q\|_{H_1} > \frac{\mu_p + \mu_{p+1}}{2} = b_p \tag{15}$$

From (13), (14) and (15)

$$\frac{1}{2\pi i} \int_{|\lambda|=b_p} \lambda \operatorname{tr}(R_\lambda - R_\lambda^0)\, d\lambda = \sum_{m=1}^{p} \sum_{n=1}^{\infty} (\mu_m - \lambda_{mn}). \tag{16}$$

On the other hand, from the formula $R_\lambda = R_\lambda^0 - R_\lambda Q R_\lambda^0$, the equality

$$R_\lambda - R_\lambda^0 = \sum_{j=1}^{N} (-1)^j R_\lambda^0 (Q R_\lambda^0)^j + (-1)^{N+1} R_\lambda (Q R_\lambda^0)^{N+1} \tag{17}$$

is obtained for every natural number N. From (16) and (17), we have

$$\begin{aligned} \sum_{m=1}^{p} \sum_{n=1}^{\infty} (\mu_m - \lambda_{mn}) &= \sum_{j=1}^{N} \frac{(-1)^j}{2\pi i} \int_{|\lambda|=b_p} \lambda \operatorname{tr}\big[R_\lambda^0 (Q R_\lambda^0)^j\big]\, d\lambda \\ &\quad + \frac{(-1)^N}{2\pi i} \int_{|\lambda|=b_p} \lambda \operatorname{tr}\big[R_\lambda (Q R_\lambda^0)^{N+1}\big]\, d\lambda. \end{aligned} \tag{18}$$

Let

$$M_p^j = \frac{(-1)^{j+1}}{2\pi i} \int_{|\lambda|=b_p} \lambda \operatorname{tr}\big[R_\lambda^0 (Q R_\lambda^0)^j\big]\, d\lambda, \tag{19}$$

$$M_{pN} = \frac{(-1)^N}{2\pi i} \int_{|\lambda|=b_p} \lambda \operatorname{tr}\big[R_\lambda (Q R_\lambda^0)^{N+1}\big]\, d\lambda. \tag{20}$$

Then from (18), (19) and (20), we have

$$\sum_{m=1}^{p} \sum_{n=1}^{\infty} (\lambda_{mn} - \mu_m) = \sum_{j=1}^{N} M_p^j + M_{pN} \tag{21}$$

Now we shall compute the right side of (21). Since the operator function QR_λ^0 in the domain $C \setminus \{\mu_m\}_{m=1}^{\infty}$ is analytic with respect to the norm in $\sigma_1(H_1)$, we can show that for M_p^j the following formula is true

$$M_p^j = \frac{(-1)^j}{2\pi i j} \int_{|\lambda|=b_p} \operatorname{tr}(QR_\lambda^0)^j \, d\lambda. \tag{22}$$

From (2) and (22), we have

$$\begin{aligned}
M_p^1 &= -\frac{1}{2\pi i} \int_{|\lambda|=b_p} \operatorname{tr}(QR_\lambda^0) \, d\lambda \\
&= -\frac{1}{2\pi i} \int_{|\lambda|=b_p} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (QR_\lambda^0 \psi_{mn}^0, \psi_{mn}^0)_{H_1} \, d\lambda \\
&= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (Q\psi_{mn}^0, \psi_{mn}^0)_{H_1} \cdot \frac{1}{2\pi i} \int_{|\lambda|=b_p} \frac{d\lambda}{\lambda - \mu_m} \\
&= \sum_{m=1}^{p} \sum_{n=1}^{\infty} (Q\psi_{mn}^0, \psi_{mn}^0)_{H_1} \\
&= \sum_{m=1}^{p} \sum_{n=1}^{\infty} \int_0^1 (Q(x)\varphi_n, \varphi_n) \alpha_m^2 \sin^2 \sqrt{\mu_m} x \, dx \\
&= \sum_{m=1}^{p} \int_0^1 \operatorname{tr} Q(x) \alpha_m^2 \sin^2 \sqrt{\mu_m} x \, dx.
\end{aligned} \tag{23}$$

Let

$$T_p(x) = \sum_{m=1}^{p} \alpha_m^2 \sin^2 \sqrt{\mu_m} x, \qquad x \in [0, 1] \tag{24}$$

and

$$F(z) = \frac{4a \sin^2 xz}{2z \cos^2 z + a \sin 2z}.$$

The function $F(z)$ may not be analytic only at the points $z = \sqrt{\mu_m}$, $(m = 1, 2, \ldots)$ and $z = \left(k - \frac{1}{2}\right)\pi$, $(k = \mp 1, \mp 2, \ldots)$ of the complex plane. It can be easily shown that if the point $z = \sqrt{\mu_m}$ or $z = \left(k - \frac{1}{2}\right)\pi$ is the singular point of $F(z)$, then this point is the simple pole point and

$$\operatorname{Res}[F(z)]_{z=\sqrt{\mu_m}} = \frac{4a \sin^2 \sqrt{\mu_m} x}{2(a + \cos^2 \sqrt{\mu_m})} = \alpha_m^2 \sin^2 \sqrt{\mu_m} x. \tag{25}$$

$$\operatorname{Res}[F(z)]_{z=\left(k-\frac{1}{2}\right)\pi} = \frac{4a\sin^2\sqrt{\mu_m}x}{-2a} = -2\sin^2\left(k-\tfrac{1}{2}\right)\pi x. \tag{26}$$

If the function $F(z)$ is analytic at the point $z = \sqrt{\mu_m}$ and $z = \left(k - \frac{1}{2}\right)\pi$, then it is clear that the formulas (25) and (26) are satisfied.

Let us denote by Γ the contour of the rectangle, whose corners are $\mp Ei$, $D_p \mp Ei$, where B is a positive variable and $D_p = p\pi$. Here we shall assume that p is a natural number such that $\sqrt{\mu_p} < D_p < \sqrt{\mu_{p+1}}$. Hence, as known,

$$\frac{1}{2\pi i}\int_\Gamma F(z)\,dz = \sum_{m=1}^{p}\operatorname{Res}[F(z)]_{z=\sqrt{\mu_m}} + \sum_{m=1}^{p}\operatorname{Res}[F(z)]_{z=\left(m-\frac{1}{2}\right)\pi}.$$

From the formulas (23), (25), (26) and the last equality, we obtain

$$T_p(x) = 2\sum_{m=1}^{p}\sin^2\left(m-\tfrac{1}{2}\right)\pi x + \frac{1}{2\pi i}\int_\Gamma F(z)\,dz. \tag{27}$$

LEMMA 4. *For every $x \in [0, 1)$, we have*

$$\int_\Gamma F(z)\,dz = \int_{D_p-i\infty}^{D_p+i\infty} F(z)\,dz. \tag{28}$$

PROOF. We have

$$\int_\Gamma F(z)\,dz = \lim_{E\to\infty}\sum_{j=1}^{4}\int_{\Gamma_j} F(z)\,dz, \tag{29}$$

where Γ_j $(j = 1, 2, 3, 4)$ are the edges of the rectangle whose contour is Γ. The integral

$$\int_{\Gamma_1} F(z)\,dz = \int_{iE}^{-iE} F(z)\,dz$$

may be shown as follows:

$$\int_{\Gamma_1} F(z)\,dz = \lim_{r\to 0}\left[\int_{iE}^{ir} F(z)\,dz + \int_{\substack{|z|=r\\ \operatorname{Re} z\geq 0}} F(z)\,dz + \int_{-ir}^{-iE} F(z)\,dz\right].$$

Since $F(z)$ is an odd function, we have

$$\int_{iE}^{ir} F(z)\,dz + \int_{-ir}^{-iE} F(z)\,dz = 0.$$

Moreover, since $\lim_{z\to 0} F(z) = 0$, from the last relations, we obtain

$$\int_{\Gamma_1} F(z)\,dz = 0. \tag{30}$$

For large values of $|z|$ and $u \geq 0$, we have

$$|F(z)| \leq \text{const. } e^{-(2-2x)|v|},$$

where $z = u + iv$. So, for every constant value of D_p and every $x \in [0, 1)$, the integrals over the upper and the lower edges of the above mentioned rectangle approaches zero as $E \to \infty$, i.e.,

$$\lim_{E\to\infty} \int_{\Gamma_2} F(z)\,dz = \lim_{E\to\infty} \int_{\Gamma_4} F(z)\,dz = 0.$$

From (29), (30) and the last equalities, the formula (28) is obtained. Lemma 4 is proved.

From (27) and (28) we find

$$T_p(x) = 2\sum_{m=1}^{p} \sin^2\!\left(m - \tfrac{1}{2}\right)\pi x + T_p^1(x), \qquad x \in [0, 1] \tag{31}$$

where $T_p^1(x) = \int_{D_p - i\infty}^{D_p + i\infty} F(z)\,dz$. For large values of p, it can be shown that the function $T_p^1(x)$ satisfies the equalities

$$|T_p^1(x)| < \text{const. } p^{\varepsilon - 1}, \qquad x \in [0, 1 - p^{-\varepsilon}), \tag{32}$$

$$|T_p^1(x)| < \text{const. } p^{1-\varepsilon}, \qquad x \in [1 - p^{-\varepsilon}, 1], \tag{33}$$

where ε is a constant number belonging to the interval $\left(\frac{1}{2}, 1\right)$.

From (32) and (33) we have

$$\begin{aligned}
&\lim_{p\to\infty}\left|\sum_{n=1}^{\infty}\int_0^1 (Q(x)\varphi_n, \varphi_n) T_p^1(x)\,dx\right| \\
&\quad = \lim_{p\to\infty}\left|\int_0^{1-p^{-\varepsilon}} \operatorname{tr} Q(x) T_p^1(x)\,dx + \int_{1-p^{-\varepsilon}}^1 \operatorname{tr} Q(x) T_p^1(x)\,dx\right| \\
&\quad \leq \text{const. } \lim_{p\to\infty}\left[\int_0^{1-p^{-\varepsilon}} p^{\varepsilon-1}\,dx + \int_{1-p^{-\varepsilon}}^1 p^{1-\varepsilon}\,dx\right] \\
&\quad = \text{const. } \lim_{p\to\infty}\left[p^{\varepsilon-1} - p^{-1} + p^{1-2\varepsilon}\right] = 0
\end{aligned} \tag{34}$$

THEOREM 5. *If the operator function $Q(x)$ satisfies conditions* (1°)–(3°) *then*

$$\lim_{p\to\infty} M_p^j = 0, \qquad j \geq 2, \tag{35}$$

$$\lim_{p\to\infty} M_{pN} = 0, \qquad N \geq 4. \tag{36}$$

PROOF. For $j = 2$ from (22), we write

$$\begin{aligned} M_p^2 &= \frac{1}{4\pi i}\int_{|\lambda|=b_p} \operatorname{tr}(QR_\lambda^0)^2 d\lambda \\ &= \frac{1}{4\pi i}\int_{|\lambda|=b_p} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}((QR_\lambda^0)^2\psi_{mn}^0, \psi_{mn}^0)_{H_1}\, d\lambda. \end{aligned} \tag{37}$$

Moreover, we know that

$$QR_\lambda^0\psi_{mn}^0 = \frac{Q\psi_{mn}^0}{\mu_m - \lambda}$$

and

$$\begin{aligned} (QR_\lambda^0)^2\psi_{mn}^0 &= (\mu_m - \lambda)^{-1} QR_\lambda^0 Q\psi_{mn}^0 \\ &= (\mu_m - \lambda)^{-1} QR_\lambda^0\left\{\sum_{r=1}^{\infty}\sum_{q=1}^{\infty}(Q\psi_{mn}^0, \psi_{rq}^0)_{H_1}\psi_{rq}^0\right\} \\ &= (\mu_m - \lambda)^{-1}\sum_{r=1}^{\infty}\sum_{q=1}^{\infty}(\mu_r - \lambda)^{-1}(Q\psi_{mn}^0, \psi_{rq}^0)_{H_1} Q\psi_{rq}^0. \end{aligned} \tag{38}$$

From (37) and (38), we have

$$M_p^2 = \frac{1}{4\pi i}\int_{|\lambda|=b_p}\left[\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{r=1}^{\infty}\sum_{q=1}^{\infty}\frac{(Q\psi_{mn}^0, \psi_{rq}^0)_{H_1}(Q\psi_{rq}^0, \psi_{mn}^0)_{H_1}}{(\lambda - \mu_m)(\lambda - \mu_r)}\right] d\lambda. \tag{39}$$

It is easy to verify that, for $m, r \leq p$

$$\int_{|\lambda|=b_p} \frac{d\lambda}{(\lambda - \mu_m)(\lambda - \mu_r)} = 0. \tag{40}$$

This formula is true also for $m, r > p$. Then from (39) and (40), we have

$$M_p^2 = \frac{1}{2\pi i}\sum_{m=1}^{p}\sum_{n=1}^{\infty}\sum_{r=p+1}^{\infty}\sum_{q=1}^{\infty}\left|(Q\psi_{mn}^0, \psi_{rq}^0)_{H_1}\right|^2 \int_{|\lambda|=b_p}(\lambda-\mu_m)^{-1}(\lambda-\mu_r)^{-1}\,d\lambda$$

$$= \sum_{m=1}^{p}\sum_{n=1}^{\infty}\sum_{r=p+1}^{\infty}\sum_{q=1}^{\infty}(\mu_m-\mu_r)^{-1}\left|(Q\psi_{mn}^0, \psi_{rq}^0)_{H_1}\right|^2$$

which implies that

$$\begin{aligned} |M_p^2| &\le \sum_{r=p+1}^{\infty}\sum_{q=1}^{\infty}(\mu_r-\mu_p)^{-1}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left|(Q\psi_{rq}^0, \psi_{mn}^0)_{H_1}\right|^2 \\ &= \sum_{r=p+1}^{\infty}(\mu_r-\mu_p)^{-1}\sum_{q=1}^{\infty}\|Q\psi_{rq}^0\|_{H_1}^2. \end{aligned} \tag{41}$$

Using (2), (3) and condition (3°), we estimate the expression $\sum_{q=1}^{\infty}\|Q\psi_{rq}^0\|_{H_1}^2$:

$$\begin{aligned} \sum_{q=1}^{\infty}\|Q\psi_{rq}^0\|_{H_1}^2 &= \sum_{q=1}^{\infty}\int_0^1 \|Q(x)\alpha_m \sin\sqrt{\mu_m}x\varphi_n\|_H^2\,dx \\ &\le 2\sum_{q=1}^{\infty}\int_0^{\pi}\|Q(x)\varphi_q\|_H^2\,dx \\ &= 2\sum_{q=1}^{\infty}\|Q(x)\varphi_q\|_{H_1}^2 < C, \end{aligned} \tag{42}$$

where C is a positive constant. From (5), (41) and (42), we have

$$|M_p^2| \le C\sum_{r=p+1}^{\infty}(\mu_r-\mu_p)^{-1} \le C_1\sum_{r=p+1}^{\infty}\left(\left(r-\tfrac{1}{2}\right)^2-\left(p-\tfrac{1}{2}\right)^2\right)^{-1}$$

where C_1 is a positive constant.

It can be shown that the inequality

$$\sum_{r=p+1}^{\infty}\left(\left(r-\tfrac{1}{2}\right)^2-\left(p-\tfrac{1}{2}\right)^2\right)^{-1} < 2\left(p-\tfrac{1}{2}\right)^{-1/2} \tag{43}$$

is true. From the last two inequalities, we have

$$\lim_{p\to\infty} M_p^2 = 0.$$

In a similar form it can be proved that

$$\lim_{p\to\infty} M_p^3 = 0.$$

We shall now prove formula (35). For this we estimate the expression $\|QR_\lambda^0\|_{\sigma_1(H_1)}$ on the circle $|\lambda| = b_p$. As shown in [8]

$$\|QR_\lambda^0\|_{\sigma_1(H_1)} \le \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \|QR_\lambda^0 \psi_{mn}^0\|_{H_1}.$$

By (4) and condition (3°), we get

$$\begin{aligned}
&\|QR_\lambda^0\|_{\sigma_1(H_1)} \\
&\quad \le C_2 \sum_{m=1}^{\infty} |\mu_m - \lambda|^{-1} \\
&\quad \le C_2 \Bigg[\sum_{m=1}^{p} (|\lambda| - \mu_m)^{-1} + \sum_{m=p+1}^{\infty} (\mu_m - |\lambda|)^{-1}\Bigg] \\
&\quad < C_2 \Bigg[\sum_{m=1}^{p} (\mu_p + \mu_{p+1} - 2\mu_m)^{-1} + \sum_{m=p+1}^{\infty} (2\mu_m - \mu_p - \mu_{p+1})^{-1}\Bigg] \\
&\quad < C_2 \Bigg[\sum_{m=1}^{p} (\mu_{p+1} - \mu_m)^{-1} + \sum_{m=p+1}^{\infty} (\mu_m - \mu_p)^{-1}\Bigg] \\
&\quad < C_2 \Bigg[p(\mu_{p+1} - \mu_m)^{-1} + \sum_{m=p+1}^{\infty} (\mu_m - \mu_p)^{-1}\Bigg], \qquad C_2 > 0.
\end{aligned} \tag{44}$$

It is easy to see that the inequality

$$|\mu_m - \mu_p| \ge C_3 \left|\left(m - \tfrac{1}{2}\right)^2 - \left(p - \tfrac{1}{2}\right)^2\right|, \qquad C_3 > 0 \tag{45}$$

is true. From (43), (44) and (45), we have

$$\begin{aligned}
\|QR_\lambda^0\|_{\sigma_1(H_1)} &\le C_4 \Bigg[1 + \sum_{m=p+1}^{\infty} \left(\left(m - \tfrac{1}{2}\right)^2 - \left(p - \tfrac{1}{2}\right)^2\right)^{-1}\Bigg] \\
&\le C_5, \quad C_4 > 0, \quad C_5 > 0, \quad (|\lambda| = b_p).
\end{aligned} \tag{46}$$

Now we estimate $\|R_\lambda^0\|_{H_1}$ on the circle $|\lambda| = b_p$. For $m \le p$,

$$|\mu_m - \lambda| \ge |\lambda| - \mu_m = \tfrac{1}{2}(\mu_{p+1} - \mu_p) - \mu_m > \tfrac{1}{2}(\mu_{p+1} - \mu_p) \ge \text{const.}\ p$$

and for $m \geq p+1$

$$|\mu_m - \lambda| \geq \mu_m - |\lambda| > \tfrac{1}{2}(\mu_{p+1} - \mu_p) \geq \text{const. } p.$$

Hence

$$|\mu_m - \lambda|^{-1} \leq \text{const. } p^{-1}, \qquad |\lambda| = b_p \tag{47}$$

On the other hand

$$\| R_\lambda^0 \|_{H_1} = \max_m \{ \, |\mu_m - \lambda|^{-1} \}.$$

From here and (47), we have

$$\| R_\lambda^0 \|_{H_1} < C_6 \cdot p^{-1}, \qquad C_6 > 0. \tag{48}$$

Using Theorem 2 and condition ($3°$) it can be shown that, on the circle $|\lambda| = b_p$, for sufficiently large p,

$$\| R_\lambda \|_{H_1} < C_7 p^{-1}, \qquad C_7 > 0. \tag{49}$$

From (19) and (48) and since $Q(x)$ satisfies the condition ($2°$), we have

$$\begin{aligned}
|M_p^j| &= \frac{1}{2\pi j} \left| \int_{|\lambda| = b_p} \operatorname{tr}(Q R_\lambda^0)^j \, d\lambda \right| \\
&\leq \frac{1}{2\pi j} \int_{|\lambda| = b_p} \| (Q R_\lambda^0)^j \|_{H_1} \, |d\lambda| \\
&\leq \frac{1}{2\pi j} \int_{|\lambda| = b_p} \| Q R_\lambda^0 \|_{\sigma_1(H_1)} \| (Q R_\lambda^0)^{j-1} \|_{H_1} \, |d\lambda| \\
&\leq \text{const.} \int_{|\lambda| = b_p} \| Q \|_{H_1}^{j-1} \| R_\lambda^0 \|_{H_1}^{j-1} \, |d\lambda| \\
&\leq \text{const.} \int_{|\lambda| = b_p} p^{1-j} \, |d\lambda| < \text{const. } p^{3-j}.
\end{aligned}$$

From here, we get

$$\lim_{p \to \infty} M_p^j = 0, \qquad j \geq 4$$

and so formula (34) is proved.

We now prove formula (35). From (20), (46), (48) and (49), we have

$$\begin{aligned}|M_{pN}| &= \frac{1}{2\pi}\left|\int_{|\lambda|=b_p} \lambda\,\mathrm{tr}[R_\lambda(QR_\lambda^0)^{N+1}]\,d\lambda\right| \\ &\le b_p \int_{|\lambda|=b_p} \|R_\lambda(QR_\lambda^0)^{N+1}\|_{\sigma_1(H_1)}\,|d\lambda| \\ &\le b_p \int_{|\lambda|=b_p} \|R_\lambda\|_{H_1}\|(QR_\lambda^0)^{N+1}\|_{\sigma_1(H_1)}\,|d\lambda| \\ &\le C_7 b_p p^{-1}\int_{|\lambda|=b_p} \|QR_\lambda^0\|_{H_1}^N\|QR_\lambda^0\|_{\sigma_1(H_1)}\,|d\lambda| \le C_8 p^{3-N}, \quad C_8>0.\end{aligned}$$

From here, we get

$$\lim_{p\to\infty} M_{pN} = 0, \qquad N \ge 4.$$

Theorem 5 is proved.

The main result of this article is given by the following theorem.

THEOREM 6. *If the operator function $Q(x)$ satisfies conditions* (1°)–(3°) *then*

$$\sum_{m=1}^{\infty}\left[\sum_{n=1}^{\infty}\left(\lambda_{mn}-\left(m-\tfrac{1}{2}\right)^2\right)-\int_0^1 \mathrm{tr}\,Q(x)\,dx\right] = \frac{1}{4}[\mathrm{tr}\,Q(1)-\mathrm{tr}\,Q(0)].$$

The series on the left side of this equality is called the regularized trace of the operator L.

PROOF. From (21), (23), (24), (31), (34), (35) and (36), we obtain

$$\begin{aligned}&\lim_{p\to\infty}\left[\sum_{m=1}^{p}\sum_{n=1}^{\infty}(\lambda_{mn}-\mu_n) - p\int_0^1 \mathrm{tr}\,Q(x)\,dx\right] \\ &= \lim_{p\to\infty}\left[\sum_{m=1}^{p}\sum_{n=1}^{\infty}(\lambda_{mn}-\mu_n) - \sum_{m=1}^{p}\int_0^1 \mathrm{tr}\,Q(x)\,dx\right] \\ &= -\lim_{p\to\infty}\sum_{m=1}^{p}\int_0^1 \mathrm{tr}\,Q(x)\cos(2m-1)\pi x\,dx \\ &= -\frac{1}{2}\sum_{m=1}^{\infty}\left[\int_0^1 \mathrm{tr}\,Q(x)\cos m\pi x\,dx - (-1)^m\int_0^1 \mathrm{tr}\,Q(x)\cos m\pi x\,dx\right] \\ &= -\frac{1}{4}\sum_{m=1}^{\infty}\left\{\left[\int_0^1 \mathrm{tr}\,Q(x)\sqrt{2}\cos m\pi x\,dx\right]\sqrt{2}\cos m\pi\cdot 0\right.\end{aligned}$$

$$
- \left[\int_0^1 \operatorname{tr} Q(x) \sqrt{2} \cos m\pi x \, dx \right] \sqrt{2} \cos m\pi \cdot 1 \Bigg\}
$$

$$
= -\frac{1}{4} [\operatorname{tr} Q(0) - \operatorname{tr} Q(1)]
$$

This proves Theorem 6.

Note that this formula is also valid for $a \leq 0$.

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