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COMPOSITION OPERATORS AND CLOSURES OF SOME MÖBIUS INVARIANT SPACES IN THE BLOCH SPACE

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Abstract

Boundedness and compactness of composition operators between the Bloch space and the closure of some Möbius invariant subspaces in the Bloch space are characterized.

1. Introduction

Let $D = \{z : |z| < 1\}$ be the unit disk in the complex plane. Let $H(D)$ be the space of all analytic functions on the unit disk D. Every analytic self map φ of D induces through composition a linear *composition operator* C_φ from $H(D)$ to itself. Thus C_φ is defined by $C_\varphi(f) = f \circ \varphi$ for $f \in H(D)$.

Composition operators have been extensively studied recently. One of the main themes for studying composition operators is to relate operator theoretical problems for C_φ with the function theoretical properties of the inducing map φ. Much work on composition operators focused on classical function spaces such as Hardy spaces, Bergman spaces and the Bloch space. More recently, composition operators on some newly introduced spaces such as Q_p spaces and the general function spaces $F(p, q, s)$ were also studied. In this note we continue this line of study, by investigating composition operators mapping into the closures of some Möbius invariant spaces in the Bloch space B.

Here it comes to the second ingredient of the story. For the classical space *BMOA*, Jones found a nice formula from the Bloch functions to this space. It was recorded by Anderson in [1] and a proof was given in [8]. From this formula, one immediately obtains a characterization of the closure of *BMOA* in the Bloch space B. The Jones' formula was recently generalized by the second author from *BMOA* to some more general Möbius function space $F(p, p - 2, s)$ in [23]. Similarly, characterizations of the closure of $F(p, p - 2, s)$ immediately follow. These closures lie between the Bloch space B and the

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little Bloch space B_0. While much has been known on composition operators from the Bloch space and the little Bloch space to *BMOA*, Q_s and $F(p, q, s)$ spaces (see, for example, [15], [19], [11] and [9]), it is natural to consider what are the behaviors of the composition operator on those closures. This note, as the authors know, is the first attempt to study this problem.

In this note, we characterize bounded composition operators from the Bloch space B and the little Bloch space B_0 into the closure of $F(p, p-2, s)$ (which include *BMOA*) in B. We also characterize compact composition operators from B and B_0 into the closure of $F(p, p-2, s)$, as well as compact composition operators on the closure of $F(p, p-2, s)$. We will also show that, similar to the case of *BMOA*, every composition operator is bounded on the closure of *BMOA* in B.

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2. The closure of $F(p, p-2, s)$ in the Bloch space

For $a \in D$, let $g(z, a) = \log(1/|\varphi_a(z)|)$ be the Green's function for D with pole at a, where $\varphi_a(z) = (a-z)/(1-\bar{a}z)$ is a Möbius transformation on the unit disk D. Let $0 < p < \infty$, $-2 < q < \infty$, $0 < s < \infty$, $-1 < q + s < \infty$, and let f be an analytic function on D. We say that $f \in F(p, q, s)$, if

$$\|f\|_{p,q,s}^p = \sup_{a\in D} \int_D |f'(z)|^p (1-|z|^2)^q g^s(z, a)\, dA(z) < \infty;$$

$f \in F_0(p, q, s)$, if

$$\lim_{|a|\to 1} \int_D |f'(z)|^p (1-|z|^2)^q g^s(z, a)\, dA(z) = 0.$$

Here $dA(z) = dx\,dy/\pi$ is the Lebesgue area measure normalized so that $A(D) = 1$.

The spaces $F(p, q, s)$ were introduced in [22] and have been studied by any authors. See, for example, [12], [14], [16] and [17]. The class of $F(p, q, s)$ spaces is very general in the sense that, with particular choices of the parameters p, q and s, it includes many classical function spaces such as the Bloch space, *BMOA*, weighted Bergman spaces, weighted Dirichlet spaces, analytic Besov spaces and Q_s spaces. See [22] for details.

We recall also that the Bloch space B is the space of analytic functions on D satisfying

$$\|f\|_B = \sup_{z\in D} |f'(z)|(1-|z|^2) < \infty,$$

and the little Bloch space B_0 is the space of functions f analytic on D for which $|f'(z)|(1-|z|^2) \to 0$ as $|z| \to 1$. It is known that B is a Banach space under the norm

$$\|f\|_B^* = |f(0)| + \|f\|_B$$

and B_0 is the closure of polynomials in B.

It is known that for $s > 1$, $F(p, p-2, s) = B$, $F_0(p, p-2, s) = B_0$; and for $0 < s \leq 1$, $F(p, p-2, s)$ is a subspace of B, and $F_0(p, p-2, s)$ is a subspace of B_0 (see, [22, p. 13]). It is also known that $F(2, 0, s) = Q_s$ and $F_0(2, 0, s) = Q_{s,0}$, which were introduced in [3], [5] (See also [20] and [21] for the case $s = 2$). For the case $s = 1$, we have $F(2, 0, 1) = Q_1 = BMOA$ and $F_0(2, 0, 1) = Q_{1,0} = VMOA$ (see, for example, [6]). We note that, for $0 \leq s < \infty$, $F(p, p-2, s)$ and $F_0(p, p-2, s)$ are Möbius invariant function spaces (see, [2]).

We also need the notion of Carleson measures. For $0 < s < \infty$, we say that a positive measure μ defined on D is an s-Carleson measure provided $\mu(S(I)) = O(|I|^s)$ for all subarcs I of ∂D, where $|I|$ denotes the normalized arc length of I and $S(I) = \{z \in D : z/|z| \in I, 1-|z| \leq |I|\}$ denotes the usual Carleson box based on I. If $\mu(S(I)) = o(|I|^s)$, as $|I| \to 0$, then we say that μ is a vanishing s-Carleson measure (cf. [4]).

In this note, we will denote the closure of the spaces $F(p, p-2, s)$ in the Bloch space by $C_B(F(p, p-2, s))$, and equip the functions in $C_B(F(p, p-2, s))$ with the Bloch norm. In [16], the closure $C_B(F(p, p-2, s))$ is characterized as follows:

THEOREM A. *Let $0 < s \leq 1$, $1 \leq p < \infty$ and $0 \leq t < \infty$. Let f be an analytic function on D. Then the following conditions are equivalent.*

(A) $f \in C_B(F(p, p-2, s))$;

(B) $\chi_{\Omega_\varepsilon(f)}(1-|z|^2)^{s-2}\,dA(z)$ *is an s-Carleson measure for every $\varepsilon > 0$;*

(C) $\sup_{a\in D} \int_{\Omega_\varepsilon(f)} |f'(z)|^t (1-|z|^2)^{t-2}(1-|\varphi_a(z)|^2)^s\,dA(z) < \infty$ *for every $\varepsilon > 0$;*

(D) $\sup_{a\in D} \int_{\Omega_\varepsilon(f)} |f'(z)|^t (1-|z|^2)^{t-2} g^s(z, a)\,dA(z) < \infty$ *for every $\varepsilon > 0$,*

where $\Omega_\varepsilon(f) = \{z \in D : |f'(z)|(1-|z|^2) \geq \varepsilon\}$.

REMARK 1. Note that $F(2, 0, 1) = BMOA$. In this case, the result of Theorem A was first obtained by Jones, and a proof was given in [8].

REMARK 2. When $t = 0$, condition (C) and (D) in the above theorem can be written as

$$\sup_{a\in D} \int_{\Omega_\varepsilon(f)} (1-|\varphi_a(z)|^2)^s\,d\lambda(z) < \infty \tag{1}$$

and

$$\sup_{a\in D}\int_{\Omega_\varepsilon(f)} g^s(z,a)\,d\lambda(z)<\infty \tag{2}$$

for every $\varepsilon>0$, where $d\lambda(z)=dA(z)/(1-|z|^2)^2$ is the hyperbolic measure on the unit disk D.

REMARK 3. Note that, this theorem tells us that $C_B(F(p,p-2,s))$ is independent of p. So, for example, $C_B(Q_s)=C_B(F(p,p-2,s))$ and $C_B(BMOA)=C_B(F(p,p-2,1))$ for any $p\geq 1$.

3. Boundedness

We will use hyperbolic derivative in our results. Recall that if φ is an analytic self-map of D, then the hyperbolic derivative of φ is defined by

$$\varphi^h(z)=\frac{|\varphi'(z)|}{1-|\varphi(z)|^2}.$$

We first characterize bounded composition operators from B to $C_B(F(p, p-2,s))$.

THEOREM 1. *Let* $0<s\leq 1$, $1\leq p<\infty$. *Let* φ *be an analytic self-map of* D. *Then* C_φ *is bounded from the Bloch space* B *to* $C_B(F(p,p-2,s))$ *if and only if*

$$\sup_{a\in D}\int_{\Omega^h_\varepsilon(\varphi)}(1-|\varphi_a(z)|^2)^s\,d\lambda(z)<\infty \tag{3}$$

for every $\varepsilon>0$, *where* $\Omega^h_\varepsilon(\varphi)=\{z\in D:\varphi^h(z)(1-|z|^2)\geq\varepsilon\}$.

PROOF. Let (3) be true. Let $f\in B$. Then

$$\begin{aligned}|(f\circ\varphi)'(z)|(1-|z|^2)&=|f'(\varphi(z))|(1-|\varphi(z)|^2)\frac{|\varphi'(z)|(1-|z|^2)}{1-|\varphi(z)|^2}\\&\leq\|f\|_B\varphi^h(z)(1-|z|^2).\end{aligned}$$

Thus for any fixed $\varepsilon>0$, if $|(f\circ\varphi)'(z)|(1-|z|^2)\geq\varepsilon$ then $\varphi^h(z)(1-|z|^2)\geq\varepsilon/\|f\|_B=\varepsilon'$. In other words,

$$\Omega_\varepsilon(f\circ\varphi)\subset\Omega^h_{\varepsilon'}(\varphi).$$

Thus

$$\sup_{a\in D}\int_{\Omega_\varepsilon(f\circ\varphi)}(1-|\varphi_a|^2)^s\,d\lambda(z)\leq\sup_{a\in D}\int_{\Omega^h_{\varepsilon'}(\varphi)}(1-|\varphi_a|^2)^s\,d\lambda(z)<\infty.$$

By Theorem A, $f\circ\varphi \in C_B(F(p, p-2, s))$. The Schwarz-Pick Lemma implies that $\|C_\varphi(f)\|_B \le \|f\|_B$. Thus C_φ maps B boundedly into $C_B(F(p, p-2, s))$.

Conversely, let C_φ be bounded from B into $C_B(F(p, p-2, s))$. By Proposition 5.4 of [15], there exist two functions $f_1, f_2 \in B$ such that

$$|f_1'(z)| + |f_2'(z)| \ge \frac{1}{1-|z|^2}. \tag{4}$$

By our assumption, both $f_1\circ\varphi$ and $f_2\circ\varphi$ are in $C_B(F(p, p-2, s))$. Given any $\varepsilon > 0$. Let $z \in \Omega_\varepsilon^h(\varphi)$. Then $\varphi^h(z)(1-|z|^2) \ge \varepsilon$. By (4),

$$\begin{aligned}(|f_1'(\varphi(z))| + |f_2'(\varphi(z))|)|\varphi'(z)|(1-|z|^2) &\ge \frac{|\varphi'(z)|(1-|z|^2)}{1-|\varphi(z)|^2}\\ &= \varphi^h(z)(1-|z|^2) \ge \varepsilon.\end{aligned}$$

Thus

$$(|(f_1\circ\varphi)'(z)| + |(f_2\circ\varphi)'(z)|)(1-|z|^2) \ge \varepsilon.$$

Therefore, either

$$|(f_1\circ\varphi)'(z)|(1-|z|^2) \ge \frac{\varepsilon}{2}$$

or

$$|(f_2\circ\varphi)'(z)|(1-|z|^2) \ge \frac{\varepsilon}{2}.$$

Hence $z \in \Omega_{\varepsilon/2}(f_1\circ\varphi) \cup \Omega_{\varepsilon/2}(f_2\circ\varphi)$. So

$$\begin{aligned}&\sup_{a\in D}\int_{\Omega_\varepsilon^h(\varphi)} (1-|\varphi_a(z)|^2)^s\,d\lambda(z)\\ &\quad\le \sup_{a\in D}\int_{\Omega_{\varepsilon/2}(f_1\circ\varphi)\cup\Omega_{\varepsilon/2}(f_2\circ\varphi)} (1-|\varphi_a(z)|^2)^s\,d\lambda(z)\\ &\quad\le \sup_{a\in D}\int_{\Omega_{\varepsilon/2}(f_1\circ\varphi)} (1-|\varphi_a(z)|^2)^s\,d\lambda(z) + \sup_{a\in D}\int_{\Omega_{\varepsilon/2}(f_2\circ\varphi)} (1-|\varphi_a(z)|^2)^s\,d\lambda(z)\\ &\quad< \infty.\end{aligned}$$

For the last inequality we have used the fact that $f_1\circ\varphi$ and $f_2\circ\varphi$ are both in $C_B(F(p, p-2, s))$ and Theorem A. Thus (3) is true. The proof is complete.

Our second result gives a characterization of bounded composition operators from the little Bloch space B_0 to $C_B(F(p, p-2, s))$.

THEOREM 2. *Let* $0 < s \le 1$, $1 \le p < \infty$. *Let* φ *be an analytic self-map of* D. *Then* C_φ *is bounded from the little Bloch space* B_0 *to* $C_B(F(p, p-2, s))$ *if and only if* $\varphi \in C_B(F(p, p-2, s))$.

PROOF. Suppose C_φ is bounded from B_0 to $C_B(F(p, p-2, s))$. Since $\mathrm{Id}(z) = z \in B_0$ we get $\varphi = C_\varphi(\mathrm{Id}) \in C_B(F(p, p-2, s))$.

Conversely, suppose $\varphi \in C_B(F(p, p-2, s))$. Then, by Theorem A,

$$\sup_{a\in D} \int_{\Omega_\varepsilon(\varphi)} (1 - |\varphi_a(z)|^2)^s \, d\lambda(z) < \infty$$

for any $\varepsilon > 0$. Let $f \in B_0$. Then, for every $\varepsilon > 0$, there is a constant r, $0 < r < 1$, such that

$$|f'(w)|(1 - |w|^2) < \varepsilon/2 \tag{5}$$

whenever $|w| > r$. Let $z \in \Omega_\varepsilon(f \circ \varphi)$. Then, by the Schwarz-Pick Lemma,

$$\varepsilon \le |f'(\varphi(z))||\varphi'(z)|(1 - |z|^2) \le |f'(\varphi(z))|(1 - |\varphi(z)|^2).$$

Thus, by (5), $|\varphi(z)| \le r$. Hence,

$$\begin{aligned}
\varepsilon &\le |f'(\varphi(z))||\varphi'(z)|(1 - |z|^2) \\
&\le \|f\|_B \frac{|\varphi'(z)|}{1 - |\varphi(z)|^2}(1 - |z|^2) \\
&\le \frac{\|f\|_B}{1 - r^2}|\varphi'(z)|(1 - |z|^2).
\end{aligned}$$

Let $\varepsilon' = (1 - r^2)\varepsilon/\|f\|_B$. Then

$$|\varphi'(z)|(1 - |z|^2) \ge \varepsilon'.$$

Thus $\Omega_\varepsilon(f \circ \varphi) \subset \Omega_{\varepsilon'}(\varphi)$. Therefore

$$\begin{aligned}
\sup_{a\in D} \int_{\Omega_\varepsilon(f\circ\varphi)} (1 - |\varphi_a(z)|^2)^s \, d\lambda(z) &\le \sup_{a\in D} \int_{\Omega_{\varepsilon'}(\varphi)} (1 - |\varphi_a(z)|^2)^s \, d\lambda(z) \\
&< \infty.
\end{aligned}$$

By Theorem A again, we know that $f \circ \varphi \in C_B(F(p, p-2, s))$. Again, the Schwarz-Pick Lemma shows that C_φ is bounded from B_0 to $C_B(F(p, p-2, s))$. The proof is complete.

To close this section, we show that every composition operator is bounded on $C_B(BMOA)$.

PROPOSITION 3. *For any analytic self-map φ of D, C_φ is a bounded operator on $C_B(BMOA)$.*

PROOF. Since every composition operator C_φ is continuous on B and maps *BMOA* into itself. It is clear from this that C_φ maps $C_B(BMOA)$ into itself as well. Thus C_φ is bounded on C_B(*BMOA*) by the Closed Graph Theorem. The proof is complete.

4. Compactness

For studying compactness, we need the following auxiliary result, which may be of independent interest.

PROPOSITION 4. *Let φ be an analytic self-map of D. Then C_φ is compact from B_0 to B if and only if*

$$\lim_{r\to 1} \sup_{z\in E_r(\varphi)} \varphi^h(z)(1-|z|^2) = 0, \tag{6}$$

where $E_r(\varphi) = \{z \in D : |\varphi(z)| \geq r\}$.

PROOF. Let (6) be true. By Theorem 2 in [13], we know that C_φ is compact from B to B. Since $B_0 \subset B$, obviously C_φ is also compact from B_0 to B.

Conversely, let C_φ be compact from B_0 to B. Suppose, on the contrary, that

$$\lim_{r\to 1} \sup_{z\in E_r(\varphi)} \varphi^h(z)(1-|z|^2) \neq 0.$$

Then there exists a sequence of points $\{a_n\}$ in D such that $\varphi(a_n) \to \zeta \in \partial D$, and

$$\lim_{n\to\infty} \varphi^h(a_n)(1-|a_n|^2) \neq 0. \tag{7}$$

Let

$$f_n(z) = \log \frac{2}{1-\overline{\varphi(a_n)}z}.$$

Then it is easy to see that $\|f_n\|_B \leq 2$. Let

$$g_n(z) = f_n^2(z)/f_n(\varphi(a_n)).$$

Then

$$g_n'(z) = 2f_n(z)f_n'(z)/f_n(\varphi(a_n)).$$

It is an easy exercise to show that for each integer $n \geq 1$, $g_n \in B_0$, $g_n(z) \to 0$ uniformly on any compact subset of D as $n \to \infty$, and $\|g_n\|_B \leq 8(1 + 2\pi/\log 2)$ for all n. Therefore, from the compactness of $C_\varphi : B_0 \to B$, we get that

$$\lim_{n\to\infty} \|C_\varphi(g_n)\|_B = \lim_{n\to\infty} \sup_{z\in D} |g_n'(\varphi(z))||\varphi'(z)|(1-|z|^2) = 0.$$

Since

$$\begin{aligned}\sup_{z\in D}|g_n'(\varphi(z))||\varphi'(z)|(1-|z|^2) &\geq |g_n'(\varphi(a_n))||\varphi'(a_n)|(1-|a_n|^2)\\ &= 2|f_n'(\varphi(a_n))||\varphi'(a_n)|(1-|a_n|^2)\\ &= \frac{2|\varphi(a_n)||\varphi'(a_n)|}{1-|\varphi(a_n)|^2}(1-|a_n|^2),\end{aligned}$$

we get that

$$\lim_{n\to\infty}\frac{2|\varphi(a_n)||\varphi'(a_n)|}{1-|\varphi(a_n)|^2}(1-|a_n|^2)=0.$$

Since

$$\lim_{n\to\infty}|\varphi(a_n)|=1$$

we get that

$$\lim_{n\to\infty}\frac{|\varphi'(a_n)|}{1-|\varphi(a_n)|^2}(1-|a_n|^2)=0,$$

which contradicts to (7). Therefore (6) has to be true. The proof is complete.

REMARK 4. By Theorem 2 in [13], (6) characterizes compact composition operators on the Bloch space B. Thus the above result says that a composition operator C_φ is compact from B_0 to B if and only if C_φ is compact from B to B.

Our next result characterizes compact composition operators between the Bloch space or the little Bloch space to $C_B(F(p,p-2,s))$, as well as compact composition operators on the closure of $F(p,p-2,s)$

THEOREM 5. *Let $0<s\leq 1$, $1\leq p<\infty$. Let φ be an analytic self-map of D. Then the following conditions are equivalent.*

(i) C_φ *is compact from B to $C_B(F(p,p-2,s))$;*

(ii) C_φ *is compact from B_0 to $C_B(F(p,p-2,s))$;*

(iii) C_φ *is compact on $C_B(F(p,p-2,s))$;*

(iv) $\varphi\in C_B(F(p,p-2,s))$ *and* (6) *holds, i.e.,* $\lim_{r\to 1}\sup_{z\in E_r(\varphi)}\varphi^h(z)(1-|z|^2)=0$,

where $E_r(\varphi)=\{z\in D:|\varphi(z)|\geq r\}$.

PROOF. We prove (iv)⇒(i) ⇒ (iii) ⇒ (ii) ⇒ (iv).

(iv) ⇒ (i): Suppose (iii) holds. Fix any $\varepsilon>0$. By (6), we know that there exists an r, $0<r<1$, such that $\varphi^h(z)(1-|z|^2)<\varepsilon/2$ whenever $|\varphi(z)|\geq r$. Let $z\in\Omega_\varepsilon^h(\varphi)$. Then $\varphi^h(z)(1-|z|^2)\geq\varepsilon$. Thus $|\varphi(z)|<r$. Hence

$$\varepsilon\leq\varphi^h(z)(1-|z|^2)\leq\frac{|\varphi'(z)|}{1-r^2}(1-|z|^2).$$

So

$$\varepsilon(1-r^2) \leq |\varphi'(z)|(1-|z|^2).$$

Let $\varepsilon' = \varepsilon(1-r^2)$, we see that $z \in \Omega_{\varepsilon'}(\varphi)$. Therefore $\Omega_\varepsilon^h(\varphi) \subset \Omega_{\varepsilon'}(\varphi)$. Since $\varphi \in C_B(F(p, p-2, s))$ we get that

$$\sup_{a \in D} \int_{\Omega_\varepsilon^h(\varphi)} (1-|\varphi_a(z)|^2)^s \, d\lambda(z) \leq \sup_{a \in D} \int_{\Omega_{\varepsilon'}(\varphi)} (1-|\varphi_a(z)|^2)^s \, d\lambda(z) < \infty.$$

By Theorem 1, this implies that C_φ is bounded from B to $C_B(F(p, p-2, s))$. By [13], we know that (6) implies that C_φ is compact from B into B. Combining these two results we know that C_φ is compact from B to $C_B(F(p, p-2, s))$.

(i) $\Rightarrow$ (iii): This is obvious since $C(F(p, p-2, s)) \subset B$.

(iii) $\Rightarrow$ (ii): Since B_0 is the closure of all polynomials in B, and the space $F(p, p-2, s)$ contains all polynomials, clearly $B_0 \subset C(F(p, p-2, s))$. The implication (iii) $\Rightarrow$ (ii) is an obvious consequence of this inclusion.

(ii) $\Rightarrow$ (iv): Suppose C_φ is compact from B_0 to $C_B(F(p, p-2, s))$. Then C_φ is bounded from B_0 to $C_B(F(p, p-2, s))$. Since $f(z) = z \in B_0$, we get that $\varphi = C_\varphi(\mathrm{Id}) \in C_B(F(p, p-2, s))$.

Also, since C_φ is compact from B_0 to $C_B(F(p, p-2, s))$ and $C_B(F(p, p-2, s)) \subset B$, we know that C_φ is compact from B_0 to B. By Proposition 4 we get that (6) is true. The proof is complete.

Remark 5. From the above proof and Theorem 1, it is easy to see that the result of Theorem 5 remains true if we replace the condition $\varphi \in C_B(F(p, p-2, s))$ in (iii) by the following condition:

$$\sup_{a \in D} \int_{\Omega_\varepsilon^h(\varphi)} (1-|\varphi_a(z)|^2)^s \, d\lambda(z) < \infty.$$

Letting $p = 1$, $s = 1$ in Theorem 5, we get the following corollary.

Corollary 6. *Let φ be an analytic self-map of D. Then C_φ is compact on $C_B(BMOA)$ if and only if $\varphi \in C_B(BMOA)$ and* $\lim_{r \to 1} \sup_{z \in E_r(\varphi)} \varphi^h(z)(1-|z|^2) = 0$, *where $E_r(\varphi) = \{z \in D : |\varphi(z)| \geq r\}$.*

We note here that compact composition operators on *BMOA* have been characterized by Bourdon, Cima and Matheson in [7] and Smith in [18]. We finish this paper with the following problem.

An Open Problem. Characterize those composition operators that are bounded on $C_B(F(p, p-2, s))$ for $0 < s < 1$.

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